Homework 6 Solutions

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2.5 - The Change of Coordinate Matrix

2. For each of the following pairs of ordered bases β and β' for $\mathbb{R}^2$, find the change of coordinate matrix that changes β'-coordinates into β coordinates.

b. $\beta = \{(\text{-}1, 3), (2, \text{-}1)\}$ and $\beta' = \{(0, 10), (5, 0)\}$.

Solution: We want to find $Q = [\mathsf{I}_{\mathbb{R}^2}]_{\beta'}^{\beta}$. The usual procedure:

$$\mathsf{I}_{\mathbb{R}^2}(0,10) = (0,10) = a(\text{-}1,3) + b(2,\text{-}1), \qquad \mathsf{I}_{\mathbb{R}^2}(5,0) = (5,0) = c(\text{-}1,3) + d(2,\text{-}1).$$

We can write these two systems as matrix equations

$$\begin{pmatrix} \text{-}1 & 2 \\ 3 & \text{-}1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \end{pmatrix}, \qquad \begin{pmatrix} \text{-}1 & 2 \\ 3 & \text{-}1 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix},$$

or together as

$$\begin{pmatrix} \text{-}1 & 2 \\ 3 & \text{-}1 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ 10 & 0 \end{pmatrix}.$$

The matrix $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ will be our change-of-coordinate matrix. We solve by taking inverses:

$$Q = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} \text{-}1 & 2 \\ 3 & \text{-}1 \end{pmatrix}^{\text{-}1}\begin{pmatrix} 0 & 5 \\ 10 & 0 \end{pmatrix} = \frac{1}{\text{-}5}\begin{pmatrix} \text{-}1 & \text{-}2 \\ \text{-}3 & \text{-}1 \end{pmatrix}\begin{pmatrix} 0 & 5 \\ 10 & 0 \end{pmatrix} = \frac{1}{\text{-}5}\begin{pmatrix} \text{-}20 & \text{-}5 \\ \text{-}10 & \text{-}15 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}.$$

Note: In general, let V be some finite-dimensional vector space, let α the standard basis of V, and let β and β' be two other bases of V. Then

$$[I_{\mathsf{V}}]_{\beta'}^{\beta} = [I_{\mathsf{V}}]_{\alpha}^{\beta}[\mathsf{I}_{\mathsf{V}}]_{\beta'}^{\alpha} = ([I_{\mathsf{V}}]_{\beta}^{\alpha})^{\text{-}1}[\mathsf{I}_{\mathsf{V}}]_{\beta'}^{\alpha}$$

The matrices $[\mathsf{I}_{\mathsf{V}}]_{\beta'}^{\alpha}$ and $[I_{\mathsf{V}}]_{\beta}^{\alpha}$ are easily computed; one can simply read off the coefficients from the basis vectors.

d. $\beta = \{(\text{-}4, 3), (2, \text{-}1)\}$ and $\beta' = \{(2, 1), (\text{-}4, 1)\}$.

Solution: Proceeding as above,

$$Q = \begin{pmatrix} \text{-}4 & 2 \\ 3 & \text{-}1 \end{pmatrix}^{\text{-}1}\begin{pmatrix} 2 & \text{-}4 \\ 1 & 1 \end{pmatrix} = \frac{1}{\text{-}2}\begin{pmatrix} \text{-}1 & \text{-}2 \\ \text{-}3 & \text{-}4 \end{pmatrix}\begin{pmatrix} 2 & \text{-}4 \\ 1 & 1 \end{pmatrix} = \frac{1}{\text{-}2}\begin{pmatrix} \text{-}4 & 2 \\ \text{-}10 & 8 \end{pmatrix} = \begin{pmatrix} 2 & \text{-}1 \\ 5 & \text{-}4 \end{pmatrix}.$$

3. For each of the following pairs of ordered bases β and β' for $\mathsf{P}_2(\mathbb{R})$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

b. $\beta = \{1, x, x^2\}$, and $\beta' = \{a_2x^2 + a_1x + a_0, b_2x^2 + b_1x + b_0, c_2x^2 + c_1x + c_0\}$

