# Homework 8 Solutions

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November 18, 2009

## 6.2 - Gram-Schmidt Orthogonalization Process

**2.** Apply the Gram-Schmidt process to the given subset $S$ of the inner product space $\mathsf{V}$. Normalize the vectors in the resulting basis to obtain an orthonormal basis $\beta$, and compute the Fourier coefficients of the given vector relative to $\beta$.

**d.** $\mathsf{V} = \operatorname{span}(S)$, where $S = \{v_1 = (1, i, 0), v_2 = (1-i, 2, 4i)\}$, and $x = (3+i, 4i, \text{-}4)$.

*Solution:*

Orthogonalization: Define $v_1' = v_1$. We compute the inner products

$$\begin{array}{rclclcl} \|v_1'\|^2 & = & \|(1,i,0)\|^2 & = & |1|^2+|1|^2+|0|^2 & = & 2 \\ \langle v_2, v_1'\rangle & = & \langle (1-i,2,4i),(1,i,0)\rangle & = & (1-i)\,\overline{1}+2\,\overline{i}+4i\,\overline{0} & = & 1-3i, \end{array}$$

and compute the next orthogonal vector

$$v_2' = v_2 - \operatorname{proj}_{v_1'}(v_2) = v_2 - \frac{\langle v_2, v_1'\rangle}{\langle v_1', v_1'\rangle} v_1' = (1-i, 2, 4i) - \tfrac{1-3i}{2}(1,i,0) = \tfrac{1}{2}(1+i, 1-i, 8i),$$

where

$$\|v_2'\|^2 \quad = \quad \|\tfrac{1}{2}(1+i,1-i,8i)\|^2 \quad = \quad \tfrac{1}{4}(|1+i|^2+|1-i|^2+|8|^2) \quad = \quad 17.$$

Normalization:

$$w_1 = \frac{v_1'}{\|v_1'\|} = \tfrac{\sqrt{2}}{2}(1,i,0), \qquad\qquad w_2 = \frac{v_2'}{\|v_2'\|} = \tfrac{\sqrt{17}}{34}(1+i,1-i,8i).$$

So $\beta = \{\frac{\sqrt{2}}{2}(1,i,0), \frac{\sqrt{17}}{34}(1+i,1-i,8i)\}$ is an orthonormal basis of $\operatorname{span}(S)$. Now we can compute the Fourier coefficients for $x = (3+i, 4i, \text{-}4)$:

$$\begin{aligned} c_1 &= \langle x, w_1\rangle = \left\langle (3+i,4i,\text{-}4), \tfrac{\sqrt{2}}{2}(1,i,0)\right\rangle = \tfrac{\sqrt{2}}{2}\left[(3+i)\,\overline{1}+4i\,\overline{i}-4\,\overline{0}\right] = \tfrac{\sqrt{2}}{2}(7+i), \\ c_2 &= \langle x, w_2\rangle = \left\langle (3+i,4i,\text{-}4), \tfrac{\sqrt{17}}{34}(1+i,1-i,8i)\right\rangle = \tfrac{\sqrt{17}}{34}\left[(3+i)\,(\overline{1+i})+4i\,(\overline{1-i})-4\,\overline{8i}\right] \\ &= \tfrac{\sqrt{17}}{34}\left[(3+i)(1-i)+4i(1+i)-4(\text{-}8i)\right] = \tfrac{\sqrt{17}}{34}((3+1-4)+(\text{-}3+1+4+32)i) = \sqrt{17}i. \end{aligned}$$

Checking that these coefficients work:

$$\begin{aligned} x = c_1w_1 + c_2w_2 &= \tfrac{\sqrt{2}}{2}(7+i)\,\tfrac{\sqrt{2}}{2}(1,i,0) + \sqrt{17}i\,\tfrac{\sqrt{17}}{34}(1+i,1-i,8i) \\ &= \tfrac{1}{2}(7+i,\text{-}1+7i,0) + \tfrac{1}{2}(\text{-}1+i,1+i,\text{-}8) = (3+i,4i,\text{-}4). \end{aligned}$$

**f.** $\mathsf{V} = \mathbb{R}^4$, $S = \{v_1 = (1,\text{-}2,\text{-}1,3), v_2 = (3,6,3,\text{-}1), v_3 = (1,4,2,8)\}$, and $x = (\text{-}1,2,1,1)$.

