Homework 7 Solutions

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5.2 - Diagonalizability

2. For each of the following matrices $A \in \mathsf{M}_{n\times n}(\mathbb{R})$, test A for diagonalizability, and if A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

b. $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

Solution: Computing eigenvalues:

$$\chi_A(\lambda) = \det \begin{pmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 3^2$$

This polynomial has roots $1 \pm 3 = 4, \text{-}2$. Two distinct eigenvalues mean that A is diagonalizable. Then

$$\mathsf{E}_4 = \mathsf{N}\begin{pmatrix} 1-4 & 3 \\ 3 & 1-4 \end{pmatrix} = \mathsf{N}\begin{pmatrix} \text{-}3 & 3 \\ 3 & \text{-}3 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$
$$\mathsf{E}_{\text{-}2} = \mathsf{N}\begin{pmatrix} 1-(\text{-}2) & 3 \\ 3 & 1-(\text{-}2) \end{pmatrix} = \mathsf{N}\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ \text{-}1 \end{pmatrix}\right\}.$$

Our diagonalization is therefore

$$A = \begin{pmatrix} 1 & 1 \\ 1 & \text{-}1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & \text{-}2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & \text{-}1 \end{pmatrix}^{\text{-}1} =: QDQ^{\text{-}1}.$$

d. $A = \begin{pmatrix} 7 & \text{-}4 & 0 \\ 8 & \text{-}5 & 0 \\ 6 & \text{-}6 & 3 \end{pmatrix}$

Solution: Computing eigenvalues:

$$\chi_A(\lambda) = \det \begin{pmatrix} 7-\lambda & \text{-}4 & 0 \\ 8 & \text{-}5-\lambda & 0 \\ 6 & \text{-}6 & 3-\lambda \end{pmatrix} = (7-\lambda)(\text{-}5-\lambda)(3-\lambda) - (\text{-}4\cdot 8(3-\lambda))$$
$$= (3-\lambda)(\lambda^2 - 2\lambda - 3) = (3-\lambda)(\lambda-3)(\lambda+1).$$

This polynomial has roots 3 and -1 (a repeated root means that A might not be diagonalizable).

$$\mathsf{E}_3 = \mathsf{N}\begin{pmatrix} 7-3 & \text{-}4 & 0 \\ 8 & \text{-}5-3 & 0 \\ 6 & \text{-}6 & 3-3 \end{pmatrix} = \mathsf{N}\begin{pmatrix} 4 & \text{-}4 & 0 \\ 8 & \text{-}8 & 0 \\ 6 & \text{-}6 & 0 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

$$\mathsf{E}_{-1} = \mathsf{N}\begin{pmatrix} 7-(-1) & -4 & 0 \\ 8 & -5-(-1) & 0 \\ 6 & -6 & 3-(-1) \end{pmatrix} = \mathsf{N}\begin{pmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 6 & -6 & 4 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}\right\}.$$

The two eigenspaces have a total dimension of 3, so A is diagonalizable. Our diagonalization is therefore

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix}^{-1} =: QDQ^{-1}.$$

f. $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

Solution: Since A is an upper-triangular matrix, we can read its eigenvalues off the diagonal: $\lambda = 1, 3$. Then

$$\mathsf{E}_1 = \mathsf{N}\begin{pmatrix} 1-1 & 1 & 0 \\ 0 & 1-1 & 2 \\ 0 & 0 & 3-1 \end{pmatrix} = \mathsf{N}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}.$$

We needn't bother to compute E_3. The dimension of E_1 is 1, although the root $\lambda = 1$ has multiplicity 2. Therefore A is not diagonalizable.

3b. Let $\mathsf{V} = \mathsf{P}_2(\mathbb{R})$. Define $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ by the mapping $\mathsf{T}(ax^2 + bx + c) = cx^2 + bx + a$. If diagonalizable, find a basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

Solution: If $\alpha = \{1, x, x^2\}$ is the standard basis on $\mathsf{P}_2(\mathbb{R})$, then

$$A := [\mathsf{T}]_\alpha = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Computing eigenvalues of A:

$$\chi_A(\lambda) = \det\begin{pmatrix} 0-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 0-\lambda \end{pmatrix} = (0-\lambda)^2(1-\lambda) - (1-\lambda) = (1-\lambda)(\lambda^2-1)$$
$$= -(1-\lambda)^2(1+\lambda).$$

The roots of this polynomial are $\lambda = \pm 1$. Now,

$$\mathsf{E}_1 = \mathsf{N}\begin{pmatrix} 0-1 & 0 & 1 \\ 0 & 1-1 & 0 \\ 1 & 0 & 0-1 \end{pmatrix} = \mathsf{N}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$$
$$\mathsf{E}_{-1} = \mathsf{N}\begin{pmatrix} 0-(-1) & 0 & 1 \\ 0 & 1-(-1) & 0 \\ 1 & 0 & 0-(-1) \end{pmatrix} = \mathsf{N}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\}.$$

