Math 32A, Fall 2005
Sample Final Problems
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1. Let $ABCD$ be a general, not necessarily planar, quadrilateral in space. Show that the two segments joining the midpoints of opposite sides of $ABCD$ bisect each other. (Hint: Show that the segments have the same midpoint.)

   [from Thomas and Finney, *Calculus*, 9^{th} edition, page 806]

2. Suppose that A, B, and C are vertices of a triangle and that a, b, and c are, respectively, the midpoints of the opposite sides. Show that $\vec{Aa} + \vec{Bb} + \vec{Cc} = 0$.

   [from Thomas and Finney, *Calculus*, 9^{th} edition, page 806]

3. Suppose that $\vec{v} = v_1\hat{u_1} + v_2\hat{u_2} + v_3\hat{u_3}$, where $\hat{u_1}, \hat{u_2}, \hat{u_3}$ are mutually orthogonal unit vectors and $v_1, v_2, v_3 \in \mathbb{R}$. Determine an expression for v_i in terms of $\vec{v}$ and $\hat{u_i}$, $i = 1, 2, 3$.

4. Show that, in the plane $\mathbb{R}^2$, if $\vec{a}$ is orthogonal to $\vec{b}$ and $\vec{a}'$ is any vector orthgonal to $\vec{a}$ and $\vec{b}'$ is any vector orthogonal to $\vec{b}$, then $\vec{a}'$ is orthogonal to $\vec{b}'$. Provide a counterexample to the claim if $\vec{a}, \vec{b} \in \mathbb{R}^3$ instead.

5. * Determine necessary and sufficient conditions for the line $y = mx + b$ and the circle $x^2 + y^2 = r^2$ to intersect orthogonally. Determine necessary and sufficient conditions for the parabola $y = ax^2$ and the circle $x^2 + y^2 = r^2$ to intersect orthogonally. Determine necessary and sufficient conditions for the line $y = mx + b$ and the parabola $y = ax^2$ to intersect orthogonally.

   Recall that two planar curves intersect orthogonally if and only if the tangent vectors at the point of intersection are orthogonal if and only if the normal vectors at the point of intersection are orthogonal.

6. If $\vec{a} \neq 0$, and if $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ and $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$, then does $\vec{b} = \vec{c}$? Prove or give a counterexample.

   [from Thomas and Finney, *Calculus*, 9^{th} edition, page 821]

7. Show that if we have a parametrized curve $\alpha : (a, b) \to \mathbb{R}^3$, where a may take the value $-\infty$ and b may take the value $+\infty$, we can always reparametrize the curve to obtain a $\beta : (0, 1) \to \mathbb{R}^3$ such that $\alpha((a, b)) = \beta((0, 1))$ (that is, the images of α and β coincide). Specifically, show we can always find a smooth one-to-one and onto function $\gamma : (0, 1) \to (a, b)$, as then we can set $\beta = \alpha \circ \gamma$.

   Hint: The function $x \mapsto \tan(x)$ maps $(-\pi/2, \pi/2)$ *bijectively* to $(-\infty, \infty)$, while the function $x \mapsto 1/(1 - x)$ maps $(0, 1)$ *bijectively* to $(1, \infty)$. A *bijection* is a one-to-one and onto function. One-to-one means that if you start with two different points in the domain space, you end up with two different points in the range space (so the function passes the horizontal line test). Onto means that you hit every point in the range space with some point in the domain space.

8. * Give conditions for two planes $P_1 : A_1x + B_1y + C_1z = D_1$ and $P_2 : A_2x + B_2y + C_2z = D_2$ to (a) coincide (P_1 and P_2 are the same plane), (b) never intersect (P_1 parallel to P_2), (c) intersect orthogonally (normal vectors are orthogonal), or (d) none of the above.

9. A particle moves on a cycloid in the xy-plane in such a way that its position at time t is

   $$\vec{r}(t) = (t - \sin t)\hat{\imath} + (1 - \cos t)\hat{\jmath}.$$

   Find the maximum and minimum values of $|\vec{v}|$ and $|\vec{a}|$. (Hint: Find the extreme values of $|\vec{v}|^2$ and $|\vec{a}|^2$ first and take square roots later.)

   [from Thomas and Finney, *Calculus*, 9^{th} edition, page 866]

10. Show that if $\vec{u}$, $\vec{v}$, and $\vec{w}$ are differentiable vector functions of t, then

$$\frac{d}{dt}(\vec{u}\cdot\vec{v}\times\vec{w}) = \frac{d\vec{u}}{dt}\cdot\vec{v}\times\vec{w} + \vec{u}\cdot\frac{d\vec{v}}{dt}\times\vec{w} + \vec{u}\cdot\vec{v}\times\frac{d\vec{w}}{dt}.$$

[from Thomas and Finney, *Calculus*, 9^{th} edition, page 867]

11. * Find the arc length of one turn of the helix

$$\vec{r}(t) = (\cos t, \sin t, t).$$

Why does it make sense geometrically for this distance to be equal to the diagonal of a square of side length 2π?