*Solution:*
Orthogonalization: Define $v_1' = v_1$. We compute the inner products

$$\begin{array}{rclclcl} \langle v_1', v_1' \rangle &=& \langle (1,\text{-}2,\text{-}1,3),(1,\text{-}2,\text{-}1,3)\rangle &=& 1^2 + (\text{-}2)^2 + (\text{-}1)^2 + 3^2 &=& 15 \\ \langle v_2, v_1' \rangle &=& \langle (3,6,3,\text{-}1),(1,\text{-}2,\text{-}1,3)\rangle &=& 3(1) + 6(\text{-}2) + 3(\text{-}1) - 1(3) &=& \text{-}15, \end{array}$$

and compute the next orthogonal vector

$$v_2' = (3,6,3,\text{-}1) - \tfrac{\text{-}15}{15}(1,\text{-}2,\text{-}1,3) = (4,4,2,2).$$

Computing more inner products:

$$\begin{array}{rclclcl} \langle v_2', v_2' \rangle &=& \langle (4,4,2,2),(4,4,2,2)\rangle &=& 4^2 + 4^2 + 2^2 + 2^2 &=& 40, \\ \langle v_3, v_1' \rangle &=& \langle (1,4,2,8),(1,\text{-}2,\text{-}1,3)\rangle &=& 1(1) + 4(\text{-}2) + 2(\text{-}1) + 8(3) &=& 15, \\ \langle v_3, v_2' \rangle &=& \langle (1,4,2,8),(4,4,2,2)\rangle &=& 1(4) + 4(4) + 2(2) + 8(2) &=& 40, \end{array}$$

So

$$v_3' = v_3 - \text{proj}_{v_1'}(v_3) - \text{proj}_{v_2'}(v_3) = (1,4,2,8) - \tfrac{15}{15}(1,\text{-}2,\text{-}1,3) - \tfrac{40}{40}(4,4,2,2) = (\text{-}4,2,1,3),$$

and

$$\langle v_2', v_2' \rangle = \langle (\text{-}4,2,1,3),(\text{-}4,2,1,3)\rangle = (\text{-}4)^2 + 2^2 + 1^2 + 3^2 = 30.$$

Normalization:

$$w_1 = \frac{v_1'}{\|v_1'\|} = \tfrac{\sqrt{15}}{15}(1,\text{-}2,\text{-}1,3), \qquad w_2 = \frac{v_1'}{\|v_1'\|} = \tfrac{\sqrt{10}}{20}(4,4,2,2), \qquad w_2 = \frac{v_1'}{\|v_1'\|} = \tfrac{\sqrt{30}}{30}(\text{-}4,2,1,3).$$

So $\beta = \{\tfrac{\sqrt{15}}{15}(1,\text{-}2,\text{-}1,3), \tfrac{\sqrt{10}}{20}(4,4,2,2), \tfrac{\sqrt{30}}{30}(\text{-}4,2,1,3)\}$ is an orthonormal basis of span$(S)$. Now we can compute the Fourier coefficients for $x = (\text{-}1,2,1,1)$:

$$\begin{aligned} c_1 &= \langle x, w_1 \rangle = \tfrac{\sqrt{15}}{15}\langle (\text{-}1,2,1,1),(1,\text{-}2,\text{-}1,3)\rangle = \text{-}3\tfrac{\sqrt{15}}{15} = \tfrac{\text{-}\sqrt{15}}{5}, \\ c_2 &= \langle x, w_2 \rangle = \tfrac{\sqrt{10}}{20}\langle (\text{-}1,2,1,1),(4,4,2,2)\rangle = 8\tfrac{\sqrt{10}}{20} = \tfrac{2\sqrt{10}}{5}, \\ c_3 &= \langle x, w_3 \rangle = \tfrac{\sqrt{30}}{30}\langle (\text{-}1,2,1,1),(\text{-}4,2,1,3)\rangle = 12\tfrac{\sqrt{30}}{30} = \tfrac{2\sqrt{30}}{5}. \end{aligned}$$

Checking that these coefficients work:

$$\begin{aligned} c_1w_1 + c_2w_2 + c_3w_3 &= \tfrac{\text{-}\sqrt{15}}{5}\tfrac{\sqrt{15}}{15}(1,\text{-}2,\text{-}1,3) + \tfrac{2\sqrt{10}}{5}\tfrac{\sqrt{10}}{20}(4,4,2,2) + \tfrac{2\sqrt{30}}{5}\tfrac{\sqrt{30}}{30}(\text{-}4,2,1,3) \\ &= \tfrac{\text{-}1}{5}(1,\text{-}2,\text{-}1,3) + \tfrac{1}{5}(4,4,2,2) + \tfrac{2}{5}(\text{-}4,2,1,3) = (\text{-}1,2,1,1). \end{aligned}$$

**6.** Let $\mathsf{V}$ be an inner product space, and let $\mathsf{W}$ be a finite-dimensional subspace of $\mathsf{V}$. If $x \notin \mathsf{W}$, prove that there exists $y \in \mathsf{V}$ such that $y \in \mathsf{W}^\perp$, but $\langle x, y \rangle \neq 0$.

