Homework 2 Solutions

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1.4 - Linear Combinations and Systems of Linear Equations

2. Solve the following systems of linear equations.

b.

$$\begin{array}{rcrcrcr} 2x_1 & - & 7x_2 & + & 4x_3 & = & 10 \\ x_1 & - & 2x_2 & + & x_3 & = & 3 \\ 2x_1 & - & x_2 & - & 2x_3 & = & 6 \end{array}$$

Solution:

1. Scaling down from first pivot:

$$\begin{array}{rrcrcrcr} 1 \cdot (& 2x_1 & - & 7x_2 & + & 4x_3 & = & 10) \\ -2 \cdot (& x_1 & - & 2x_2 & + & x_3 & = & 3) \\ -1 \cdot (& 2x_1 & - & x_2 & - & 2x_3 & = & 6) \end{array}$$

2. Summing down from first row:

$$\begin{array}{rcrcrcr} 2x_1 & - & 7x_2 & + & 4x_3 & = & 10 \\ & & -3x_2 & + & 2x_3 & = & 4 \\ & & -6x_2 & + & 6x_3 & = & 4 \end{array}$$

3. Scaling down from second pivot:

$$\begin{array}{rrcrcrcr} & 2x_1 & - & 7x_2 & + & 4x_3 & = & 10 \\ 2(& & & -3x_2 & + & 2x_3 & = & 4) \\ -1(& & & -6x_2 & + & 6x_3 & = & 4) \end{array}$$

4. Summing down from second row:

$$\begin{array}{rcrcrcr} 2x_1 & - & 7x_2 & + & 4x_3 & = & 10 \\ & & -3x_2 & + & 2x_3 & = & 4 \\ & & & & -2x_3 & = & 4 \end{array}$$

5. Scaling up from second pivot:

$$\begin{array}{rrcrcrcr} -3 \cdot (& 2x_1 & - & 7x_2 & + & 4x_3 & = & 10) \\ 7 \cdot (& & & -3x_2 & + & 2x_3 & = & 4) \\ & & & & & -2x_3 & = & 4 \end{array}$$

6. Summing up from second pivot:

$$\begin{array}{rcrcrcr} -6x_1 & & & + & 2x_3 & = & -2 \\ & & -3x_2 & + & 2x_3 & = & 4 \\ & & & & -2x_3 & = & 4 \end{array}$$

7. Summing up from third pivot:

$$\begin{array}{rrrcr} -6x_1 & & & = & 2 \\ & -3x_2 & & = & 8 \\ & & -2x_3 & = & 4 \end{array}$$

8. Solution:

$$\begin{array}{rcr} x_1 & = & -1/3 \\ x_2 & = & -8/3 \\ x_3 & = & -2 \end{array}$$

d.

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & 2x_3 & & & = & 2 \\ x_1 & & & + & 8x_3 & + & 5x_4 & = & -6 \\ x_1 & + & x_2 & + & 5x_3 & + & 5x_4 & = & 3 \end{array}$$

Solution:

1. Scaling down from first pivot:

$$\begin{array}{rrcrcrcrcr}
 & x_1 & + & 2x_2 & + & 2x_3 & & & = & 2 \\
\text{-1} \cdot (& x_1 & & & + & 8x_3 & + & 5x_4 & = & \text{-6}) \\
\text{-1} \cdot (& x_1 & + & x_2 & + & 5x_3 & + & 5x_4 & = & 3)
\end{array}$$

2. Summing down from first row:

$$\begin{array}{rcrcrcrcr}
x_1 & + & 2x_2 & + & 2x_3 & & & = & 2 \\
 & & 2x_2 & + & \text{-6}x_3 & - & 5x_4 & = & 8 \\
 & & x_2 & + & \text{-3}x_3 & - & 5x_4 & = & \text{-1}
\end{array}$$

3. Scaling down from second pivot:

$$\begin{array}{rrcrcrcrcr}
 & x_1 & + & 2x_2 & + & 2x_3 & & & = & 2 \\
 & & & 2x_2 & + & \text{-6}x_3 & - & 5x_4 & = & 8 \\
\text{-2} \cdot (& & & x_2 & + & \text{-3}x_3 & - & 5x_4 & = & \text{-1})
\end{array}$$

4. Summing down from second row:

$$\begin{array}{rcrcrcrcr}
x_1 & + & 2x_2 & + & 2x_3 & & & = & 2 \\
 & & 2x_2 & + & \text{-6}x_3 & - & 5x_4 & = & 8 \\
 & & & & & & 5x_4 & = & 10
\end{array}$$