12. * Evaluate or show nonexistence of the following limits.

(a)

$$\lim_{(x,y)\to(0,0), x\neq y} \frac{x - y + 2\sqrt{x} - 2\sqrt{y}}{\sqrt{x} - \sqrt{y}}$$

(b)

$$\lim_{(x,y)\to(0,0), x\neq 0} y \sin\frac{1}{x}$$

(c)

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - xy^2}{x^2 + y^2}$$

(d)

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4 + y^2}$$

[from Thomas and Finney, *Calculus*, 9^{th} edition, page 920 - 922]

13. Find $\partial x/\partial u$ and $\partial y/\partial u$ (in terms of x and y) if the equations $u = x^2 - y^2$ and $v = x^2 - y$ define x and y as functions of the independent variables u and v, adn the partial derivatives exist. Then let $s = x^2 + y^2$ and find $\partial s/\partial u$.

[from Thomas and Finney, *Calculus*, 9^{th} edition, page 932]

14. * If $|a|$ is much greater than $|b|$, $|c|$, and $|d|$, to which of a, b, c, and d is the value of the determinant

$$f(a,b,c,d) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

most sensitive?

[from Thomas and Finney, *Calculus*, 9^{th} edition, page 943]

15. If $f(u,v,w)$ is differentiable and $u = x - y$, $v = y - z$, and $w = z - x$, show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0.$$

[from Thomas and Finney, *Calculus*, 9^{th} edition, page 951]

16. Suppose that $f(x,y,z,w) = 0$ and $g(x,y,z,w) = 0$ determine z and w as differentiable functions of the independent variables x and y and suppose that

$$\frac{\partial f}{\partial z}\frac{\partial g}{\partial w} \neq \frac{\partial f}{\partial w}\frac{\partial g}{\partial z}.$$

Find $\partial z/\partial x$ and $\partial w/\partial y$, assuming that x is independent of y and y is independent of x.

[from Thomas and Finney, *Calculus*, 9^{th} edition, page 957]

17. * A flat circular plate has the shape of the region $x^2 + y^2 \leq 1$. The plate, including the boundary where $x^2 + y^2 = 1$, is heated so that the temperature at the point (x,y) is

$$T(x,y) = x^2 + 2y^2 - x.$$

Find the temperature at the hottest and coldest points on the plate.

[from Thomas and Finney, *Calculus*, 9^{th} edition, page 976]

Note that this involves finding the critical points of f (and determining their nature) *as well as* maximizing or minimizing f constrained to the unit circle.

18. Use Lagrange multipliers to determine the maximum value of $\prod_{i=1}^{n} a_i = a_1 \cdot \cdots \cdot a_n$ subject to the constraint $\sum_{i=1}^{n} a_i = a_1 + \cdots + a_n = 1$ and each $a_i > 0$. Show that a corollary to this result is the Arithmetic Mean-Geometric Mean Inequality Theorem, which states that the arithmetic mean of a set of a positive real numbers is greater than or equal to the geometric mean:

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geq \left(\prod_{i=1}^{n} a_i\right)^{1/n}.$$

19. * Let $a_1, \ldots, a_n$ be n positive numbers. Find the maximum of $\sum_{i=1}^{n} a_i x_i$ subject to the constraint $\sum_{i=1}^{n} x_i^2 = 1$.

[from Thomas and Finney, *Calculus*, 9^{th} edition, page 989]

20. * Recall that if $f : \mathbb{R}^2 \to \mathbb{R}$ is a real-valued function of two variables, the nature of the critical points of f can be found by analyzing the determinant of the matrix of second partials. Lecture provided one explanation of why this works; this problem gives another way of looking at the situation.

    (a) Let $P \in \mathbb{R}^2$, and let P_θ be the image of P rotated about the origin an angle θ in the counterclockwise direction. For example, if $P = (1,0)$, then $P_{\pi/2} = (0,1)$. If $P = (x,y)$, find the Cartesian coordinates of $P_\theta = (x', y')$ in terms of x, y, and θ. Hint: Express (x,y) in polar coordinates (r,α), then determine the polar coordinates of (r', α') in terms of r and α after rotating by θ, then transform back to Cartesian coordinates (x', y'). You may want to find the inverse transformation as well to use in (b) (but that's just rotation by $-\theta$).

