

Homework 1 Solutions

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1 2.1 - Linear Transformations, Null Spaces, and Ranges

For 3 and 6, prove that T is a linear transformation, find bases for both $N(T)$ and $R(T)$, compute the nullity and rank of T , and verify the dimension theorem. Finally, determine whether T is one-to-one or onto:

3 $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$.

Solution: Linearity:

$$\begin{aligned} T((a_1, a_2) + c(b_1, b_2)) &= T((a_1 + cb_1, a_2 + cb_2)) \\ &= ((a_1 + cb_1) + (a_2 + cb_2), 0, 2(a_1 + cb_1) - (a_2 + cb_2)) \\ &= ((a_1 + a_2) + c(b_1 + b_2), 0, (2a_1 - a_2) + c(2b_1 - b_2)) \\ &= (a_1 + a_2, 0, 2a_1 - a_2) + c(b_1 + b_2, 0, 2b_1 - b_2) \\ &= T(a_1, a_2) + cT(b_1, b_2). \end{aligned}$$

As for the nullspace,

$$N(T) = \{(a_1, a_2) : a_1 + a_2 = 2a_1 - a_2 = 0\}$$

The linear system

$$\begin{aligned} a_1 + a_2 &= 0 \\ 2a_1 - a_2 &= 0 \end{aligned}$$

has only the trivial solution. Therefore, the nullspace of T is $\{0\}$. The range of T is

$$\text{span}\{(1, 0, 2), (1, 0, -1)\} = \text{span}\{(1, 0, 0), (0, 0, 1)\};$$

each of these sets is also a basis of $R(T)$. Thus $\text{rank}(T) = 2$. This verifies the rank-nullity theorem:

$$2 = \dim(V) = \text{nullity}(T) + \text{rank}(T) = 0 + 2.$$

This function is one-to-one, since its nullity is zero. It is not onto, since the second coordinate of any vector in the range must be zero.

6 $T : M_{n \times n}(F) \rightarrow F$ defined by $T(A) = \text{tr}(A)$.

Solution: Linearity:

$$\begin{aligned} T(A + cB) &= \sum_{i=1}^n (A + cB)_{ii} = \sum_{i=1}^n (A_{ii} + cB_{ii}) \\ &= \sum_{i=1}^n A_{ii} + c \sum_{i=1}^n B_{ii} = T(A) + cT(B). \end{aligned}$$

In Section 1.6, problem 15 (Homework 2), we computed a basis for the nullspace, and concluded that it had dimension $n^2 - 1$. The range of \mathbb{T} is all of \mathbb{F} (any $r \in \mathbb{F}$ can be the trace of a matrix - take the matrix whose only nonzero entry is r , in the upper-left corner). Thus $\text{rank}(T) = \dim(\mathbb{F}) = 1$. The rank-nullity theorem holds:

$$n^2 = \dim(\mathbb{V}) = \text{nullity}(\mathbb{T}) + \text{rank}(\mathbb{T}) = (n^2 - 1) + 1$$

This function is one-to-one if and only if $n = 1$. It is always onto, since $\text{R}(T) = \mathbb{F}$.

9 In this exercise, $\mathbb{T} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a function. For each of the following parts, state why \mathbb{T} is not linear.

a $\mathbb{T}(a_1, a_2) = (1, a_2)$.

Solution:

$$0 \cdot \mathbb{T}(0, 0) = (0, 0) \neq (1, 0) = \mathbb{T}(0, 0) = \mathbb{T}(0 \cdot (0, 0))$$

Thus \mathbb{T} fails linearity.

c $\mathbb{T}(a_1, a_2) = (a_1, a_1^2)$.

Solution:

$$-1 \cdot \mathbb{T}(1, 0) = (-1, -1) \neq (1, 1) = \mathbb{T}(-1, 0) = \mathbb{T}(-1 \cdot (1, 0))$$

Thus \mathbb{T} fails linearity.

e $\mathbb{T}(a_1, a_2) = (a_1 + 1, a_2)$.

Solution:

$$0 \cdot \mathbb{T}(0, 0) = (0, 0) \neq (1, 0) = \mathbb{T}(0, 0) = \mathbb{T}(0 \cdot (0, 0))$$

Thus \mathbb{T} fails linearity.

14c Let \mathbb{V} and \mathbb{W} be vector spaces and $\mathbb{T} : \mathbb{V} \rightarrow \mathbb{W}$ be linear. Suppose $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for \mathbb{V} and \mathbb{T} is one-to-one and onto. Prove that

$$\mathbb{T}(\beta) := \{\mathbb{T}(v_1), \mathbb{T}(v_2), \dots, \mathbb{T}(v_n)\}$$

is a basis for \mathbb{W} .