*Solution:* By Theorem 6.6, we may write $x = u + y$, with $u \in W$ and $y \in W^\perp$.

$$\langle x, y \rangle = \langle u + y, y \rangle = \langle u, y \rangle + \langle y, y \rangle = \langle y, y \rangle.$$

Note that $y \neq 0$, otherwise $x = u \in W$. Coercivity of the inner product forces $\langle x, y \rangle = \langle y, y \rangle > 0$.

**11.** Let $A$ be an $n \times n$ matrix with complex entries. Prove that $AA^* = I$ if and only if the rows of $A$ form an orthonormal basis for $\mathbb{C}^n$.

*Solution:* Denote by $r_i$ the $i$th row of $A$. Observe that

$$(AA^*)_{ij} = \sum_{k=1}^{n} A_{ik}(A^*)_{kj} = \sum_{k=1}^{n} A_{ik}\,\overline{A_{jk}} = \langle r_i, r_j \rangle .$$

so

$$AA^* = I \iff (AA^*)_{ij} = \langle r_i, r_j \rangle = \delta_{ij} \iff \{r_i\}_{i=1}^n \text{ forms an orthonormal basis.}$$

**13c.** Let $\mathsf{V}$ be an inner product space, and $\mathsf{W}$ be a finite-dimensional subspace of $\mathsf{V}$. Prove that $\mathsf{W} = (\mathsf{W}^\perp)^\perp$.

*Solution:*
$\boxed{\subseteq}$ If $x \in W$ and $y \in \mathsf{W}^\perp$, then $\langle x, y \rangle = \langle y, x \rangle = 0$, so $x \in (\mathsf{W}^\perp)^\perp$.
$\boxed{\supseteq}$ If $x \notin \mathsf{W}$, then, by problem 13, there exists $y \in \mathsf{W}^\perp$ such that $\langle x, y \rangle \neq 0$, so $x \notin (\mathsf{W}^\perp)^\perp$.

**16. a.** Let $\mathsf{V}$ be an inner-product space, and let $S = \{v_1, \ldots, v_n\}$ be an orthonormal subset of $\mathsf{V}$. Prove that for any $x \in \mathsf{V}$ we have

$$\|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i \rangle|^2.$$

*Solution:* Define $\mathsf{W} = \operatorname{span}(S)$. This is a finite-dimensional subspace of $\mathsf{V}$. Given $x \in \mathsf{V}$, we can write $x = u + y$, where $u \in \mathsf{W}$ and $y \in \mathsf{W}^\perp$ (Theorem 6.6). Since $S$ is an orthonormal basis of $\mathsf{W}$, we can write $u = \sum_{i=1}^{n} \langle u, v_i \rangle v_i$, so that

$$\|x\|^2 = \|u\|^2 + \|y\|^2 = \Big\|\sum_{i=1}^{n} \langle u, v_i \rangle v_i\Big\|^2 + \|y\|^2.$$

Applying problem 6.1.12,

$$= \sum_{i=1}^{n} |\langle u, v_i \rangle|^2 \|v_i\|^2 + \|y\|^2$$
$$= \sum_{i=1}^{n} |\langle u, v_i \rangle|^2 + \|y\|^2 \geq \sum_{i=1}^{n} |\langle u, v_i \rangle|^2$$

**b.** In the context of (a), prove that Bessel's inequality is an equality if and only if $x \in \operatorname{span}(S)$.

*Solution:* In the context of (a), we have equality if and only if $\|y\|^2 = 0$, which occurs if and only if $y = 0$, meaning $x = u \in \mathsf{W} = \operatorname{span}(S)$.

**18.** Let $\mathsf{V} = \mathsf{C}([-1, 1])$. Suppose that $\mathsf{W}_e$ and $\mathsf{W}_o$ denote the subspaces of $\mathsf{V}$ consisting of even and odd functions, respectively. Prove that $\mathsf{W}_e \perp \mathsf{W}_o$, where the inner product on $\mathsf{V}$ is defined by

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\,dt.$$

*Solution:* Given $f \in \mathsf{W}_e$ and $g \in \mathsf{W}_o$,

$$\begin{aligned}\langle f(t), g(t)\rangle &= \int_{-1}^{1} f(t)g(t)\,dt = \int_{-1}^{0} f(t)g(t)\,dt + \int_{0}^{1} f(t)g(t)\,dt \\ &= -\int_{1}^{0} f(-t)g(-t)\,dt + \int_{0}^{1} f(t)g(t)\,dt = \int_{0}^{1} \big(f(-t)g(-t) + f(t)g(t)\big)\,dt.\end{aligned}$$

