Homework 5 Solutions

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2.4 - Invertibility and Isomorphisms

4. Let A and B be $n \times n$ invertible matrices. Prove that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Solution: Using the associativity of matrix multiplication,

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and

$$(B^{-1}A^{-1})(AB) = B(AA^{-1})B^{-1} = BIB^{-1} = BB^{-1} = I.$$

Thus AB is invertible and $B^{-1}A^{-1}$ is its inverse

5. Let A be invertible. Prove that A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

Lemma: *Given* $A \in \mathsf{M}_{m\times n}(F)$ *and* $B \in \mathsf{M}_{n\times p}(F)$*, we have the identity* $(AB)^t = B^tA^t$.
Proof:

$$((AB)^t)_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk}B_{ki} = \sum_{k=1}^{n}(A^t)_{kj}(B^t)_{ik} = \sum_{k=1}^{n}(B^t)_{ik}(A^t)_{kj} = (B^tA^t)_{ij}.$$

The entries of $(AB)^t$ and B^tA^t all agree, so the matrices are identical.

Solution: Applying the Lemma above,

$$(A^t)(A^{-1})^t = (A^{-1}A)^t = I^t = I$$

and

$$(A^{-1})^t(A^t) = (AA^{-1})^t = I^t = I.$$

Thus $(A^{-1})^t = (A^t)^{-1}$

9. Let A and B be $n \times n$ matrices such that AB are invertible. Prove that A and B are invertible. Give an example to show that arbitrary matrices A and B need not be invertible if AB is invertible.

Lemma: *Given vector spaces* $\mathsf{V}, \mathsf{W}, \mathsf{Z}$ *and linear transformations* $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ *and* $\mathsf{U} : \mathsf{W} \to \mathsf{Z}$*,*

$$\mathsf{N}(\mathsf{T}) \subseteq \mathsf{N}(\mathsf{U} \circ \mathsf{T}) \quad \textit{and} \quad \mathsf{R}(\mathsf{U} \circ \mathsf{T}) \subseteq \mathsf{R}(\mathsf{U}).$$

Proof: Given $v \in \mathsf{N}(\mathsf{T})$,

$$(U \circ \mathsf{T})(v) = \mathsf{U}(\mathsf{T}(v)) = \mathsf{U}(0) = 0,$$

so $v \in \mathsf{N}(\mathsf{U} \circ \mathsf{T})$. Thus $\mathsf{N}(\mathsf{T}) \subseteq \mathsf{N}(\mathsf{U} \circ \mathsf{T})$. Given $z \in \mathsf{R}(\mathsf{U} \circ \mathsf{T})$, there is some v in V such that

$$z = (\mathsf{U} \circ \mathsf{T})(v) = \mathsf{U}(\mathsf{T}(v)) = \mathsf{U}(w) \quad \text{for} \quad w = \mathsf{T}(v) \in \mathsf{W},$$

so $z \in \mathsf{R}(\mathsf{U})$. Thus $\mathsf{R}(\mathsf{U} \circ \mathsf{T}) \subseteq \mathsf{R}(\mathsf{U})$.

Corollary:

$$\text{rank}(\mathsf{U} \circ \mathsf{T}) \leq \min\{\text{rank}(\mathsf{U}), \text{rank}(\mathsf{T})\}$$

Proof: By the rank-nullity theorem,

$$\text{rank}(\mathsf{U} \circ T) = \dim(V) - \text{nullity}(\mathsf{U} \circ \mathsf{T}) \leq \dim(V) - \text{nullity}(T) = \text{rank}(T).$$

Also,

$$\text{rank}(\mathsf{U} \circ T) = \dim(R(U \circ T)) \leq \dim(R(U)) = \text{rank}(U).$$

Solution: We will prove this by considering the linear transformations L_A and L_B. If AB is invertible, L_{AB} is an isomorphism, and $\text{rank}(\mathsf{L}_{AB}) = n$; By the above,

$$\text{rank}(\mathsf{L}_{AB}) = \text{rank}(\mathsf{L}_A \circ \mathsf{L}_B) \leq \min\{\text{rank}(\mathsf{L}_A), \text{rank}(\mathsf{L}_B)\},$$

so $\text{rank}(\mathsf{L}_A) = \text{rank}(L_B) = n$. Thus L_A and L_B are isomorphisms, and A and B are invertible.

To find a counterexample, we need to relax the condition that A and B be square matrices. Observe that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where the RHS is certainly invertible (in fact, it is its own inverse). However, neither of the multiplicands are invertible - they are not even square.

10. Let A and B be $n \times n$ matrices such that $AB = I_n$.

a. Use Exercise 9 to conclude that A and B are invertible.