Solution: We must prove that (1) $\mathbb{T}(\beta)$ is linearly independent and (2) $\mathbb{T}(\beta)$ spans \mathbb{W}

(1) Given a linear combination in $\mathbb{T}(\beta)$

$$c_1 \mathbb{T}(v_1) + c_2 \mathbb{T}(v_2) + \dots + c_n \mathbb{T}(v_n) = 0, \tag{1}$$

we can pull T out of the sum

$$\mathbb{T}(c_1 v_1 + c_2 v_2 + \dots + c_n v_n) = 0.$$

Since \mathbb{T} is one-to-one, we conclude that the argument (the input to the function) must be zero:

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0.$$

However, $\beta = \{v_1, \dots, v_n\}$ is a linearly independent set, so $c_1 = c_2 = \dots = c_n = 0$. Therefore (1) must be trivial. Therefore $\mathbb{T}(\beta)$ is linearly independent.

(2) Given $w \in W$, there exists, by surjectivity (“onto-ness”) of T , some $v \in V$ such that $w = T(v)$. Since β spans v , there must be some c_1, \dots, c_n such that

$$w = T(v) = T(c_1v_1 + c_2v_2 + \dots + c_nv_n) = c_1T(v_1) + c_2T(v_2) + \dots + c_nT(v_n)$$

Thus $w \in \text{span}\{T(v_1), T(v_2), \dots, T(v_n)\}$.

The moral of this exercise is that one-to-one functions preserve the linear independence of sets of vectors, and onto functions preserve surjectivity of sets of vectors. Functions which are both one-to-one and onto therefore preserve bases.

17 Let V and W be finite-dimensional vector spaces and $T : V \rightarrow W$ be linear.

(a) Prove that if $\dim(V) < \dim(W)$, then T cannot be onto.

Solution: The rank-nullity theorem gives that $\text{rank}(T) + \text{nullity}(T) = \dim(V)$. Specifically, $\text{rank}(T) \leq \dim(V)$. In this case, $\dim(V) < \dim(W)$, so $\dim(\text{R}(T)) = \text{rank}(T) < \dim(W)$. Thus $\text{R}(T) \neq W$, and T is not onto.

(b) Prove that if $\dim(V) > \dim(W)$, then T cannot be one-to-one.

Solution: The rank-nullity theorem gives that $\text{nullity}(T) = \dim(V) - \text{rank}(T)$. Now, $\text{rank}(T) \leq \dim(W)$, so $\text{nullity}(T) \geq \dim(V) - \dim(W)$. If $\dim(V)$ is strictly larger than $\dim(W)$, then the nullspace must be nontrivial, and T cannot be one-to-one.

18 Give an example of a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\text{N}(T) = \text{R}(T)$.

Solution: Consider the linear transform $T : \mathbb{R} \rightarrow \mathbb{R}$ with mapping

$$T(a, b) = (0, a)$$

Then, $\text{N}(T) = \{(a, b) \in \mathbb{R}^2 : a = 0\} = \{(0, b) : b \in \mathbb{R}\} = \text{R}(T)$.

24 Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Include figures for each of the following parts.

1. Find a formula for $T(a, b)$, where T represents the projection on the y -axis along the x -axis.

Solution: If T projects on the y -axis, then $T(a, b)$ must have the form $(0, s)$. If it projects along the x -axis, the displacement vector $T(a, b) - (a, b)$ must have the form $(t, 0)$. Thus

$$(0, s) = T(a, b) = (a, b) + (t, 0).$$

The linear system

$$\begin{array}{rcl} -t & = & a \\ s & = & b \end{array}$$

has solution $t = -a$, $s = b$, and so $T(a, b) = (0, b)$. Observe that, when applied to a point on the y -axis, the point is not moved.

$$T(0, b) = (0, b).$$

2. Find a formula for $T(a, b)$, where T represents the projection on the y axis along the line $L = \{(s, s) : s \in \mathbb{R}\}$.

Solution: Here, $\mathbb{T}(a, b)$ must have the form $(0, t)$, and the displacement vector must have the form (s, s) . Thus

$$(0, t) = \mathbb{T}(a, b) = (a, b) + (s, s).$$

The linear system

$$\begin{array}{rcl} & -s & = a \\ t + & -s & = b \end{array}$$

has solution $s = -a$, $t = b - a$, so $\mathbb{T}(a, b) = (a, b - a)$. Again, observe

$$\mathbb{T}(0, b) = (0, b - 0) = (0, b).$$