Solution: We can easily write vectors in β' as linear combinations in β:

$$\begin{array}{rcrcrcr} a_2x^2+a_1x+a_0 &=& a_0\,1 &+& a_1\,x &+& a_2\,x^2 \\ b_2x^2+b_1x+b_0 &=& b_0\,1 &+& b_1\,x &+& b_2\,x^2 \\ c_2x^2+c_1x+c_0 &=& c_0\,1 &+& c_1\,x &+& c_2\,x^2 \end{array}$$

Thus

$$Q = \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}.$$

d. $\beta = \{x^2 - x + 1, x + 1, x^2 + 1\}$ and $\beta' = \{x^2 + x + 4, 4x^2 - 3x + 2, 2x^2 + 3\}$

Solution: Let $\gamma = \{1, x, x^2\}$, the standard ordered basis of $\mathsf{P}_2(\mathbb{R})$. We compute $Q = [\mathsf{I}_{\mathsf{P}_2(\mathbb{R})}]_{\beta'}^{\beta}$ using the identity $Q = ([\mathsf{I}_{\mathsf{P}_2(\mathbb{R})}]_{\beta}^{\gamma})^{-1}[\mathsf{I}_{\mathsf{P}_2(\mathbb{R})}]_{\beta'}^{\gamma}$:

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 & 2 & 3 \\ 1 & -3 & 0 \\ 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 4 & 2 & 3 \\ 1 & -3 & 0 \\ 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ 1 & 3 & 1 \end{pmatrix}$$

(I found the inverse matrix using Cramer's rule, but you could also find it by solving the systems

$$a_i(1,-1,1) + b_i(1,1,0) + c_i(1,0,1) = e_i,$$

(where e_i is the ith standard basis vector of $\mathbb{R}^3$) and writing those coefficients down the ith column.

6. For each matrix A and ordered basis β, find $[\mathsf{L}_A]_\beta$. Also, find an invertible matrix such that $[\mathsf{L}_A]_\beta = Q^{-1}AQ$.

Solution: Let α be the standard ordered basis of the relevant $\mathbb{R}^n$ (i.e. the basis in which $[L_A]_\alpha^\alpha = A$). We can write

$$[\mathsf{L}_A]_\beta^\beta = [\mathsf{I}_{\mathbb{R}^n}\mathsf{L}_A\mathsf{I}_{\mathbb{R}^n}]_\beta^\beta = [\mathsf{I}_{\mathbb{R}^n}]_\alpha^\beta[\mathsf{L}_A]_\alpha^\alpha[I_{\mathbb{R}^n}]_\beta^\alpha = ([\mathsf{I}_{\mathbb{R}^n}]_\beta^\alpha)^{-1}A[I_{\mathbb{R}^n}]_\beta^\alpha$$

Letting $Q = [I_{\mathbb{R}^n}]_\beta^\alpha$ (this is just the matrix whose column vectors are the corresponding elements of β), we have our matrix such that $[\mathsf{L}_A]_\beta = Q^{-1}AQ$.

b. $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

Solution: Here $Q = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ and

$$[\mathsf{L}_A]_\beta = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 3 & 1 \end{pmatrix} = \frac{1}{-2} \begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1. \end{pmatrix}$$

d. $A = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

Solution: Now, to avoid another instance of Cramer's rule, I'll try to compute $[\mathsf{L}_A]_\beta$ directly - that is, send each of the basis vectors through the matrix, then try to express it as a linear combination in β. If the previous problem is any indication, this should be easier.

$$\begin{pmatrix}13 & 1 & 4\\ 1 & 13 & 4\\ 4 & 4 & 10\end{pmatrix}\begin{pmatrix}1\\1\\-2\end{pmatrix} = \begin{pmatrix}6\\6\\-12\end{pmatrix} = 6\begin{pmatrix}1\\1\\-2\end{pmatrix} + 0\begin{pmatrix}1\\-1\\0\end{pmatrix} + 0\begin{pmatrix}1\\1\\1\end{pmatrix},$$

$$\begin{pmatrix}13 & 1 & 4\\ 1 & 13 & 4\\ 4 & 4 & 10\end{pmatrix}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}12\\-12\\0\end{pmatrix} = 0\begin{pmatrix}1\\1\\-2\end{pmatrix} + 12\begin{pmatrix}1\\-1\\0\end{pmatrix} + 0\begin{pmatrix}1\\1\\1\end{pmatrix},$$

$$\begin{pmatrix}13 & 1 & 4\\ 1 & 13 & 4\\ 4 & 4 & 10\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}18\\18\\18\end{pmatrix} = 0\begin{pmatrix}1\\1\\-2\end{pmatrix} + 0\begin{pmatrix}1\\-1\\0\end{pmatrix} + 18\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