5. Scaling up from second pivot:

$$\begin{array}{rrcrcrcrcr}
\text{-1} \cdot (& x_1 & + & 2x_2 & + & 2x_3 & & & = & 2) \\
 & & & 2x_2 & + & \text{-6}x_3 & - & 5x_4 & = & 8 \\
 & & & & & & & 5x_4 & = & 10
\end{array}$$

6. Summing up from second pivot:

$$\begin{array}{rcrcrcrcr}
\text{-}x_1 & & & + & \text{-8}x_3 & - & 5x_4 & = & 6 \\
 & & 2x_2 & + & \text{-6}x_3 & - & 5x_4 & = & 8 \\
 & & & & & & 5x_4 & = & 10
\end{array}$$

7. Summing up from third pivot:

$$\begin{array}{rcrcrcrcr}
\text{-}x_1 & & & + & \text{-8}x_3 & & & = & 16 \\
 & & 2x_2 & + & \text{-6}x_3 & & & = & 18 \\
 & & & & & & 5x_4 & = & 10
\end{array}$$

8. Solution:

$$\begin{array}{rcrcr}
x_1 & = & \text{-16} & + & \text{-}x_3 \\
x_2 & = & 9 & + & 3x_3 \\
x_3 & = & & & x_3 \\
x_4 & = & 2 & &
\end{array}$$

This solution set is the line through the point (-16, 9, 0, 2), in the direction (-1, 3, 1, 0).

3. For each of the following lists of vectors in $\mathbb{R}^3$, determine whether the first vector can be expressed as a linear combination of the other two.

b. (1, 2, -3), (-3, 2, 1), (2, -1, -1).

Solution: We can find such a linear combination i.e. $a, b \in \mathbb{R}$ such that

$$a(1,2,\text{-}3) + b(\text{-}3,2,1) = (2,\text{-}1,\text{-}1)$$

if we can find a solution to the linear system

$$\begin{array}{rcrcr} a & + & \text{-}3b & = & 2 \\ 2a & + & 2b & = & \text{-}1 \\ \text{-}3a & + & b & = & \text{-}1 \end{array}$$

1. Scaling down from first pivot:

$$\begin{array}{rrcrcr} 6\cdot(& a & + & \text{-}3b & = & 2) \\ \text{-}3\cdot(& 2a & + & 2b & = & \text{-}1) \\ 2\cdot(& \text{-}3a & + & b & = & \text{-}1) \end{array}$$

2. Summing down from first row:

$$\begin{array}{rcrcr} a & + & \text{-}3b & = & 2 \\ & & \text{-}24b & = & 15 \\ & & \text{-}16b & = & 10 \end{array}$$

The bottom two rows are equivalent.
So these vectors are indeed linearly dependent.

3. Scaling up from the second pivot:

$$\begin{array}{rrcrcr} 8\cdot(& a & + & \text{-}3b & = & 2) \\ 3\cdot(& & & 8b & = & \text{-}5) \end{array}$$

3. Summing up from the second row:

$$\begin{array}{rrcr} a & & = & 1 \\ & 8b & = & \text{-}5 \end{array}$$

4. Solution:

$$\begin{array}{rcr} a & = & 1 \\ b & = & \text{-}5/8 \end{array}$$

d. $(2,\text{-}1,0)$, $(1,2,\text{-}3)$, $(1,\text{-}3,2)$.

Solution: Suppose there existed such a linear combination,

$$a(2,\text{-}1,0) + b(1,2,\text{-}3) = (1,\text{-}3,2)$$

Then there must be a solution to the linear system

$$\begin{array}{rcrcr} 2a & + & b & = & 1 \\ \text{-}a & + & 2b & = & \text{-}3 \\ & & \text{-}3b & = & 2 \end{array}$$

So $b = \text{-}2/3$. Plugging this into the two first equations,

$$\begin{array}{rcrcr} 2a & + & \text{-}2/3 & = & 1 \\ \text{-}a & + & 2(\text{-}2/3) & = & \text{-}3 \end{array}$$

Solving for a gives conflicting results: $a = 5/6$, and $5/3$. This set of equations has no solution, so the three vectors are linearly independent.

