Homework 4 Solutions

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2.2 - The Matrix Representation of a Linear Transformation

2. Let β and γ be the standard ordered bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. For each linear transformation $\mathsf{T} : \mathbb{R}^n \to \mathbb{R}^m$, compute $[\mathsf{T}]_\beta^\gamma$

b. $\mathsf{T} : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $\mathsf{T}(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$.

Solution: A transformation from a 2- to a 3- dimensional space has a 3×2 matrix:

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}.$$

c. $\mathsf{T} : \mathbb{R}^3 \to \mathbb{R}$ defined by $\mathsf{T}(a_1, a_2, a_3) = 2a_1 + a_2 - 3a_3$.

Solution:

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 2 & 1 & -3 \end{pmatrix}.$$

4. Define

$$\mathsf{T} : \mathsf{M}_{\times 2\times 2}(\mathbb{R}) \to \mathsf{P}_2(\mathbb{R}) \quad \text{by} \quad \mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2.$$

Let

$$\beta = \left\{ v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, v_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \quad \text{and} \quad \gamma = \{1, x, x^2\}.$$

Solution: $\mathsf{T}(v_1) = 1$, $\mathsf{T}(v_2) = 1 + x^2$, $\mathsf{T}(v_3) = 0$, $\mathsf{T}(v_4) = 2x$. Thus

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

5. Let

$$\alpha \left\{ v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, v_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \quad \text{and} \quad \beta = \{1, x, x^2\}, \quad \text{and} \quad \gamma = \{1\}.$$

c. Define $\mathsf{T} : \mathsf{M}_{\times 2\times 2}(F) \to F$ by $\mathsf{T}(A) = \text{trace}(A)$. Compute $[\mathsf{T}]_\alpha^\gamma$

Solution: $\mathsf{T}(v_1) = 1$, $\mathsf{T}(v_2) = 0$, $\mathsf{T}(v_3) = 0$, $T(v_4) = 1$. Thus

$$[\mathsf{T}]_\alpha^\gamma = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}.$$

d. Define $\mathsf{T} : P_2(\mathbb{R}) \to \mathbb{R}$ by $\mathsf{T}(f(x)) = f(2)$. Compute $[\mathsf{T}]_\beta^\gamma$.

Solution: $\mathsf{T}(1) = 1|_{x=2} = 1$, $\mathsf{T}(x) = x|_{x=2} = 2$, $T(x^2) = x^2|_{x=2} = 4$. Thus

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}.$$

8. Let V be an n-dimensional vector space with an ordered basis β. Define $\mathsf{T} : \mathsf{V} \to \mathsf{F}^n$ by the mapping $\mathsf{T}(x) = [x]_\beta$. Prove that T is linear.

Solution: Denote $\beta = \{v_1, \ldots, v_n\}$. Given $v = c_1v_1 + \cdots + c_nv_n$, and $w = d_1v_1 + \cdots + d_nv_n$ in V, and some scalar a,

$$\begin{aligned}\mathsf{T}(v + aw) &= \mathsf{T}\big((c_1 + ad_1)v_1 + \cdots + (c_n + ad_n)v_n\big) = (c_1 + ad_1, \ldots, c_n + ad_n) \\ &= (c_1, \ldots, c_n) + a(d_1, \ldots, d_n) = \mathsf{T}(v) + a\mathsf{T}(w).\end{aligned}$$

14. Let $\mathsf{V} = \mathsf{P}(\mathbb{R})$ and for $j \geq 1$, define $\mathsf{T}_j(f(x)) = f^{(j)}(x)$, where $f^(j)(x)$ is the j-th derivative of $f(x)$. Prove that the set $\{\mathsf{T}_1, \ldots, \mathsf{T}_n\}$ is a linearly independent subset of $\mathcal{L}$.

Solution: Suppose there existed a linear combination $c_1, \ldots, c_n$ such that

$$c_1\mathsf{T}_1 + c_2\mathsf{T}_2 + \cdots + c_n\mathsf{T}_n = \mathsf{T}_0 = \mathbf{0}_{\mathcal{L}(\mathsf{V})}. \tag{1}$$

That is, for every polynomial p,

$$0 = (c_1\mathsf{T}_1 + \cdots + c_n\mathsf{T}_n)(p) = c_1\mathsf{T}_1(p) + \cdots + c_n\mathsf{T}_n(p) = c_1p' + c_2p'' + \cdots + c_np^{(n)}.$$

If $p = x^n$, this becomes

$$c_1(nx^{n-1}) + c_2(n(n-1)x^{n-2}) + \cdots + c_n(n!) = 0.$$

This is a linear combination of polynomials, all of different degrees. We showed in a previous problem that this combination must be trivial. Thus (1) is trivial, and we have shown that $\mathsf{T}_1, \ldots, \mathsf{T}_n$ are linearly independent.

16. Let V and W be vector spaces such that $\dim(V) = \dim(W)$, and let $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be linear. Show that there exist ordered bases β and γ for V and W respectively, such that $[T]_\beta^\gamma$ is a diagonal matrix.

Solution: I'll assume this means only finite-dimensional vector spaces - the book doesn't define matrix representations for infinite bases. Let $\alpha = \{v_1, \ldots, v_k\}$ be a basis of $\mathsf{N}(\mathsf{T})$. Extend this to $\beta = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$, a basis of V.