Thus $\beta = \{(1,0,1),(0,1,0),(1,0,\text{-}1)\}$ is a diagonalizing basis of L_A. Noting that

$$[\mathsf{L}_A]_\beta = [\mathsf{L}_{[\mathsf{T}]_\alpha}]_\beta = [\phi_\alpha \circ \mathsf{T} \circ \phi_\alpha^{\text{-}1}]_\beta = [\mathsf{T}]_{\phi_\alpha^{\text{-}1}(\beta)}, \tag{1}$$

we know that $\phi_\alpha^{\text{-}1}(\beta) = \{1+x^2, x, 1-x^2\}$ is a diagonalizing basis of T (equation (1) justifies the obvious final step of converting the vectors of β into their corresponding polynomials).

8. Suppose that $A \in \mathsf{M}_{n\times n}n(F)$ has two distinct eigenvalues, λ_1 and λ_2, and that $\dim(\mathsf{E}_{\lambda_1}) = n-1$. Prove that A is diagonalizable.

Solution: Distinct eigenspaces intersect trivially, and any eigenspace has dimension ≥ 1, so

$$\dim(\mathsf{E}_{\lambda_1} + \mathsf{E}_{\lambda_2}) = \dim(\mathsf{E}_{\lambda_1} \oplus \mathsf{E}_{\lambda_2}) = \dim(\mathsf{E}_{\lambda_1}) + \dim(\mathsf{E}_{\lambda_2}) \geq (n-1)+1 = n.$$

The eigenspaces of A span F^n, and so A is diagonalizable.

11. Let A be an $n \times n$ matrix that is similar to an upper triangular matrix, and has the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with corresponding multiplicities $m_1, m_2, \ldots, m_k$. Prove the following statements.

Lemma: *If A and B are similar matrices, then $\chi_A(\lambda) = \chi_B(\lambda)$.*
Let $A = Q^{\text{-}1}BQ$ for some matrix $Q \in \mathsf{M}_{n\times n}(\mathsf{F})$. By the multiplicative property of determinants,

$$\chi_A(\lambda) = \det(A - \lambda I) = \det(Q^{\text{-}1}BQ - \lambda I) = \det(Q^{\text{-}1}(B - \lambda I)Q) = \det(B - \lambda I) = \chi_B(\lambda).$$

Solution: Let M be an upper-triangular matrix such that $A = QMQ^{\text{-}1}$. It was proved (5.4:9) that the eigenvalues of M coincide with the diagonal elements M_{ii}. Since similar transformations have the same characteristic polynomials (lemma, above), they share eigenvalues λ_i and multiplicities m_i.

a. $\operatorname{trace}(A) = \sum_{i=1}^k m_i\lambda_i$

Solution: By (2.5:10),

$$\operatorname{trace}(A) = \operatorname{trace}(M) = \sum_{i=1}^n M_{ii} = \sum_{i=1}^k m_i\lambda_i. \tag{2}$$

b. $\det(A) = (\lambda_1)^{m_1}(\lambda_2)^{m_2}\cdots(\lambda_n)^{m_n}$.

Solution: The determinant of an upper-triangular matrix is the product of its diagonal entries M_{ii} (determinant property 4). By the multiplicative property of determinants,

$$\det(A) = \det(QMQ^{\text{-}1}) = \det(M) = \prod_{i=1}^n M_{ii} = \prod_{i=1}^n \lambda_i^{m_i}. \tag{3}$$

Finally, one can show (the proof is a little too complicated to show here, but see pp.370,385 in the text) that *every* matrix is similar, over some field, to an upper-triangular matrix. The identities (2) and (3) above are therefore universal properties of matrices.

14a. Find the general solution to the system of differential equations

$$x' = x + y, \qquad y' = 3x - y. \tag{4}$$

Solution: Let $\mathsf{V} = \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^2)$ be the space of smooth curves in $\mathbb{R}^2$. We can consider the derivative as a linear operator $\mathsf{D} : \mathsf{V} \to \mathsf{V}$. Then

$$\mathsf{D}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x+y \\ 3x-y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & \text{-}1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} =: A\begin{pmatrix} x \\ y \end{pmatrix}.$$

We diagonalize A in the usual fashion:

$$\chi_A(\lambda) = \det\begin{pmatrix} 1-\lambda & 1 \\ 3 & \text{-}1-\lambda \end{pmatrix} = (1-\lambda)(\text{-}1-\lambda) - 3 = \lambda^2 - 4.$$