Solution: Observe that $AB = I_n$ is invertible (in fact, it its own inverse): $I_n I_n = I_n$. Thus, by 9 above, A and B are both invertible.

b. Prove that $A = B^{-1}$

Solution: For any $v \in \mathbb{R}^n$ (applying associativity of matrix multiplication),

$$(BA)Bv = B(AB)v = B(I_n v) = Bv.$$

Thus BA acts as the identity matrix I_n (and thus L_{BA} acts as the identity transform $\mathsf{I}_{\mathbb{R}^n}$) on all $w = Bv \in \mathsf{R}(\mathsf{L}_B)$. By (a), B is invertible, so L_B is onto, and L_{BA} acts as $\mathsf{I}_{\mathbb{R}^n}$ everywhere. This mapping determines the identity $BA = I_n$.

c. State and prove analogous results for linear transformations defined on finite-dimensional vector spaces

Solution: Let W *and* V *be* n*-dimensional vector fields, and let* $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ *and* $\mathsf{U} : \mathsf{W} \to \mathsf{V}$ *be linear transformations such that* $U \circ T = \mathsf{I}_\mathsf{V}$. *Then* $T \circ \mathsf{U} = \mathsf{I}_\mathsf{W}$.

Proof: For any $v \in \mathsf{V}$, applying associativity of linear transformations,

$$(\mathsf{T} \circ \mathsf{U})(\mathsf{T}(v)) = \mathsf{T}(\mathsf{U} \circ \mathsf{T}(v)) = \mathsf{T}(v).$$

Thus $\mathsf{T} \circ \mathsf{U}$ acts as the identity on $\mathsf{R}(\mathsf{T})$. By the Corollary above, $\text{rank}(\mathsf{U}) = \text{rank}(\mathsf{T}) = n$, so $\mathsf{R}(\mathsf{T}) = \mathsf{W}$. Therefore $\mathsf{T} \circ \mathsf{U}$ acts as the identity everywhere.

17. Let V and W be finite-dimensional vector spaces and $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be an isomorphism. Let V_0 be a subspace of V.

a Prove that $\mathsf{T}(\mathsf{V}_0)$ is a subspace of W.

Solution: Given $v, w \in \mathsf{T}(\mathsf{V}_0)$, and $k \in \mathsf{F}$, there exist $v_0, w_0 \in \mathsf{V}_0$ such that $\mathsf{T}(v_0) = v$ and $\mathsf{T}(w_0) = w$. Then $v_0 + kw_0 \in \mathsf{V}_0$, and

$$v + kw = T(v_0) + kT(w_0) = T(v_0 + kw_0) \in T(\mathsf{V}_0).$$

Thus $\mathsf{T}(\mathsf{V}_0)$ is a subspace of W.

b Prove that $\dim(\mathsf{V}_0) = \dim(\mathsf{T}(\mathsf{V}_0))$.

Definition: Given vector spaces V, W and $\mathsf{V}_0 \subseteq \mathsf{V}$, and linear transformation $\mathsf{T} : \mathsf{V} \to \mathsf{W}$, we may define the linear transformation $\mathsf{T}|_{\mathsf{V}_0} : \mathsf{V}_0 \to \mathsf{W}$, the "restriction of T to V_0", by the mapping $\mathsf{T}|_{\mathsf{V}_0}(v_0) = T(v_0)$ for $v_0 \in \mathsf{V}_0$.

Lemma: *In the situation above,* $\mathsf{N}(\mathsf{T}|_{\mathsf{V}_0}) = \mathsf{N}(\mathsf{T}) \cap \mathsf{V}_0$*, and* $\mathsf{R}(\mathsf{T}|_{\mathsf{V}_0}) \subseteq \mathsf{R}(\mathsf{T})$.
Proof: Notice that

$$\begin{aligned} \mathsf{N}(\mathsf{T}|_{\mathsf{V}_0}) &= \{v_0 \in \mathsf{V}_0 : \mathsf{T}|_{\mathsf{V}_0}(v_0) = 0\} = \{v_0 \in \mathsf{V}_0 : \mathsf{T}(v_0) = 0\} \\ &= \{v_0 \in \mathsf{V}_0 : v_0 \in \mathsf{N}(\mathsf{T})\} = \mathsf{V}_0 \cap \mathsf{N}(\mathsf{T}). \end{aligned}$$

Next, if $w \in \mathsf{R}(\mathsf{T}|_{\mathsf{V}_0})$, then $w = \mathsf{T}(v_0)$ for some $v_0 \in \mathsf{V}_0 \subseteq \mathsf{V}$, so $w \in \mathsf{R}(\mathsf{T})$.