Applying the definition of odd and even functions,

$$= \int_{0}^{1} \big(-f(t)g(t) + f(t)g(t)\big)\,dt = \int_{0}^{1} 0\,dt = 0.$$

**19c.** Let $\mathsf{V} = \mathsf{P}(\mathbb{R})$ with the inner product $\langle f(x), g(x)\rangle = \int_0^1 f(t)g(t)\,dt$ and let $\mathsf{W} = \mathsf{P}_1(\mathbb{R}) \leq \mathsf{V}$. Find the orthogonal projection of $h(x) = 4 + 3x - 2x^2$ onto $\mathsf{W}$.

*Solution:* Let $\beta = \{v_1 = 1, v_2 = x\}$, a basis of $\mathsf{W}$. Let $v_1' = v_1$. Then

$$\begin{aligned}\|v_1'\|^2 &= \langle 1, 1\rangle = \int_0^1 1^2\,dx = 1 \\ \langle v_2, v_1'\rangle &= \langle x, 1\rangle = \int_0^1 x(1)\,dx = \left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{2}.\end{aligned}$$

The next orthogonal basis vector:

$$v_2' = v_2 - \operatorname{proj}_{v_1'}(v_2) = x - \frac{1/2}{1}1 = -\frac{1}{2} + x.$$

where

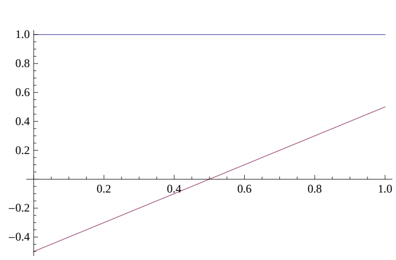
$$\|v_2\|^2 = \left\langle -\tfrac{1}{2} + x, -\tfrac{1}{2} + x\right\rangle = \int_0^1 (-\tfrac{1}{2} + t)^2\,dt = \tfrac{1}{3}\left[(-\tfrac{1}{2} + x)^3\right]_0^1 = \tfrac{1}{3}(\tfrac{1}{8} - \tfrac{1}{8}) = \tfrac{1}{12}.$$

Now we compute inner products with $h(x) = 4 + 3x - 2x^2$,

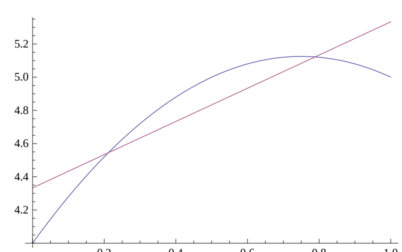
$$\begin{aligned}\langle h(x), v_1'\rangle &= \left\langle 4 + 3x - 2x^2, 1\right\rangle = \int_0^1 (4 + 3x - 2x^2)\,dx = \left[4x + \tfrac{3}{2}x^2 - \tfrac{2}{3}x^3\right]_0^1 = \tfrac{29}{6} \\ \langle h(x), v_2'\rangle &= \left\langle 4 + 3x - 2x^2, -\tfrac{1}{2} + x\right\rangle = \int_0^1 (4 + 3x - 2x^2)(-\tfrac{1}{2} + x)\,dx \\ &= \int_0^1 (-2 + \tfrac{5}{2}x + 4x^2 - 2x^3)\,dx = \left[-2x + \tfrac{5}{4}x^2 + \tfrac{4}{3}x^3 - \tfrac{1}{2}x^4\right]_0^1 = \tfrac{1}{12}.\end{aligned}$$

So

$$\operatorname{proj}_{\mathsf{W}}(h) = \operatorname{proj}_{v_1'}(h) + \operatorname{proj}_{v_2'}(h) = \frac{29/6}{1}1 + \frac{1/12}{1/12}(-\tfrac{1}{2} + x) = x + \tfrac{13}{3}.$$




(a) Orthogonal Basis




(b) Orthogonal Approximation

**20c.** In the context of (19c), find the distance from $h(x)$ to $\mathsf{W}$.

*Solution:* The distance from $h(x)$ to $\mathsf{W}$ is the distance between $h(x)$ and $\text{proj}_{\mathsf{W}}(h)$.