This has roots $\lambda = \pm 2$. Computing eigenspaces:

$$\mathsf{E}_2 = \mathsf{N}\begin{pmatrix} 1-2 & 1 \\ 3 & \text{-}1-2 \end{pmatrix} = \mathsf{N}\begin{pmatrix} \text{-}1 & 1 \\ 3 & \text{-}3 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$
$$\mathsf{E}_{\text{-}2} = \mathsf{N}\begin{pmatrix} 1-(\text{-}2) & 1 \\ 3 & \text{-}1-(\text{-}2) \end{pmatrix} = \mathsf{N}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ \text{-}3 \end{pmatrix}\right\}.$$

We have a basis of eigenvectors $\beta = \{\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ \text{-}3 \end{smallmatrix}\right)\}$. Let α be the standard basis of $\mathbb{R}^2$. Changing basis,

$$\begin{pmatrix} x \\ y \end{pmatrix} = [\mathsf{I}]_\beta^\alpha \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & \text{-}3 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

where and $f_1(t)$ and $f_2(t)$ satisfy

$$f_1'(t) = 2f_1(t) \quad \text{and} \quad f_2'(t) = \text{-}2f_2(t).$$

These differential equations have solutions $f_1(t) = c_1e^{2t}$ and $f_2(t) = c_2e^{\text{-}2t}$. Observe, then,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & \text{-}3 \end{pmatrix}\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} c_1e^{2t} + c_2e^{\text{-}2t} \\ c_1e^{2t} - 3c_2e^{\text{-}2t} \end{pmatrix}.$$

18. Two linear operators T and U on an n-dimensional vector space V are called **simultaneously diagonalizable** if there exists some basis β of V such that $[\mathsf{T}]_\beta$ and $[\mathsf{U}]_\beta$ are diagonal matrices.

a. Prove that if T and U are simultaneously diagonalizable operators, then T and U commute.

Lemma: *If $D_1, D_2 \in \mathsf{M}_{n\times n}$ are two diagonal matrices, then $D_1D_2 = D_2D_1$.*

$$(D_1D_2)_{ij} = \sum_{k=1}^n (D_1)_{ik}(D_2)_{kj} = \sum_{k=1}^n \delta_{ik}(D_1)_{ik}\delta_{kj}(D_2)_{kj} = \delta_{ij}(D_1)_{ii}(D_2)_{ii}$$
$$= \delta_{ij}(D_2)_{ii}(D_1)_{ii} = (D_2D_1)_{ij}.$$

Solution: Let β be a basis of V that diagonalizes both T and U. Since diagonal matrices commute with each other (lemma, above),

$$[\mathsf{TU}]_\beta = [\mathsf{T}]_\beta[\mathsf{U}]_\beta = [\mathsf{U}]_\beta[\mathsf{T}]_\beta = [\mathsf{UT}]_\beta.$$

Now we can relate the two operators in the same way:

$$\mathsf{TU} = \phi_\beta^{-1} \circ \mathsf{L}_{[\mathsf{TU}]_\beta} \circ \phi_\beta = \phi_\beta^{-1} \circ \mathsf{L}_{[\mathsf{UT}]_\beta} \circ \phi_\beta = \mathsf{UT}.$$

6.1 - Inner Products and Norms

5. In $\mathbb{C}^2$, show that $\langle x, y\rangle = xAy^*$ is an inner product, where

$$A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}.$$

Solution: We test $\langle \cdot, \cdot \rangle$ for the various properties of an inner product

1. Linearity (in the first position) follows from linearity of matrix multiplication:

$$\langle x_1 + cx_2, y\rangle = (x_1 + cx_2)Ay^* = (x_1A + cx_2A)y^* = x_1Ay^* + x_2Ay^* = \langle x_1, y\rangle + c\langle x_2, y\rangle$$

2. Symmetry: Observe that

$$A^* = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}^* = \begin{pmatrix} \bar{1} & \bar{-i} \\ \bar{i} & \bar{2} \end{pmatrix} = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} = A.$$

Thus (observing that the Hermitian adjoint of a scalar is just its complex conjugate),

$$\langle y, x\rangle = yAx^* = ((x^*)^*A^*y^*)^* = (xAy^*)^* = \overline{xAy^*} = \overline{\langle x, y\rangle}.$$

3. Coercivity:

$$\langle (x_1, x_2), (x_1, x_2)\rangle = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} \begin{pmatrix} \bar{x_1} \\ \bar{x_2} \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \bar{x_1} + i\bar{x_2} \\ -\bar{x_1} + 2\bar{x_2} \end{pmatrix} = |x_1|^2 + 2|x_2|^2$$

If $(x_1, x_2) \neq (0, 0)$, then the RHS is a positive real number.