So $[\mathsf{L}_A]_\beta = \begin{pmatrix}6 & 0 & 0\\ 0 & 12 & 0\\ 0 & 0 & 18\end{pmatrix}$. As before, $Q = \begin{pmatrix}1 & 1 & 1\\ 1 & -1 & 1\\ -2 & 0 & 1\end{pmatrix}$.

I hasten to add that $[\mathsf{L}_A]_\beta$ is not usually a diagonal matrix. In this case, the basis β was carefully picked by the textbook authors.

10. Prove that if A and B are similar $n \times n$ matrices, then $\text{trace}(A) = \text{trace}(B)$.

Lemma: *If A and B are $n \times n$ matrices, then* $\text{trace}(AB) = \text{trace}(BA)$.

$$\text{trace}(AB) = \sum_{i=1}^{n}(AB)_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik}B_{ki} = \sum_{k=1}^{n}\sum_{i=1}^{n} B_{ki}A_{ik} = \sum_{k=1}^{n}(BA)_{kk} = \text{trace}(BA).$$

Solution: Suppose that $B = Q^{-1}AQ$, for some (invertible $n \times n$) matrix Q. Then

$$\text{trace}(B) = \text{trace}(Q^{-1}(AQ)) = \text{trace}((AQ)Q^{-1}) = \text{trace}(A(QQ^{-1})) = \text{trace}(A).$$

Here Q^{-1} takes the place of A, and AQ takes the place of B in the lemma above.

13. Let V be a finite-dimensional vector space over a field F, and let $\beta = \{x_1, \ldots, x_n\}$ be an ordered basis for V. Let Q be an $n \times n$ invertible matrix with entries from F. Define

$$x'_j = \sum_{i=1}^{n} Q_{ij}x_i, \quad \text{for } 1 \leq j \leq n \tag{1}$$

and set $\beta' = \{x'_1, \ldots, x'_n\}$. Prove that β' is a basis for V and hence that Q is the change of coordinate matrix changing β' coordinates into β coordinates.

Solution: Applying ϕ_β to both sides of (1),

$$[x'_j]_\beta = \sum_{i=1}^{n} Q_{ij}[x_i]_\beta, \quad \text{for } 1 \leq j \leq n.$$

Equating the kth entries of these vectors,

$$([x'_j]_\beta)_k = \sum_{i=1}^{n} Q_{ij}([x_i]_\beta)_k, \quad \text{for } 1 \leq j \leq n.$$

Now, since we're taking our vectors x_i from the ordered basis β, the coordinate vectors $[x_i]_\beta$ should be just the standard basis vectors e_i. So

$$([x_i]_\beta)_k = (e_i)_k = \delta_{ik},$$

and our sum becomes

$$([x'_j]_\beta)_k = \sum_{i=1}^{n} Q_{ij}\delta_{ik} = Q_{kj}.$$

In matrix form, we have

$$\begin{pmatrix}[x'_1]_\beta & \cdots & [x'_n]_\beta\end{pmatrix} = Q. \tag{2}$$

The RHS of (2) is an invertible matrix, so its column vectors C_j must be linearly independent. So $[x'_j]_\beta$, and consequently $x'_j = \phi_\beta^{-1}([x'_j]_\beta)$ must be linearly independent. Thus β' is a basis.

Equation (2) also shows that $Q = [\mathsf{I_V}]_{\beta'}^{\beta}$, so it is the change-of-basis matrix from β' variables to β variables.