$$\begin{aligned}\|h(x)-\text{proj}_{\mathsf{W}}(h)\| &= \left(\int_0^1 \left((4+3x-2x^2)-(\tfrac{13}{3}+x)\right)^2\,dx\right)^{1/2} = \left(\int_0^1 \left(\text{-}\tfrac{1}{3}+2x-2x^2\right)^2\,dx\right)^{1/2} \\ &= \left(\int_0^1 \left(\tfrac{1}{9}-\tfrac{4}{3}x+\tfrac{16}{3}x^2-8x^3+4x^4\right)\,dx\right)^{1/2} = \left(\left[\tfrac{1}{9}x-\tfrac{2}{3}x^2+\tfrac{16}{9}x^3-2x^4+\tfrac{4}{5}x^5\right]_0^1\right)^{1/2} \\ &= \sqrt{(5-30+80-90+36)/45} = \sqrt{5}/15.\end{aligned}$$

**22.** Let $\mathsf{V}=\mathsf{C}([0,1])$ with the inner product $\langle f,g\rangle=\int_0^1 f(t)g(t)\,dt$. Let $\mathsf{W}$ be the subspace spanned by the linearly independent set $\{t,\sqrt{t}\}$.

**a.** Find an orthonormal basis for $\mathsf{W}$

*Solution:* Let $\beta=\{v_1=t, v_2=\sqrt{t}\}$, a basis of $\mathsf{W}$. Let $v_1'=v_1$. Then

$$\begin{aligned}\|v_1'\|^2 &= \langle t,t\rangle &= \textstyle\int_0^1 t^2\,dx &= \left[\tfrac{1}{3}t^3\right]_0^1 = \tfrac{1}{3} \\ \langle v_2,v_1'\rangle &= \left\langle \sqrt{t},t\right\rangle &= \textstyle\int_0^1 \sqrt{t}t\,dt &= \left[\tfrac{2}{5}x^{\frac{5}{2}}\right]_0^1 = \tfrac{2}{5}.\end{aligned}$$

The next orthogonal basis vector:

$$v_2' = v_2-\text{proj}_{v_1'}(v_2) = \sqrt{t}-\tfrac{2/5}{1/3}t = \sqrt{t}-\tfrac{6}{5}t.$$

where

$$\begin{aligned}\|v_2\|^2 &= \textstyle\int_0^1(\sqrt{t}-\tfrac{6}{5}t)^2\,dt = \int_0^1(t-\tfrac{12}{5}t^{\frac{3}{2}}+\tfrac{36}{25}t^2)\,dt \\ &= \left[(\tfrac{1}{2}t^2-\tfrac{24}{25}t^{\frac{5}{2}}+\tfrac{12}{25}t^3\right]_0^1 = \tfrac{1}{50}(25-48+24) = \tfrac{1}{50}.\end{aligned}$$

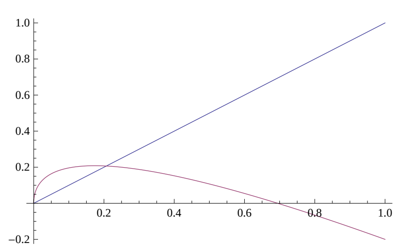
**b.** Let $h(t)=t^2$. Use the orthonormal basis obtained in (a) to obtain the closest approximation of $h$ in $\mathsf{W}$.

*Solution:* Now we compute inner products with $h(t)=t^2$,

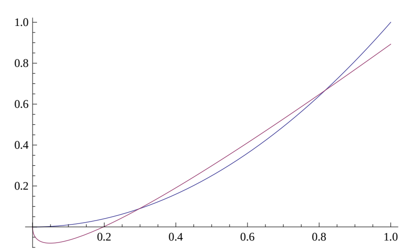
$$\begin{aligned}\langle h(t),v_1'\rangle &= \left\langle t^2,t\right\rangle = \textstyle\int_0^1 t^3\,dx = \left[\tfrac{1}{4}t^4\right]_0^1 = \tfrac{1}{4} \\ \langle h(t),v_2'\rangle &= \left\langle t^2,\sqrt{t}-\tfrac{6}{5}t\right\rangle = \textstyle\int_0^1(t^{\frac{5}{2}}-\tfrac{6}{5}t^3)\,dx = \left[\tfrac{2}{7}t^{\frac{7}{2}}-\tfrac{3}{10}t^4\right]_0^1 = \tfrac{\text{-}1}{70}.\end{aligned}$$

So

$$\text{proj}_{\mathsf{W}}(h) = \text{proj}_{v_1'}(h)+\text{proj}_{v_2'}(h) = \tfrac{1/4}{1/3}t+\tfrac{\text{-}1/70}{1/50}(\sqrt{t}-\tfrac{6}{5}t) = \tfrac{\text{-}5}{7}\sqrt{t}+\tfrac{45}{28}t.$$



(c) Orthogonal Basis



(d) Orthogonal Approximation