5h. Determine whether the given vector is in the span of S:

$$v = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S = \left\{ v_1 = \begin{pmatrix} 1 & 0 \\ \text{-}1 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

Solution: Suppose there existed $a, b, c \in \mathbb{R}$ such that $av_1 + bv_2 + cv_2 = v$. A quick check of the set reveals that only v_1 has a nonzero element in the bottom-left position. Since v has a zero in that position, then $a = 0$. Likewise, considering the bottom-right element, we see that $b = 1$. This leaves us with the much simpler linear combination,

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} c & 1+c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Considering the elements, we get the conflicting solutions $c = 0$ and $c =$ -1.

10. Show that if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices.

Solution: The generic symmetric 2×2 matrix satisfies the equation $A = A^t$, or

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

This gives the solutions

$$a_{11} = a_{11}, \qquad a_{12} = a_{12}, \qquad a_{21} = a_{12}, \qquad a_{22} = a_{22}$$

and the solution set

$$\left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} : a_{11}, a_{12}, a_{22} \in \mathbb{R} \right\}.$$

This generic symmetric matrix can be generated from the given matrices,

$$a_{11} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_{12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + a_{22} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so they do indeed span the symmetric matrices.

14. Show that if S_1 and S_2 are arbitrary subsets of a vector space V, then

$$\operatorname{span}(S_1 \cup S_2) = \operatorname{span}(S_1) + \operatorname{span}(S_2).$$

Solution:

$$\begin{aligned} \operatorname{span}(S_1 \cup S_2) &= \left\{ \sum_{i=1}^{n} c_i u_i :_i \in S_1 \cup S_2, c_i \in \mathbb{R} \right\} \\ &= \left\{ \sum_{i=1}^{n} c_i u_i : u_i \in S_1 \text{ or } u_i \in S_2, c_i \in \mathbb{R} \right\}. \end{aligned}$$

Now, we can divide the u_i into those vectors from S_1 and those from S_2. Conversely, we can combine the v_i and w_i into a sum of vectors from $S_1 \cup S_2$.

$$\begin{aligned} &= \left\{ \sum_{i=1}^{m} c_i v_i + \sum_{i=1}^{m} d_i w_i : v_i \in S_1, w_i \in S_2, c_i, d_i \in \mathbb{R} \right\} \\ &= \{ v + w : v \in \operatorname{span}(S_1), w \in \operatorname{span}(S_2) \} \\ &= \operatorname{span}(S_1) + \operatorname{span}(S_2). \end{aligned}$$

1.5 - Linear Dependence and Linear Independence

2. Determine whether the following sets are linearly dependent or independent.

   d. $\{x^3 - x, 2x^2 + 4, \text{-}2x^3 + 3x^2 + 2x + 6\}$ in $\mathsf{P}_3(\mathbb{R})$

   Solution: The linear equation

$$a(x^3 - x) + b(2x^2 + 4) + c(\text{-}2x^3 + 3x^2 + 2x + 6) = 0$$

can be rearranged, combining like terms,

$$x^3(a - 2c) + x^2(b + 3c) + x(\text{-}a + 2c) + 1(4b + 6c) = 0.$$

Since different powers of x are linearly independent*, this linear combination must be trivial, i.e.

$$\begin{array}{rcrcrcl} a & & & - & 2c & = & 0 \\ & & b & + & 3c & = & 0 \\ \text{-}a & & & + & 2c & = & 0 \\ & & 4b & + & c & = & 0 \end{array}$$

The first and third equations are obviously equivalent. Dropping the third, we have a system of three equations, the first of which is independent of the other two (it contains a variable not found in the other two), which are independent of one another (they are not multiples of one another). Such a system (three independent equations in three unknowns) has a unique solution. Since (0,0,0) is obviously a solution, this system is linearly independent.

* (This comes from the definition of the “ring” of polynomials. However, it can also be derived when one considers polynomials as real functions. If)

   f. $\{(1, \text{-}1, 2), (2, 0, 1), (\text{-}1, 2, \text{-}1)\}$ in $\mathbb{R}^3$

   Solution:

$$\begin{array}{rcrcrcl} a & + & 2b & + & \text{-}c & = & 0 \\ \text{-}a & & & + & 2c & = & 0 \\ 2a & + & b & + & \text{-}c & = & 0 \end{array}$$

Matlab gives only the trivial solution. These vectors are independent.

   h. $\left\{ \begin{pmatrix} 1 & 0 \\ \text{-}2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \text{-}1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \text{-}1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & \text{-}2 \end{pmatrix} \right\}$ in $\mathsf{M}_{2\times 2}(\mathbb{R})$.