4.4 - Summary–Important Facts about Determinants

4h Evaluate the determinant by any legitimate method

Solution: Working down the first column,

$$\begin{aligned}
\det\begin{pmatrix} 1 & -2 & 3 & -12 \\ -5 & 12 & -14 & 19 \\ -8 & 22 & -20 & 31 \\ -4 & 8 & -14 & 15 \end{pmatrix} &= \det\begin{pmatrix} 12 & -14 & 19 \\ 22 & -20 & 31 \\ 8 & -14 & 15 \end{pmatrix} - (-5)\det\begin{pmatrix} -2 & 3 & -12 \\ 22 & -20 & 31 \\ 8 & -14 & 15 \end{pmatrix} \\
&\quad + (-8)\det\begin{pmatrix} -2 & 3 & -12 \\ 12 & -14 & 19 \\ 8 & -14 & 15 \end{pmatrix} - (-4)\det\begin{pmatrix} -2 & 3 & -12 \\ 12 & -14 & 19 \\ 22 & -20 & 31 \end{pmatrix} \\
&= \left[12\det\begin{pmatrix} -20 & 31 \\ -14 & 15 \end{pmatrix} - 22\det\begin{pmatrix} -14 & 19 \\ -14 & 15 \end{pmatrix} + 8\det\begin{pmatrix} -14 & 19 \\ -20 & 31 \end{pmatrix}\right] \\
&\quad - (-5)\left[-2\det\begin{pmatrix} -20 & 31 \\ -14 & 15 \end{pmatrix} - 22\det\begin{pmatrix} 3 & -12 \\ -14 & 15 \end{pmatrix} + 8\det\begin{pmatrix} 3 & -12 \\ -20 & 31 \end{pmatrix}\right] \\
&\quad + (-8)\left[-2\det\begin{pmatrix} -14 & 19 \\ -14 & 15 \end{pmatrix} - 12\det\begin{pmatrix} 3 & -12 \\ -14 & 15 \end{pmatrix} + 8\det\begin{pmatrix} 3 & -12 \\ -14 & 19 \end{pmatrix}\right] \\
&\quad - (-4)\left[-2\det\begin{pmatrix} -14 & 19 \\ -20 & 31 \end{pmatrix} - 12\det\begin{pmatrix} 3 & -12 \\ -20 & 31 \end{pmatrix} + 22\det\begin{pmatrix} 3 & -12 \\ -14 & 19 \end{pmatrix}\right] \\
&= [12(134) - 22(56) + 8(-54)] - (-5)[-2(134) - 22(-123) + 8(-147)] \\
&\qquad + (-8)[-2(56) - 12(-123) + 8(-111)] - (-4)[-2(-54) - 12(-147) + 22(-111)] \\
&= 166.
\end{aligned}$$

5. Suppose $M \in \mathsf{M}_{n\times n}(F)$ can be written in the form

$$M = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}$$

where A is a square matrix. Prove that $\det(M) = \det(A)$.

Solution: This is a special case of problem 6, with $C = I$. By property 4, $\det(I) = 1$, so

$$\det(M) = \det(A)\det(I) = \det(A).$$

6. Prove that if $M \in \mathsf{M}_{n\times n}(F)$ can be written in the form

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where A and C are square matrices, then $\det(M) = \det(A)\det(C)$.

Solution: Assume $A \in \mathsf{M}_{k\times k}$. Let $m = n - k$. We will prove by induction on k.

For the case $k = 1$, we have the picture

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \left(\begin{array}{c|ccc} a & b_1 & \cdots & b_m \\ \hline 0 & & & \\ \vdots & & C & \\ 0 & & & \end{array}\right).$$

Expanding the determinant down the first column,

$$\begin{aligned} \det(M) &= a\det(\bar{M}_{11}) + 0 + \cdots + 0 \\ &= a\det(C) = \det(A)\det(C). \end{aligned}$$

Now, suppose we've proven this fact for matrices of dimensions up to $k-1$. We have the picture

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \left(\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1k} & b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} & b_{k1} & \cdots & b_{km} \\ \hline 0 & \cdots & 0 & & & \\ \vdots & \ddots & \vdots & & C & \\ 0 & \cdots & 0 & & & \end{array}\right).$$