Define $\eta = \{\mathsf{T}(v_{k+1}, \ldots, \mathsf{T}(v_n)\}$. I claim that this set is linearly independent. Suppose

$$c_{k+1}\mathsf{T}(v_{k+1}) + \cdots + c_n\mathsf{T}(v_n) = 0 \tag{2}$$

for some scalars $c_1, \ldots, c_n$. We can pull T out of the sum:

$$\mathsf{T}(c_{k+1}v_{k+1} + \cdots + c_nv_n) = 0.$$

Thus $c_{k+1}v_{k+1} + \cdots + c_nv_n \in \mathsf{N}(\mathsf{T})$. However, as is the case whenever a basis is partitioned,

$$\operatorname{span}\{v_1, \ldots, v_k\} \cap \operatorname{span}\{v_{k+1}, \ldots, v_n\} = \{0\}.$$

So $c_{k+1}v_{k+1} + \cdots + c_nv_n = 0$. Linear independence of $v_{k+1}, \ldots, v_n$ implies that $c_{k+1} = \ldots = c_n = 0$, so (2) is a trivial linear combination.

Extend η to $\gamma = \{w_1, \ldots, w_k, \mathsf{T}(v_{k+1}), \ldots \mathsf{T}(v_n)\}$, a basis of W. Now consider the matrix $[\mathsf{T}]_\beta^\gamma$. First of all, $\mathsf{T}(v_1) = \cdots = \mathsf{T}(v_k) = 0$. Next, the images of the remaining basis vectors in β, are exactly the corresponding vectors in γ. That is,

$$\begin{aligned}
\mathsf{T}(v_{k+1}) &= 0w_1 + \cdots + 0w_k + 1\mathsf{T}(v_{k+1}) + 0\mathsf{T}(v_{k+2}) + \cdots + 0\mathsf{T}(v_n) \\
\mathsf{T}(v_{k+2}) &= 0w_1 + \cdots + 0w_k + 0\mathsf{T}(v_{k+1}) + 1\mathsf{T}(v_{k+2}) + \cdots + 0\mathsf{T}(v_n) \\
&\vdots \\
\mathsf{T}(v_n) &= 0w_1 + \cdots + 0w_k + 0\mathsf{T}(v_{k+1}) + 0\mathsf{T}(v_{k+2}) + \cdots + 1\mathsf{T}(v_n).
\end{aligned}$$

Thus $([\mathsf{T}]_\beta^\gamma)_{ij} = 1_{\{i>k\}}\delta_{ij}$, that is, a diagonal matrix with first k diagonal entries set to 0, and remaining entries set to 1.

2.3 - Composition of Linear Transformations and Matrix Multiplication

1. *A*nswers are given in the book

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5. Complete the proof of Theorem 2.12 and its corollary: Suppose A to be an $m \times n$ matrix, B to be an $n \times p$ matrix, and C to be a $q \times m$ matrix.

(b) $a(AB) = (aA)B = A(aB)$ for any scalar a.

Solution: Considering the ijth entries,

$$(a(AB))_{ij} = a(AB)_{ij} = a\sum_{k=1}^{n} A_{ik}B_{kj}.$$

Next,

$$((aA)B)_{ij} = \sum_{k=1}^{n}(aA)_{ik}B_{kj} = \sum_{k=1}^{n} aA_{ik}B_{kj} = a\sum_{k=1}^{n} A_{ik}B_{kj},$$

the same as the above. Next,

$$(A(aB))_{ij} = \sum_{k=1}^{n} A_{ik}(aB)_{kj} = \sum_{k=1}^{n} A_{ik}aB_{kj} = a\sum_{k=1}^{n} A_{ik}B_{kj},$$

the same as the others.

(d) If V is an n-dimensional vector space with an ordered basis β, then $[\mathsf{I_V}]_\beta = I_n$.

Solution: Denote $\beta = \{v_1, \ldots, v_n\}$. Since

$$\mathsf{I_V}(v_i) = v_i = \sum_{j=1}^{n} \delta_{ij}v_j,$$

we have $([\mathsf{I_V}])_{ij} = \delta_{ij} = (I_n)_{ij}$.

Cor. Let A be an $m \times n$ matrix, $B_1, \ldots, B_k$ be $n \times p$ matrices, $C_1, \ldots, C_k$ be $q \times m$ matrices, and $a_1, \ldots, a_k$ be scalars. Then

$$A\left(\sum_{i=1}^{k} a_i B_i\right) = \sum_{i=1}^{k} a_i A B_i$$

and

$$\left(\sum_{i=1}^{k} a_i C_i\right) A = \sum_{i=1}^{k} a_i C_i A.$$

Solution: I will only prove the first equality. The proof of the second is almost identical. Assume that this fact is known for $k < N$ (if $k = 1$, this is just the situation in part (b)). Now, consider

$$A\left(\sum_{i=1}^{N} a_i B_i\right) = A\left(\sum_{i=1}^{N-1} a_i b_i + a_N B_N\right).$$

By distributivity,

$$= A\left(\sum_{i=1}^{N-1} a_i B_i\right) + A(a_N B_N).$$

By our assumption for $k < N$,

$$= \sum_{i=1}^{N-1} a_i A B_i + a_N A B_N = \sum_{i=1}^{N} a_i A B_i,$$

and that gives us the desired form.

6. Prove (b) of Theorem 2.13:

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. For each j ($1 \leq j \leq p$) let u_j and v_j denote the jth columns of AB and B, respectively. Then

(b) $v_j = Be_j$, where e_j is the jth standard vector of F^p.

Solution: We have

$$v_j = \begin{pmatrix} B_{1j} \\ B_{2j} \\ \vdots \\ B_{nj} \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{p} \delta_{kj} B_{1k} e_k \\ \sum_{k=1}^{p} \delta_{kj} B_{2k} e_k \\ \vdots \\ \sum_{k=1}^{p} \delta_{kj} B_{nk} e_k \end{pmatrix} = \begin{pmatrix} (Be_j)_1 \\ (Be_j)_2 \\ \vdots \\ (Be_j)_n \end{pmatrix} = Be_j,$$