   Solution: We can treat these matrices as 4-dimensional column vectors.

$$\begin{array}{rcrcrcrcl} a & & & + & \text{-}c & + & 2d & = & 0 \\ & & \text{-}b & + & 2c & + & d & = & 0 \\ \text{-}2a & + & b & + & c & + & d & = & 0 \\ a & + & b & & & + & \text{-}2d & = & 0 \end{array}$$

Matlab gives only the trivial solution. These vectors are independent.

4. In F^n, let e_j denote the vector whose jth coordinate is 1 and whose other coordinates are 0. Prove that $\{e_1, e_2, \ldots, e_n\}$ is linearly independent.

Solution: If $a_j \in \mathsf{F}$, then $a_j e_j$ is the vector whose jth coordinate is a_j and whose other coordinates are 0. By our rules of vector addition, the linear combination $a_1e_1 + a_2e_2 + \cdots + a_ne_n$ produces the vector $(a_1, a_2, \ldots, a_n) \in \mathsf{F}^n$. If

$$a_1e_1 + a_2e_2 + \cdots + a_ne_n = \mathbf{0} = (0, 0, \ldots, 0),$$

then $a_1 = a_2 = \cdots = a_n = 0$. The only valid linear combination is the trivial one, thus the vectors are linearly independent.

9. Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other.

Solution:
$\boxed{\Rightarrow}$ Suppose u and v are linearly dependent, that is, there exist a and b, at least one of which is not zero, such that $au+bv = 0$. Assume $a \neq 0$ (otherwise swap u and v).* Then, we can rearrange the equation, dividing both sides by a, to get

$$u = -\frac{b}{a}v.$$

Thus u is a multiple of v.

$\boxed{\Leftarrow}$ Suppose $u = kv$ or $v = ku$ for some k in $\mathbb{R}$ (if the latter case, swap u and v). Then we can rearrange the equation,

$$u + (\text{-}k)v = 0$$

so u and v are linearly independent.

* This condition is important, since any algebraic proof requiring division by zero is no good. Swapping u and v allows me to start fresh with a better arrangement.

15. Let $S = \{u_1, u_2, \ldots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots, u_k\})$.

Solution:
$\boxed{\Rightarrow}$ Suppose S is linearly dependent. Then there exist $a_1, \ldots, a_n$, at least one of which is nonzero, such that

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$$

Let k be the largest integer $1 \leq k \leq n$ such that $a_{k+1} \neq 0$. If $k = 0$, we have the equation $a_1u_1 = 0$. Then $u_1 = 0$, since $a_1 \neq 0$. If otherwise, we have the equation

$$a_1u_1 + a_2u_2 + \cdots + a_ku_k + a_{k+1}u_{k+1} = 0$$

(we can ignore the zero terms to the right of u_{k+1}). Rearranging and dividing by $a_{k+1} \neq 0$,

$$\frac{\text{-}a_1}{a_{k+1}}u_1 + \frac{\text{-}a_2}{a_{k+1}}u_2 + \cdots + \frac{\text{-}a_k}{a_{k+1}}u_k = u_{k+1}.$$

Thus u_{k+1} lies in $\operatorname{span}(\{u_1, \ldots, u_k\})$.

$\boxed{\Leftarrow}$ Suppose $u_{k+1} \in \operatorname{span}(\{u_1, u_2, \ldots, u_k\})$. Then there exist $a_1, \ldots, a_k$ such that

$$u_{k+1} = a_1u_1 + a_2u_2 + \cdots + a_ku_k.$$

Define $a_{k+1} = \text{-}1$, and $a_i = 0$ for $k + 2 \leq i \leq n$. Then

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = a_1u_1 + \cdots + a_ku_k - a_{k+1}u_{k+1} = 0,$$

so S is linearly independent.

18. Let S be a set of nonzero polynomials in $\mathsf{P}(F)$ such that no two have the same degree. Prove that S is linearly independent.

Solution: Note that S may be infinite. However, a linear combination is defined as a finite weighted sum of vectors. We need only worry about n vectors at a time.

Consider a linear combination

$$a_1f_1 + a_2f_2 + \cdots + a_nf_n = 0$$

where f_i are polynomials in S, and $a_i \in S$. These f_i must all have different degrees, so assume they are ordered from smallest to largest degree (otherwise, we can rearrange them and start the proof over). If the linear combination is nontrivial, let a_{k+1} be the last nonzero coefficient. If f_{k+1} has degree d, then $a_{k+1}f_{k+1}$ can be written

$$a_{k+1}c_0 + a_{k+1}c_1x + \cdots + a_{k+1}c_dx^d.$$

The polynomials $f_1, \ldots, f_k$ have degree strictly less than d, so a_if_i has zero x^d coefficient for all $i \neq k$.No combination of these polynomials could cancel out the x^d term of f_k. Therefore $a_k = 0$, so the linear combination must be trivial.