Solution: We may write $\mathsf{T}(\mathsf{V}_0) = \mathsf{R}(\mathsf{T}|_{\mathsf{V}_0})$, that is, the range of the restricted transformation $\mathsf{T}|_{\mathsf{V}_0} : \mathsf{V}_0 \to \mathsf{W}$. By the rank-nullity theorem,

$$\dim(\mathsf{T}(\mathsf{V}_0)) = \operatorname{rank}(\mathsf{T}|_{\mathsf{V}_0}) = \dim(\mathsf{V}_0) - \operatorname{nullity}(\mathsf{T}|_{\mathsf{V}_0}) \geq \operatorname{rank}(\mathsf{T}|_{\mathsf{V}_0}) = \dim(\mathsf{V}_0) - \operatorname{nullity}(\mathsf{T}) = \dim(\mathsf{V}_0)$$

Now, since $\dim(\mathsf{T}(\mathsf{V}_0)) = \operatorname{rank}(\mathsf{T}|_{\mathsf{V}_0}) \leq \dim(\mathsf{V}_0)$ trivially, we have the desired equality.

18. Repeat Example 7 with the polynomial $p(x) = 1 + x + 2x^2 + x^3$.

Solution: We are given $\mathsf{T}(f(x)) = f'(x)$, and standard bases $\beta \subseteq \mathsf{P}_3(\mathbb{R})$, $\gamma \subseteq \mathsf{P}_2(\mathbb{R})$. We have

$$A = [\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Now,

$$\mathsf{L}_A \phi_\beta(p(x)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}.$$

But since $\mathsf{T}(p(x)) = p'(x) = 1 + 4x + 3x^2$, we have

$$\phi_\gamma \mathsf{T}(p(x)) = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix},$$

so $\mathsf{L}_A \phi_\beta(p(x)) = \phi_\gamma \mathsf{T}(p(x))$.

20. Let $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W. Let β and γ be ordered bases for V and W, respectively. Prove that $\operatorname{rank}(\mathsf{T}) = \operatorname{rank}(\mathsf{L}_A)$

and that nullity(T) = nullity(L_A), where $A = [\mathsf{T}]_\beta^\gamma$.

Solution: Following the diagram in figure 2.2, we have $\mathsf{L}_A = \phi_\gamma \mathsf{T} \phi_\beta^{-1}$, and likewise $\mathsf{T} = \phi_\gamma^{-1} \mathsf{L}_A \phi_\beta$. Using the Corollary in problem 9 above,

$$\text{rank}(\mathsf{L}_A) \leq \min\{\text{rank}(\phi_\gamma), \text{rank}(\mathsf{T}), \text{rank}(\phi_\beta^{-1})\}$$

and

$$\text{rank}(\mathsf{T}) \leq \min\{\text{rank}(\phi_\gamma^{-1}), \text{rank}(\mathsf{L}_A), \text{rank}(\phi_\beta)\}.$$

Thus rank(T) = rank(L_A). By the rank-nullity theorem,

$$\text{nullity}(\mathsf{T}) = \dim(\mathsf{V}) - \text{rank}(\mathsf{T}) = n - \text{rank}(\mathsf{T}) = \dim(\mathbb{R}^n) - \text{rank}(\mathsf{L}_A) = \text{nullity}(\mathsf{L}_A).$$

22. Let $c_0, c_1, \ldots, c_n$ be distinct scalars from an infinite field F. Define $\mathsf{T} : \mathsf{P}_n(F) \to \mathsf{F}^{n+1}$ by $\mathsf{T}(f) = (f(c_0), f(c_1), \ldots, f(c_n))$. Prove that T is an isomorphism.

Solution: We can define an inverse transformation $\mathsf{T}^{-1} : \mathsf{F}^{n+1} \to \mathsf{P}_n(F)$ by the mapping

$$\mathsf{T}^{-1}(v_0, \ldots, v_n) = \underset{(c_0,v_0),\ldots,(c_n,v_n)}{\text{LagrangePoly}}(x),$$

where the RHS is the Lagrange polynomial passing through the points $(c_0, v_0), \ldots, (c_n, v_n)$. Since this polynomial is unique given a choice of $v_0, \ldots, v_n$, the inverse transformation is well-defined. We can check that T^{-1} is an inverse:

$$\begin{aligned}\mathsf{T}\mathsf{T}^{-1}(v_0, \ldots, v_n) &= \mathsf{T}\left(\underset{(c_0,v_0),\ldots,(c_n,v_n)}{\text{LagrangePoly}}(c_0), \ldots, \underset{(c_0,v_0),\ldots,(c_n,v_n)}{\text{LagrangePoly}}(c_n)\right) \\ &= (v_0, \ldots, v_n),\end{aligned}$$

and

$$\begin{aligned}\mathsf{T}^{-1}\mathsf{T}(f(x)) &= \mathsf{T}^{-1}(f(c_0), \ldots, f(c_n)) \\ &= \underset{(c_0,f(c_0)),\ldots,(c_n,f(c_n))}{\text{LagrangePoly}}(x) = f(x).\end{aligned}$$