    (b) A *quadratic plane curve* Q, specified by an ordered set of parameters $\{A, \ldots, F\} \subset \mathbb{R}$, is the set of $(x,y) \in \mathbb{R}^2$ satisfying

    $$q(x,y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

    If $P = (x,y) \in Q$ (so (x,y) satisfies the above equation), determine an equation which $(x', y') = P_\theta$ satisfies. Specifically, you will be able to determine the constants $A', \ldots, F'$ such that

    $$q'(x', y') = A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0,$$

    where $A', \ldots, F'$ will be expressed in terms of $A, \ldots, F$ and θ. There will be a moderate amount of algebra. Denote this new quadratic curve by $Q' = Q_\theta$; notice that, geometrically, it is essentially "the same" as Q, the only difference is that we've rotated our coordinate axes.

(c) Show that $B'^2 - 4A'C' = B^2 - 4AC$. This will required quite a bit of algebra and possibly some trigonometric identities. The specific quantity $B^2 - 4AC$ is the *discriminant* of Q, and this shows that the discriminant of quadratic plane curves is *invariant* under rotations.

(d) Note that if we happen to rotate Q a certain angle θ about the origin, we can manage to get $B' = 0$. Identify this angle (in terms of $A, \ldots, F$). Thus, we can always rotate any quadratic curve Q by *some* θ to effectively eliminate the "cross term" Bxy, and the resulting quadratic curve, Q' has a simpler equation, one in which $B' = 0$. We can now complete the squares to put Q' in "standard form". Do so, and show that the sign of the product $A'C'$ can be used to determine whether Q' is an ellipse, parabola, or hyperbola. For reference, the standard form of these quadratic curves is given in the table below. Note that standard form equations are always centered at the origin, but a general quadratic equation need not be (so instead of x's and y's in the equations below, you'd get $(x-h)$'s and $(y-k)$'s).

quadratic curve	*standard form equation(s)*
ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a, b \in \mathbb{R}$
parabola	$x^2 = 4py$ or $y^2 = 4px$, $p \in \mathbb{R}$
hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, $a, b \in \mathbb{R}$

(e) In light of (c) and (d), how would you use the discriminant $B^2 - 4AC$ to determine whether Q was an ellipse, parabola, or hyperbola?

(f) Given a quadratic polynomial

$$q(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F,$$

we can determine what the level curves $q = K$ look like from the discriminant, where K is a constant. Show that the level curve $q' = K$ corresponds precisely with the level curve $q = K$, where $q' = 0$ is the equation for the rotated quadratic curve $q = 0$. That is, (x, y) satisfies $q(x, y) = K$ if and only if (x', y') satisfies $q'(x', y') = K$ (this is much easier than it might first appear).

(g) Now we tie this back into determining the nature of critical points of $f : \mathbb{R}^2 \to \mathbb{R}$. We know that f can be approximated around (x_0, y_0) by

$$\begin{array}{rcl} q(x,y) & = & f(x_0,y_0) \\ & + & f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ & + & \frac{1}{2}f_{xx}(x_0,y_0)(x-x_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{1}{2}f_{yy}(x_0,y_0)(y-y_0)^2 \end{array}.$$

If (x_0, y_0) is a critical point of f, this simplifies to

$$q(x, y) = A(x - x_0)^2 + B(x - x_0)(y - y_0) + C(y - y_0)^2 + F,$$

where

$$\begin{aligned} A &= \frac{1}{2} f_{xx}(x_0, y_0), \\ B &= f_{xy}(x_0, y_0), \\ C &= \frac{1}{2} f_{yy}(x_0, y_0), \\ F &= f(x_0, y_0). \end{aligned}$$

It follows that the level curves of f around (x_0, y_0) "look like" the level curves of q around (x_0, y_0), and the level curves of q are either ellipses, parabolas, or hyperbolas, depending on the sign of the discriminant $B^2 - 4AC$ (which you should recognize is exactly the *negative* of the determinant of the matrix of second partials), as we've shown in (e). Further, we know we can center our coordinate system around (x_0, y_0) and rotate it in such a way that the level curves of

$$q'(x', y') = A'x'^2 + C'y'^2 + F$$

correspond to the level curves of q, as given by (f). Show that if the level curves of q' are ellipses, then q' has a maximum or minimum of F at $(0, 0)$, and conclude that q has a maximum or minimum of F at (x_0, y_0). On the other hand, if the level curves of q' are hyperbolas, then q' has a saddle at $(0, 0)$, and conclude that q has a saddle at (x_0, y_0).

The key here is that by translating and rotating our coordinate system and reexpressing q in this new coordinate system, we've made our analysis much simpler. Further, we've shown that the conclusions we draw in this new coordinate system can be transferred back to our old coordinate system.