1.6 - Bases and Dimension

3. Determine which of the following sets are bases for $\mathsf{P}_2(\mathbb{R})$.
    b. $\{1 + 2x + x^2, 3 + x^2, x + x^2\}$

Solution: Solving the linear system

$$\begin{array}{rcrcrcl} a & + & b & + & c & = & 0 \\ 2a & & & + & c & = & 0 \\ a & + & 3b & & & = & 0 \end{array},$$

Matlab gives only the trivial solution. Thus the vectors are linearly independent.

d. $\{\text{-}1 + 2x + 4x^2, \text{-}2 + 3x - x^2, 1 - x + 6x^2\}$

Solution: Solving the linear system

$$\begin{array}{rcrcrcl} 4a & + & \text{-}b & + & 6c & = & 0 \\ 2a & + & 3b & + & \text{-}c & = & 0 \\ \text{-}a & + & \text{-}2b & + & c & = & 0 \end{array},$$

Matlab gives only the trivial solution. Thus the vectors are linearly independent.

8. Let W denote the subspace of $\mathbb{R}^5$ consisting of all the vectors having coordinates that sum to zero. The vectors

$$\begin{array}{rlrl} u_1 = & (2, \text{-}3, 4, \text{-}5, 2), & u_2 = & (\text{-}6, 9, \text{-}12, 15, \text{-}6) \\ u_3 = & (3, \text{-}2, 7, \text{-}9, 1), & u_4 = & (2, \text{-}8, 2, \text{-}2, 6) \\ u_5 = & (\text{-}1, 1, 2, 1, \text{-}3), & u_6 = & (0, \text{-}3, \text{-}18, 9, 12) \\ u_7 = & (1, 0, \text{-}2, 3, \text{-}2), & u_8 = & (2, \text{-}1, 1, \text{-}9, 7) \end{array}$$

generate W. Find a subset of the set $\{u_1, u_2, \ldots, u_8\}$ that is a basis for W.

Solution: This problem is a great opportunity to exhibit a very economical linear-systems solving technique.

First, we drop the second vector, which is obviously a multiple of the first. The third vector is not, so $\{u_1, u_3\}$ is linearly independent. We'll build up from that. The system $au_1 + bu_3 = u_4$ (notice the reordering) has 5 equations and 2 unknowns. To save time, I'll first solve the first two equations. The (most likely unique) solution to that system can then be tested in the rest of the equations.

$$\begin{array}{rcrcr} 2a & + & 3b & = & 2 \\ -3a & + & -2b & = & -8 \end{array}$$

This has the unique solution $a = 4$, $b = -2$. Plugging these values in, we see that u_4 is indeed a linear combination of u_1 and u_3, so we drop it from the list. Next, try the same with u_5 :

$$\begin{array}{rcrcr} 2a & + & 3b & = & -1 \\ -3a & + & -2b & = & 1 \end{array}$$

This gives the solution $a = b = -\frac{1}{5}$. Plugging these values in, we get

$$-\tfrac{1}{5}(2, -3, 4, -5, 2) + -\tfrac{1}{5}(3, -2, 7, -9, 1) = -\tfrac{1}{5}(5, -5, 11, -14, 3) \neq (-1, 1, 2, 1, -3) = u_5$$

Thus there is no solution to the linear combination - solving the first two equations precludes solving the rest. Thus $\{u_1, u_3, u_5\}$ is linearly independent. On to u_6. Now we have a 3-by-3:

$$\begin{array}{rcrcrcr} 2a & + & 3b & + & -c & = & 0 \\ -3a & + & -2b & + & c & = & -3 \\ 4a & + & 7b & + & 2c & = & -18 \end{array}$$

This gives the solution $a = 1$, $b = -2$, $c = -4$. Plugging these values in,

$$(2, -3, 4, -5, 2) + -2(3, -2, 7, -9, 1) + -4(-1, 1, 2, 1, -3) = (0, -3, -18, 9, 12) = u_6$$