Expanding down the first column:

$$\begin{aligned} \det(M) &= a_{11}\det(\bar{M}_{11}) - \cdots \pm a_{k1}\det(\bar{M}_{k1}) + 0 + \cdots + 0 \\ &= a_{11}\det\begin{pmatrix} \bar{A}_{11} & \bar{B}_1 \\ 0 & C \end{pmatrix} - \cdots \pm a_{k1}\det\begin{pmatrix} \bar{A}_{k1} & \bar{B}_k \\ 0 & C \end{pmatrix}, \end{aligned}$$

where $\bar{B}_j$ is the $(k-1)\times m$ matrix formed by removing the jth row of B. Note that $\bar{A}_{j1}$ is a $(k-1)\times(k-1)$ matrix, so we can apply our inductive assumption:

$$\begin{aligned} &= a_{11}\det(\bar{A}_{11})\det(C) - \cdots \pm a_{k1}\det(\bar{A}_{k1})\det(C) \\ &= \big(a_{11}\det(\bar{A}_{11}) - \cdots \pm a_{k1}\det(\bar{A}_{k1})\big)\det(C) \\ &= \det(A)\det(C). \end{aligned}$$

5.1 - Eigenvalues and Eigenvectors

2d. For $\mathsf{V} = \mathsf{P}_2(\mathbb{R})$, let $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ have the mapping

$$\mathsf{T}(a + bx + cx^2) = (\text{-}4a + 2b - 2c) - (7a + 3b + 7c)x + (7a + b + 5c)x^2.$$

Let $\beta = \{x - x^2, \text{-}1 + x^2, \text{-}1 - x + x^2\}$. Find $[\mathsf{T}]_\beta$, determine whether β consists of eigenvectors of T.

Solution:

$$\begin{aligned}
\mathsf{T}(x - x^2) &= (\text{-}4(0) + 2(1) - 2(\text{-}1)) \;-\; (7(0) + 3(1) + 7(\text{-}1))x \;+\; (7(0) + 1(1) + 5(\text{-}1))x^2 \\
&= 4 + 4x - 4x^2 \\
&= 0(x - x^2) \;+\; 0(\text{-}1 + x^2) \;+\; (\text{-}4)(\text{-}1 - x + x^2).
\end{aligned}$$

$$\begin{aligned}
\mathsf{T}(\text{-}1 + x^2) &= (\text{-}4(\text{-}1) + 2(0) - 2(1)) \;-\; (7(\text{-}1) + 3(0) + 7(1))x \;+\; (7(\text{-}1) + 1(0) + 5(1))x^2 \\
&= 2 - 2x^2 \\
&= 0(x - x^2) \;+\; (\text{-}2)(\text{-}1 + x^2) \;+\; 0(\text{-}1 - x + x^2).
\end{aligned}$$

$$\begin{aligned}
\mathsf{T}(\text{-}1 - x + x^2) &= (\text{-}4(\text{-}1) + 2(\text{-}1) - 2(1)) \;-\; (7(\text{-}1) + 3(\text{-}1) + 7(1))x \;+\; (7(\text{-}1) + 1(\text{-}1) + 5(1)) \\
&= 3x - 3x^2 \\
&= 3(x - x^2) \;+\; 0(\text{-}1 + x^2) \;+\; 0(\text{-}1 - x + x^2).
\end{aligned}$$

So this transformation has matrix

$$[\mathsf{T}]_\beta = \begin{pmatrix} 0 & 0 & 3 \\ 0 & \text{-}2 & 0 \\ \text{-}4 & 0 & 0 \end{pmatrix}.$$

This matrix is not diagonal (in the usual sense), and so the basis is not a basis of eigenvectors.

3. For each of the following matrices $A \in \mathsf{M}_{n\times n}(F)$,

(i) Determine all the eigenvalues of A

(ii) For each eigenvalue λ, find a set of corresponding eigenvectors.

(iii) If possible, find a basis of F^n consisting of eigenvectors of A.

(iv) If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{\text{-}1}AQ = D$.

b. $A = \begin{pmatrix} 0 & \text{-}2 & \text{-}3 \\ \text{-}1 & 1 & \text{-}1 \\ 2 & 2 & 5 \end{pmatrix}$ for $F = \mathbb{R}$.