Compute $\langle x, y\rangle$ for $x = (1 - i, 2 + 3i)$ and $y = (2 + i, 3 - 2i)$.

Solution:

$$\begin{aligned}
\langle x, y\rangle &= \begin{pmatrix} 1-i & 2+3i \end{pmatrix} \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} \begin{pmatrix} \overline{2+i} \\ \overline{3-2i} \end{pmatrix} = \begin{pmatrix} 1-i & 2+3i \end{pmatrix} \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} \begin{pmatrix} 2-i \\ 3+2i \end{pmatrix} \\
&= \begin{pmatrix} 1-i & 2+3i \end{pmatrix} \begin{pmatrix} (2-i) + i(3+2i) \\ -i(2-i) + 2(3+2i) \end{pmatrix} = \begin{pmatrix} 1-i & 2+3i \end{pmatrix} \begin{pmatrix} 2i \\ 5+2i \end{pmatrix} \\
&= (1-i)(2i) + (2+3i)(5+2i) = (2+2i) + (4+19i) = 6 + 21i.
\end{aligned}$$

9. Let β be a basis for a finite-dimensional inner product space.

a. Prove that if $\langle x, z\rangle = 0$ for all $z \in \beta$, then $x = 0$.

Solution: Since β is a spanning set, we may write $x = \sum_{i=1}^{n} c_i z_i$, with all $z_i \in \beta$. Then

$$\langle x, x \rangle = \left\langle x, \sum\nolimits_{i=1}^{n} c_i z_i \right\rangle = \sum_{i=1}^{n} \overline{c_i} \langle x, z \rangle = 0.$$

Coercivity of the inner product implies that $x = 0$.

b. Prove that if $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in \beta$, then $x = y$.

Solution: If $\langle x, z \rangle = \langle y, z \rangle$, then $\langle x - y, z \rangle = \langle x, z \rangle - \langle y, z \rangle = 0$ for all $z \in \beta$. By the above, $x - y = 0$, so $x = y$.

10. Let V be an inner product space, and suppose that x and y are orthogonal vectors in V. Prove that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. Deduce the Pythagorean theorem in $\mathbb{R}^2$.

Solution:

$$\|x + y\|^2 = \langle x + y, x + y \rangle = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle x, y \rangle .$$

By orthogonality of x and y,

$$= \langle x, x \rangle + \langle y, y \rangle = \|x\|^2 + \|y\|^2.$$

In $\mathbb{R}^2$, if two orthogonal vectors x and y are laid head-to-tail, they form two edges of a right triangle, of which $x + y$ forms the third edge. If we denote the lengths of these edges by a, b, and c, respectively, then

$$c^2 = \|x + y\|^2 = \|x\|^2 + \|y\|^2 = a^2 + b^2.$$

Thus we prove the Pythagorean Theorem.

12. Let $\{v_1, v_2, \ldots, v_k\}$ be an orthogonal set in V, and let $a_1, a_2, \ldots, a_k$ be scalars. Prove that

$$\left\| \sum_{i=1}^{k} a_i v_i \right\|^2 = \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2.$$

Solution: Clearly, in the case that $k = 1$,

$$\left\| \sum_{i=1}^{k} a_i v_i \right\|^2 = \|a_1 v_1\|^2 = |a_1|^2 \|v_1\|^2 = \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2.$$

Now, assume the result is proven for sets of $k-1$ or fewer orthogonal vectors. Suppose $v_1, \ldots, v_k$ are orthogonal, and define $b_{k-1} = \sum_{i=1}^{k-1} a_i k_i$. Observe that

$$\langle b_{k-1}, a_k \rangle = \left\langle \sum_{i=1}^{k-1} a_i v_i, a_k v_k \right\rangle = \sum_{i=1}^{k-1} a_i \langle v_i, v_k \rangle = 0,$$

so a_k and b_{k-1} are orthogonal. Applying the previous problem,

$$\left\| \sum_{i=1}^{k} a_i v_i \right\|^2 = \|a_k v_k + b_{k-1}\|^2 = |a_k v_k\|^2 + \|b_k\|^2.$$

By assumption,

$$= |a_k|^2 \|v_k\|^2 + \sum_{i=1}^{k-1} |a_i|^2 \|v_i\|^2 = \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2.$$

17. Let T be a linear operator on an inner product space V, and suppose that $\|\mathsf{T}(x)\| = \|x\|$ for all x. Prove that T is one-to-one.

Solution: If $v \in \mathsf{N}(\mathsf{T})$, then $\|v\| = \|\mathsf{T}(v)\| = \|0\| = 0$. By coercivity of the norm, $v = 0$. Thus T is one-to-one.