So u_6 is a linear combination of $\{u_1, u_3, u_5\}$. Moving on to u_7,

$$\begin{array}{rcrcrcr} 2a & + & 3b & + & -c & = & 1 \\ -3a & + & -2b & + & c & = & 0 \\ 4a & + & 7b & + & 2c & = & -2 \end{array}$$

This gives the solution $a = \frac{-13}{21}$, $b = \frac{8}{21}$, $c = \frac{-23}{21}$. Plugging that in, we get

$$\begin{aligned} &\tfrac{-13}{21}(2, -3, 4, -5, 2) + \tfrac{8}{21}(3, -2, 7, -9, 1) + \tfrac{-23}{21}(-1, 1, 2, 1, -3) \\ &\qquad = \tfrac{1}{21}(21, 0, -42, -30, 21) \neq (1, 0, -2, 3, -2) = u_7 \end{aligned}$$

So $\{u_1, u_3, u_5, u_7\}$ is linearly independent. We can stop here, if we're smart. The space of vectors in F^5 whose entries sum to 0 is a 4-dimensional space. Supposing we've picked the first four entries, the last one must be the negative of their sum - we have at most four degrees of freedom. Another way to see this is that W is the null space of the linear transformation $T : \mathsf{F}^5 \to \mathbb{R}$, with the mapping

$$T(v_1, v_2, v_3, v_4, v_5) = v_1 + v_2 + v_3 + v_4 + v_5$$

The range of this linear transform is all of $\mathbb{R}$ (You can tweak the entries to sum to any real number), so $\text{rank}(T) = 1$. Therefore

$$\dim(W) = \text{nullity}(T) = \dim(\mathsf{F}^n) - \text{rank}(T) = 5 - 1 = 4.$$

We can design a nicer basis for this space by considering the generic vector,

$$\big(a_1, a_2, a_3, a_4, \text{-}(a_1 + a_2 + a_3 + a_4)\big).$$

Again, the first four entries are free, but the last is the negative of their sum. Breaking this apart, we get

$$a_1(1, 0, 0, 0, \text{-}1) + a_2(0, 1, 0, 0, \text{-}1) + a_3(0, 0, 1, 0, \text{-}1) + a_4(0, 0, 0, 1, \text{-}1),$$

so we have the basis

$$\big\{(1, 0, 0, 0, \text{-}1),\ (0, 1, 0, 0, \text{-}1),\ (0, 0, 1, 0, \text{-}1),\ (0, 0, 0, 1, \text{-}1)\big\}$$

10. In each part, use the Lagrange interpolation formula to construct the polynomial of smallest degree whose graph contains the following points.

    b. (-4, 24), (1, 9), (3, 3).

    Solution: These points are actually co-linear. They lie on the line $y = \text{-}3x + 12$. The Lagrange interpolator is

    $$\begin{aligned} p(x) &= 24\frac{(x-1)(x-3)}{(\text{-}4-1)(\text{-}4-3)} + 9\frac{(x-\text{-}4)(x-3)}{(1-\text{-}4)(1-3)} + 3\frac{(x-\text{-}4)(x-1)}{(3-\text{-}4)(3-1)} \\ &= x^2\left(\frac{24}{35} + \frac{9}{\text{-}10} + \frac{3}{14}\right) + x\left(\text{-}4\frac{24}{35} + 1\frac{9}{\text{-}10} + 3\frac{3}{14}\right) + \left(3\frac{24}{35} + \text{-}12\frac{9}{\text{-}10} + \text{-}4\frac{3}{14}\right) \\ &= 0 - 3x + 12, \end{aligned}$$

    so just what we expected.

    d. (-3, -30), (-2, 7), (0, 15), (1, 10).

    Solution: This has Lagrange interpolator

    $$p(x) = \text{-}30\tfrac{(x+2)x(x-1)}{(\text{-}3+2)(\text{-}3)(\text{-}3-1)} + 7\tfrac{(x+3)x(x-1)}{(\text{-}2+3)(\text{-}2)(\text{-}2-1)} + 15\tfrac{(x+3)(x+2)(x-1)}{(3)(2)(\text{-}1)} + 10\tfrac{(x+3)(x+2)x}{(1+3)(1+2)(1)}$$

    This simplifies to (knowing that the function must pass through (0,15)),

    $$\begin{aligned} &x^3(\tfrac{\text{-}30}{\text{-}12} + \tfrac{7}{6} + \tfrac{15}{\text{-}6} + \tfrac{10}{12}) + x^2(1\tfrac{\text{-}30}{\text{-}12} + 2\tfrac{7}{6} + 4\tfrac{15}{\text{-}6} + 5\tfrac{10}{12}) + x(\text{-}2\tfrac{\text{-}30}{\text{-}12} + \text{-}3\tfrac{7}{6} + 1\tfrac{15}{\text{-}6} + 6\tfrac{10}{12}) + 15 \\ &\quad = 2x^3 - x^2 - 6x + 15 \end{aligned}$$

    To test, plug in any of the x-values, and see that you get the right y-values.