Solution:

$$\begin{aligned}
\chi_A(\lambda) &= \det \begin{pmatrix} 0 - \lambda & \text{-}2 & \text{-}3 \\ \text{-}1 & 1 - \lambda & \text{-}1 \\ 2 & 2 & 5 - \lambda \end{pmatrix} \\
&= (0 - \lambda)(1 - \lambda)(5 - \lambda) + (\text{-}2)(\text{-}1)2 + (\text{-}3)(\text{-}1)2 - (0 - \lambda)(\text{-}1)2 - (\text{-}3)(1 - \lambda)2 - (\text{-}2)(\text{-}1)(5 - \lambda) \\
&= \text{-}\lambda^3 + (0 + 1 + 5)\lambda^2 + (\text{-}5 - 2 - 6 - (\text{-}2))\lambda + (0 + 4 + 6 - 0 - (\text{-}6) - 10) \\
&= \text{-}\lambda^3 + 6\lambda^2 - 11\lambda + 6.
\end{aligned}$$

Guessing correctly that 1 is a root of this polynomial, I can reduce it by long division:

$$\begin{aligned}
\text{-}\lambda^3 + 6\lambda^2 - 11\lambda + 6 \quad &= \quad (\lambda - 1)(\text{-}\lambda^2) & +5\lambda^2 - 11\lambda + 6 \\
&= \quad (\lambda - 1)(\text{-}\lambda^2 + 5\lambda) & -6\lambda + 6 \\
&= \quad (\lambda - 1)(\text{-}\lambda^2 + 5\lambda - 6) & \\
&= \quad \text{-}1(\lambda - 1)(\lambda - 2)(\lambda - 3). &
\end{aligned}$$

So my solutions are $\lambda = 1, 2, 3$. The corresponding eigenspaces must all have dimension 1. Computing eigenspaces:

$$\mathsf{E}_1 = \mathsf{N}\begin{pmatrix} 0-1 & \text{-}2 & \text{-}3 \\ \text{-}1 & 1-1 & \text{-}1 \\ 2 & 2 & 5-1 \end{pmatrix} = \mathsf{N}\begin{pmatrix} \text{-}1 & \text{-}2 & \text{-}3 \\ \text{-}1 & 0 & \text{-}1 \\ 2 & 2 & 4 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ \text{-}1 \end{pmatrix}\right\},$$

$$\mathsf{E}_2 = \mathsf{N}\begin{pmatrix} 0-2 & \text{-}2 & \text{-}3 \\ \text{-}1 & 1-2 & \text{-}1 \\ 2 & 2 & 5-2 \end{pmatrix} = \mathsf{N}\begin{pmatrix} \text{-}2 & \text{-}2 & \text{-}3 \\ \text{-}1 & \text{-}1 & \text{-}1 \\ 2 & 2 & 3 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ \text{-}1 \\ 0 \end{pmatrix}\right\},$$

$$\mathsf{E}_3 = \mathsf{N}\begin{pmatrix} 0-3 & \text{-}2 & \text{-}3 \\ \text{-}1 & 1-3 & \text{-}1 \\ 2 & 2 & 5-3 \end{pmatrix} = \mathsf{N}\begin{pmatrix} \text{-}3 & \text{-}2 & \text{-}3 \\ \text{-}1 & \text{-}2 & \text{-}1 \\ 2 & 2 & 2 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ \text{-}1 \end{pmatrix}\right\}.$$

Each of these vectors was computed by observing the (obvious) linear relations between the columns of the corresponding matrices. For instance, the first and third columns of the last matrix are identical. The diagonalizing basis and matrices are

$$\beta = \left\{\begin{pmatrix} 1 \\ 1 \\ \text{-}1 \end{pmatrix}, \begin{pmatrix} 1 \\ \text{-}1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \text{-}1 \end{pmatrix}\right\}, \qquad Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \text{-}1 & 0 \\ \text{-}1 & 0 & \text{-}1 \end{pmatrix}, \qquad \text{and} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

d. $A = \begin{pmatrix} 2 & 0 & \text{-}1 \\ 4 & 1 & \text{-}4 \\ 2 & 0 & \text{-}1 \end{pmatrix}$ for $F = \mathbb{R}$.

