Midterm Solutions

Josh Hernandez

November 4, 2009

1 (20 points) Compute the matrix representation $[\mathsf{T}]_\beta^\gamma$ of the linear transformation

$$\mathsf{T} : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{with mapping} \quad \mathsf{T}(x, y) = (x - y, 2x + y)$$

with respect to the ordered bases $\beta = \{(1,0),(0,1)\}$ and $\gamma = \{(\text{-}1,1),(1,0)\}$.

Solution: First compute $T(\beta)$:

$$T(1,0) = (1-0, 2(1)+0) = (1,2) \quad \text{and} \quad T(0,1) = (0-1, 2(0)+1) = (\text{-}1,1).$$

Now, we write each of those vectors as linear combinations in γ:

$$(1,2) = 2(\text{-}1,1) + 3(1,0) \quad \text{and} \quad (\text{-}1,1) = 1(\text{-}1,1) + 0(1,0).$$

We write these coefficients vertically as the columns of our matrix:

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}.$$

2 (20 points) Define $\mathsf{W} = \{f \in \mathsf{P}_3(\mathbb{R}) | f'(0) = 0\}$

a Prove that W is a subspace of $\mathsf{P}_3(\mathbb{R})$.

Solution: Take $f, g \in \mathsf{W}$ and $k \in \mathbb{R}$. By linearity of derivatives,

$$(f + kg)'(0) = f'(0) + kg'(0) = 0 + k0 = 0.$$

Thus $f + kg \in \mathsf{W}$, and we have proved the subspace criterion.

b Find a basis for W (and prove that it is a basis).

Solution: To find the generic element of W, we start with the generic element of $\mathsf{P}_3(\mathbb{R})$ and apply the constraints. Define $f(x) = a + bx + cx^2 + dx^3$. If $f \in \mathsf{W}$, then

$$0 = f'(0) = (b + 2cx + 3dx^2)|_{x=0} = b$$

Thus

$$\mathsf{W} = \{a + cx^2 + dx^3 : a, c, d \in \mathbb{R}\}.$$

Let $\beta = \{1, x^2, x^3\}$. Clearly this is a spanning set – given $f \in \mathsf{W}$,

$$f = a + cx^2 + dx^3 = a(1) + c(x^2) + d(x^3).$$

Since $1, x^2, x^3$ all have different degrees, they must be linearly independent. So β is indeed a basis.

3 (20 points) Let V be a vector space over $\mathbb{R}$. Let u, v, and w be distinct vectors in V. Prove that $\{u, v, w\}$ is linearly independent if and only if $\{u+v, u+w, v+w\}$ is linearly independent.

Solution:

$\boxed{\Rightarrow}$ Assume $\{u, v, w\}$ are linearly independent. Given the linear combination

$$a(u+v) + b(u+w) + c(v+w) = 0, \tag{1}$$

we can rearrange this into the combination

$$(a+b)u + (a+c)v + (b+c)w = 0.$$

Since $\{u, v, w\}$ are linearly independent, the coefficients all must equal 0:

$$\begin{array}{ccccccc} a & + & b & & & = & 0 \\ a & & & + & c & = & 0 \\ & & b & + & c & = & 0 \end{array}$$

Solving this system, we get $a = b = c = 0$, so (1) must be a trivial combination. Thus $\{u+v, u+w, v+w\}$ are linearly independent.

$\boxed{\Leftarrow}$ Assume $\{x = u+v,\ y = u+w,\ z = v+w\}$ are linearly independent. Given the linear combination

$$au + bv + cw = 0 \tag{2}$$

we can write $u = \frac{1}{2}(x+y-z)$, $v = \frac{1}{2}(x+z-y)$, $w = \frac{1}{2}(y+z-x)$, and rearrange the combination

$$\begin{aligned} & a\tfrac{1}{2}(x+y-z) + b\tfrac{1}{2}(x+z-y) + c\tfrac{1}{2}(y+z-x) \\ &= \tfrac{1}{2}(a+b-c)x + \tfrac{1}{2}(a+c-b)y + \tfrac{1}{2}(b+c-a)z = 0. \end{aligned}$$

Since $\{x, y, z\}$ were presumed to be independent, the coefficients above must all be zero, so

$$\begin{array}{ccccccc} a & + & b & - & c & = & 0 \\ a & - & b & + & c & = & 0 \\ -a & + & b & + & c & = & 0 \end{array}$$

This linear combination has only the solution $a = b = c = 0$, so (2) must be a trivial combination. Thus $\{u, v, w\}$ are linearly independent.