12. Let u, v, w be distinct vectors of a vector space V. Show that if $\{u, v, w\}$ is a basis for V, then $\{u + v + w, v + w, w\}$ is also a basis for V.

Solution: Suppose, for contradiction, that there existed some a, b, c, not all zero, such that

$$0 = a(u + v + w) + b(v + w) + cw = au + (a + b)v + (a + b + c)w.$$

Now, u, v, w are linearly independent, so all of these coefficients have to be zero, i.e.

$$\begin{array}{ccccccc} a & & & & & = & 0 \\ a & + & b & & & = & 0 \\ a & + & b & + & c & = & 0 \end{array}$$

This linear combination only has the trivial solution - in fact, swapping the a and c columns, and then the first and third rows, we have a system in row-echelon form:

$$\begin{array}{ccccccc} c & + & b & + & a & = & 0 \\ & & b & + & a & = & 0 \\ & & & & a & = & 0 \end{array}$$

Thus $a = b = c = 0$, and so $\{u + v + w, v + w, w\}$ are linearly independent.

15. The set of all $n \times n$ matrices having trace equal to zero is a subspace W of $\mathsf{M}_{n\times n}(F)$. Find a basis for W. What is the dimension of W?

Solution: If A is such a matrix, one can freely choose every entry a_{ij} up till a_{nn}. This entry must be equal to the negative sum of the diagonal entries $a_{11}, \ldots, a_{n-1,n-1}$. A schematic of such a matrix:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n\text{-}1} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,n\text{-}1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n\text{-}1,1} & a_{n\text{-}1,2} & \cdots & a_{n\text{-}1,n\text{-}1} & a_{n\text{-}1,n} \\ a_{n1} & a_{n2} & \cdots & a_{n,n\text{-}1} & \text{-}\sum_{i=1}^{n\text{-}1} a_{ii} \end{pmatrix} \tag{1}$$

Breaking this apart into a basis, we have two different sorts of matrices; first, those with zeros along the diagonal, whose basis is

$$\left\{ \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \right\}$$

There are $n^2 - n$ off-diagonal entries, so this basis contains $n^2 - n$ matrices. Second, there are the diagonal matrices. These have the basis

$$\left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \text{-}1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \text{-}1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & \text{-}1 \end{pmatrix} \right\}.$$

There are $n-1$ of these basis matrices, one for each diagonal entry except the last. Joining these two bases we have a linearly independent set (for each $(i,j) \neq (n,n)$, there is exactly one basis matrix M such that $M_{ij} \neq 0$) spanning W (any matrix of the form in (1) can be built from a combination of these vectors, where the coefficients a_{ij} correspond to the basis vectors in the obvious way). The dimension of W is exactly $(n^2-n)+(n-1)=n^2-1$.

20. Let V be a vector space having dimension n, and let S be a generating subset of V.

    (a) Prove that there is a subset of S that is a basis of V.

    Solution: Let $\beta = \{v_1, \ldots, v_n\}$ be a basis of V. We can not assume that S is finite, but we need only find a finite subset of S that generates β.

    For each $v_i \in \beta$, there is some collection $\{w_{i1}, w_{i2}, \ldots, w_{ik_i}\} \subseteq S$ and scalars $d_{i1}, \ldots, d_{ik_i}$ such that $v_i = \sum_{j=1}^{k_i} d_{ij} w_{ij}$. Define

    $$T = \{w_{ij} : 1 \leq i \leq n, 1 \leq j \leq k_i\}$$

    Then T is a finite subset of S. Observe that, for any $v \in V$, we can write $v = \sum c_i v_i$, so

    $$v = \sum_{i=1}^{n} c_i \left(\sum_{j=1}^{k_i} d_{ij} w_{ij} \right)$$

    Thus T spans V. By Theorem 1.10, there must be a subset of $T \subseteq S$ which is a basis of V.

    (b) Prove that S contains at least n vectors.

    Solution: By the above, $T \subseteq S$ contains a basis of V, which must have exactly n vectors. Therefore S must have at least n vectors.