Solution:

$$\chi_A(\lambda) = \det\begin{pmatrix} 2-\lambda & 0 & \text{-}1 \\ 4 & 1-\lambda & \text{-}4 \\ 2 & 0 & \text{-}1-\lambda \end{pmatrix} = (2-\lambda)(1-\lambda)(\text{-}1-\lambda) - (\text{-}1)(1-\lambda)2$$
$$= (1-\lambda)(\lambda^2 - (2+\text{-}1)\lambda + (2\cdot 1(\text{-}1) - (\text{-}1)2) = (1-\lambda)\lambda(\lambda-1).$$

My solutions are $\lambda = 0, 1$. Then

$$\mathsf{E}_0 = \mathsf{N}\begin{pmatrix} 2 & 0 & \text{-}1 \\ 4 & 1 & \text{-}4 \\ 2 & 0 & \text{-}1 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}\right\},$$

$$\mathsf{E}_1 = \mathsf{N}\begin{pmatrix} 2-1 & 0 & \text{-}1 \\ 4 & 1-1 & \text{-}4 \\ 2 & 0 & \text{-}1-1 \end{pmatrix} = \mathsf{N}\begin{pmatrix} 1 & 0 & \text{-}1 \\ 4 & 0 & \text{-}4 \\ 2 & 0 & \text{-}2 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

The diagonalizing basis and matrices are

$$\beta = \left\{\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}, \qquad Q = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad \text{and} \qquad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4g. Find the eigenvalues of T and an ordered basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix, where $\mathsf{V} = \mathsf{P}_3(\mathbb{R})$ and $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ has the mapping

$$T(f(x)) = xf'(x) + f''(x) - f(2)$$

Solution: Observe that

$$\mathsf{T}(a+bx+cx^2+dx^3)$$
$$= x(b+2cx+3dx^2)+(2c+6dx)-(a+2b+4c+8d)$$
$$= (\text{-}a-2b-2c-8d)+(0a+b+0c+6d)x+(0a+0b+2c+0d)x^2+(0a+0b+0c+3d)x^3$$

Let $\alpha=\{1,x,x^2,x^3\}$, the standard ordered basis for $\mathsf{P}_3(\mathbb{R})$. Then

$$[\mathsf{T}]_\alpha = \begin{pmatrix} \text{-}1 & \text{-}2 & \text{-}2 & \text{-}8 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Problem 9 will show that -1, 1, 2, and 3 (the diagonal entries of this triangular matrix) are the eigenvalues of T. Then

$$\mathsf{E}_{\text{-}1} = \mathsf{N}\begin{pmatrix} 0 & \text{-}2 & \text{-}2 & \text{-}8 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}, \quad \mathsf{E}_1 = \mathsf{N}\begin{pmatrix} \text{-}2 & \text{-}2 & \text{-}2 & \text{-}8 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ \text{-}1 \\ 0 \\ 0 \end{pmatrix}\right\},$$
$$\mathsf{E}_2 = \mathsf{N}\begin{pmatrix} \text{-}3 & \text{-}2 & \text{-}2 & \text{-}8 \\ 0 & \text{-}1 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 2 \\ 0 \\ \text{-}3 \\ 0 \end{pmatrix}\right\}, \quad \mathsf{E}_3 = \mathsf{N}\begin{pmatrix} \text{-}4 & \text{-}2 & \text{-}2 & \text{-}8 \\ 0 & \text{-}2 & 0 & 6 \\ 0 & 0 & \text{-}1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} \text{-}7 \\ 6 \\ 0 \\ 2 \end{pmatrix}\right\}.$$

The diagonalizing basis is therefore

$$\beta = \{1,\ 1-x,\ 1-3x^2,\ \text{-}7+6x+2x^3\}.$$

9. Prove that the eigenvalues of an upper-triangular matrix M are the diagonal entries of M

Solution: Assume $M \in \mathsf{M}_{n\times n}(F)$. Then

$$\chi_M(\lambda) = \det\begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & \cdots & a_2n \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}-\lambda \end{pmatrix} = (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda),$$

since the determinant of an upper-triangular matrix is the product of its diagonal entries (determinant property 4). The roots of this polynomial, i.e. the eigenvalues of M, are $\lambda = a_{11}, a_{22}, \ldots, a_{nn}$.