A second, perhaps quicker proof:

Clearly, $u+v, u+w$, and $v+w$ lie in the span of $\{u, v, w\}$:

$$u+v = 1(u) + 1(v) + 0(w) \quad u+w = 1(u) + 0(v) + 1(w) \quad v+w = 0(u) + 1(v) + 1(w).$$

Observe that u, v, w lie in the span of $\{u+v, u+w, v+w\}$:

$$\begin{aligned} u &= \tfrac{1}{2}(u+v) + \tfrac{1}{2}(u+w) - \tfrac{1}{2}(v+w) \quad v = \tfrac{1}{2}(u+v) - \tfrac{1}{2}(u+w) + \tfrac{1}{2}(v+w) \\ w &= -\tfrac{1}{2}(u+v) + \tfrac{1}{2}(u+w) + \tfrac{1}{2}(v+w) \end{aligned}$$

Thus $\text{span}\{u, v, w\} = \text{span}\{u+v, u+w, v+w\}$. If one of these sets is linearly independent, then the dimension of its span is 3, so the dimension of the span of the other set is 3, so the other set is linearly independent.

4 (20 points) Let V be a vector space, $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ be a linear transformation. Prove that $\mathsf{T}^2 = \mathbf{0}_{\mathcal{L}(\mathsf{V},\mathsf{V})}$ if and only if $\mathsf{R}(\mathsf{T}) \subseteq \mathsf{N}(\mathsf{T})$.

Solution:
$\boxed{\Rightarrow}$ Suppose $\mathsf{T}^2 = 0$. Given $w \in \mathsf{R}(\mathsf{T})$, there exists $v \in \mathsf{V}$ such that $\mathsf{T}(v) = w$. Then

$$\mathsf{T}(w) = \mathsf{T}(\mathsf{T}(v)) = \mathsf{T}^2(v) = 0,$$

so $w \in \mathsf{N}(\mathsf{T})$. Thus $\mathsf{R}(\mathsf{T}) \subseteq \mathsf{N}(\mathsf{T})$.

$\boxed{\Leftarrow}$ Suppose $\mathsf{R}(\mathsf{T}) \subseteq \mathsf{N}(\mathsf{T})$. Then given $v \in V$, $\mathsf{T}(v) \in \mathsf{N}(\mathsf{T})$ and

$$\mathsf{T}^2(v) = \mathsf{T}(\mathsf{T}(v)) = 0$$

Since v was arbitrary, $\mathsf{T}^2 = 0$.

5 For any finite dimensional vector space V of dimension n with an ordered basis β, show that the coordinate map $\phi_\beta : \mathsf{V} \to \mathbb{R}^n$ defined by $\phi_\beta(x) = [x]_\beta$ is one-to-one and onto.

Solution: Let $\beta = \{v_1, \ldots, v_n\}$, and take $v = \sum_{i=1}^n c_i v_i$ and $w = \sum_{i=1}^n d_i v_i$ in V. For any $k \in \mathbb{R}$, observe

$$\phi_\beta(v + kw) = \phi_\beta\left(\sum_{i=1}^n (c_i + kd_i)v_i\right) = \begin{pmatrix} c_1 + kd_1 \\ \vdots \\ c_n + kd_n \end{pmatrix}$$

by the usual addition properties of vectors in $\mathbb{R}^n$,

$$\begin{aligned} &= \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} + k \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = \phi_\beta\left(\sum_{i=1}^n c_i v_i\right) + k\phi_\beta\left(\sum_{i=1}^n d_i v_i\right) \\ &= \phi_\beta(v) + k\phi_\beta(w). \end{aligned}$$

Thus ϕ_β is linear.

Suppose $\phi_\beta(v) = (0, \ldots, 0)$. Then $v = \sum_{i=1}^n 0 v_i = 0$. Thus $\mathsf{N}(\phi_\beta) = \{0\}$, so ϕ_β is one-to-one.

Finally, given $(r_1, r_2, \ldots, r_n) \in \mathbb{R}^n$, we have $v = \sum_{i=1}^n r_i v_i \in \mathsf{V}$ such that

$$\phi_\beta(v) = \phi_\beta\left(\sum_{i=1}^n r_i v_i\right) = (r_1, \ldots, r_n).$$

Thus $(r_1, \ldots, r_n) \in \mathsf{R}(\phi_\beta)$, so ϕ_β is onto.

Note: Another, nontrivial fact about ϕ_β is that it is well-defined - that is, that every vector in V maps to a unique column-vector in $\mathbb{R}^n$. This comes from linear independence of β.

Suppose there were two distinct values for $\phi_\beta(v)$, i.e. distinct $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = v = b_1 v_1 + b_2 v_2 + \cdots + b_n v_n.$$

Then

$$(a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \cdots + (a_n - b_n)v_n = 0$$

is a non-trivial linear combination in β (if the representations are distinct, at least one of the coefficients above has to be nonzero). This contradicts our assumption of linear independence, so the mapping of ϕ_β must be unique.