33. Let W_1 and W_2 be subspaces of a vector space V with bases β_1 and β_2, respectively.

    (a) Assume $\mathsf{W}_1 \oplus \mathsf{W}_2 = \mathsf{V}$. Prove that $\beta_1 \cap \beta_2 = \emptyset$ and $\beta_1 \cup \beta_2$ is a basis for V.

    Solution: First, if v lies in $\beta_1 \cap \beta_2$, then $v \in \mathsf{W}_1 \cap \mathsf{W}_2 = \{0\}$. However, 0 can not be a member of any basis, neither β_1 or β_2. Therefore $\beta_1 \cap \beta_2 = \emptyset$.

    Denote the elements of β_1 and β_2 by $v_1, v_2 \ldots$ and $w_1, w_2, \ldots$, respectively. We now must prove that (1) $\beta_1 \cup \beta_2$ is linearly independent, and that (2) $\beta_1 \cup \beta_2$ spans V.

    (1) Suppose

    $$c_1 v_1 + \cdots + c_m v_m + c_{m+1} w_1 + \ldots + c_{m+n} w_n = 0 \tag{2}$$

    for some $c_1, \ldots, c_{m+n}$. We can rearrange this sum

    $$c_1 v_1 + \cdots + c_m v_m = \text{-}c_{m+1} w_1 + \ldots + \text{-}c_{m+n} w_n$$

    Now, the LHS lies in W_1, and the RHS lies in W_2, so both must lie in $\mathsf{W}_1 \cap \mathsf{W}_2 = \{0\}$ Thus

    $$c_1 v_1 + \cdots + c_m v_m = 0 \quad \text{and} \quad \text{-}c_{m+1} w_1 + \ldots + \text{-}c_{m+n} w_n = 0$$

    but both of these linear combinations must be trivial, since β_1 and β_2 are both bases. Thus $c_1 = c_2 = \cdots = c_{m+n} = 0$, so (2) must be trivial. Therefore $\beta_1 \cup \beta_2$ must be linearly independent.

(2) Now, given $u \in \mathsf{V}$, we can write $u = v + w$, with $v \in \mathsf{W}_1$ and $w \in \mathsf{W}_2$. Likewise, we can write $v = \sum_{i=1}^{n_1} c_i v_i$ and $w = \sum_{i=1}^{n_2} c_i w_i$. So

$$u = v + w = \sum_{i=1}^{n_1} c_i v_i + \sum_{i=1}^{n_2} c_i w_i,$$

which is a linear combination in $\beta_1 \cup \beta_2$. So the union of bases spans V.

(b) Conversely, if $\beta_1 \cap \beta_2 = \emptyset$ and $\beta_1 \cup \beta_2$ is a basis for V, prove that $\mathsf{W}_1 \oplus \mathsf{W}_2 = \mathsf{V}$.

Solution: Denote the elements of β_1 and β_2 by $v_1, v_2 \ldots$ and $w_1, w_2, \ldots$, respectively. We need to prove that (1) $\mathsf{W}_1 \cap \mathsf{W}_2 = \{0\}$, and (2) $\mathsf{W}_1 + \mathsf{W}_2 = \mathsf{V}$.

(1) Suppose $u \in \mathsf{W}_1 \cap \mathsf{W}_2$. Then we can write u in two ways:

$$c_1 v_1 + c_2 v_2 + \cdots + c_m v_m = v = d_1 w_1 + d_2 w_2 + \cdots + d_n w_n.$$

This gives the linear combination

$$c_1 v_1 + c_2 v_2 + \cdots + c_m + \text{-} d_1 w_1 + \text{-} d_2 w_2 + \cdots + \text{-} d_n w_n = 0$$

Observe that all vectors in the sum lie in $\beta_1 \cup \beta_2$, which is a basis, so all coefficients must be zero. Therefore $u = 0$. So $\mathsf{W}_1 \cap \mathsf{W}_2 = \{0\}$.

(2) Now, any $u \in \mathsf{V}$ can be written as a linear combination in $\beta_1 \cup \beta_2$, that is to say, of vectors from either β_1 or β_2. Rearrange this sum so that it has the form

$$u = \sum_{i=1}^{n_1} v_i + \sum_{j=1}^{n_2} w_i = v + w,$$

where v is some vector in W_1 and w is some vector in W_2. Thus $\mathsf{V} = \mathsf{W}_1 + \mathsf{W}_2$.