11. A scalar matrix is a square matrix of the form λI.

a. Prove that if a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$

Solution: Suppose $Q^{\text{-}1}AQ = \lambda I$, for some invertible matrix Q. Then

$$A = Q(\lambda I)Q^{\text{-}1} = \lambda(QIQ^{\text{-}1}) = \lambda QQ^{\text{-}1} = \lambda I.$$

b. Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.

Solution: If A is diagonalizable, and has only one eigenvalue, λ, then there exists some invertible matrix Q such that

$$Q^{-1}AQ = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{pmatrix} = \lambda I.$$

By (a) above, A must be a scalar matrix.

c. Prove that $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is not diagonalizable

Solution: Observe that $\chi_A(\lambda) = (1-\lambda)(1-\lambda)$, so A has only one eigenvalue. However, A is not a scalar matrix, so by (b) above, A must not be diagonalizable.

14. For any square matrix A, prove that A and A^t have the same characteristic polynomial, and hence the same eigenvalues.

Solution: Determinant property 8 states that if A is a square matrix, then $\det(A) = \det(A^t)$. Observe, then,

$$\chi_A(\lambda) = \det(A - \lambda I) = \det((A-\lambda)^t) = \det(A^t - \lambda I^t) = \det(A^t - \lambda I) = \chi_{A^t}(\lambda).$$

The roots of LHS and RHS are the eigenvalues of A and A^t, so these eigenvalues must be the same.

15. a. Let T be a linear operator on the vector space V, and let x be an eigenvector of T corresponding to the eigenvalue λ. For any positive integer m, prove that x is an eigenvector of T^m corresponding to the eigenvalue λ^m.

Solution: Clearly $T^1(x) = \lambda^1 x$. Now, suppose $T^{m-1}(x) = \lambda^{m-1}x$. By associativity of functions,

$$\mathsf{T}^m(x) = \mathsf{T}(\mathsf{T}^{m-1}(x)) = \mathsf{T}(\lambda^{m-1}x) = \lambda^{m-1}\mathsf{T}(x) = \lambda^{m-1}\lambda x = \lambda^m x.$$

The rest follows by induction.

b. State and prove the analogous result for matrices.

Solution: Let $A \in \mathsf{M}_{n\times n}(F)$, and let $v \in \mathsf{F}^n$ be an eigenvector of A corresponding to the eigenvalue λ. For any positive integer m, x is the eigenvector of A^m corresponding to the eigenvalue λ^m.

Proof by induction: Clearly $A^1x = \lambda^1 x$. Now, suppose $A^{m-1}x = \lambda^{m-1}x$. By associativity of matrix multiplication,

$$A^m(x) = A(A^{m-1}x) = A(\lambda^{m-1}x) = \lambda^{m-1}Ax = \lambda^{m-1}(\lambda x) = \lambda^m x.$$

17. Let T be the linear operator on $\mathsf{M}_{n\times n}(\mathbb{R})$ defined by $\mathsf{T}(A) = A^t$.

a. Show that ± 1 are the only eigenvalues of T

Solution: Suppose A is an eigenvector of T corresponding to the eigenvalue λ. Then

$$A = (A^t)^t = \mathsf{T}(\mathsf{T}(A)) = \mathsf{T}^2(A) = \lambda^2 A.$$

If $A \neq 0$, then $\lambda^2 = 1$, and so $\lambda = \pm 1$.

b. Find the eigenvectors corresponding to these eigenvectors.

Solution: The eigenvectors corresponding to 1, those which satisfy

$$A^t = \mathsf{T}(A) = A,$$

are the symmetric matrices. The eigenvectors corresponding to -1, those which satisfy

$$A^t = \mathsf{T}(A) = -A,$$

are the skew-symmetric matrices

c. Find an ordered basis β for $\mathsf{M}_{2\times 2}(\mathbb{R})$ such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

Solution: We showed, a long time ago, that $\mathsf{M}_{2\times 2}$ is the direct sum of the symmetric and skew symmetric matrices. We can therefore form a basis of $\mathsf{M}_{2\times 2}$ from the disjoint union of the bases of the two spaces, i.e.

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

Then $[\mathsf{T}]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.