Homework 1 Solutions

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1.1 - Introduction

2. Find the equations of the lines through the following pairs of points in space.

   b. (3, -2, 4) and (-5, 7, 1)

   Solution:
   $$x = (3,\text{-}2,4) + r[(\text{-}5,7,1) - (3,\text{-}2,4)] = (3,\text{-}2,4) + r(\text{-}8,9,\text{-}3).$$

   d. (-2, -1, 5) and (3, 9, 7)

   Solution:
   $$x = (\text{-}2,\text{-}1,5) + r[(3,9,7) - (\text{-}2,\text{-}1,5)] = (\text{-}2,\text{-}1,5) + r(5,10,2).$$

3. Find the equations of the planes containing the following points in space.

   b. (3, -6, 7), (-2, 0, -4), and (5, -9, -2).

   Solution:
   $$\begin{aligned} x &= (3,\text{-}6,7) + r[(\text{-}2,0,\text{-}4) - (3,\text{-}6,7)] + s[(5,\text{-}9,\text{-}2) - (3,\text{-}6,7)] \\ &= (3,\text{-}6,7) + r(\text{-}5,6,\text{-}11) + s(2,3,\text{-}9) \end{aligned}$$

   d. (1, 1, 1), (5, 5, 5), and (-6, 4, 2).

   Solution:
   $$\begin{aligned} x &= (1,1,1) + r[(5,5,5) - (1,1,1)] + s[(\text{-}6,4,2) - (1,1,1)] \\ &= (1,1,1) + r(4,4,4) + s(\text{-}7,3,1) \end{aligned}$$

7. Prove that the diagonals of a parallelogram bisect each other.

   Solution: A parallelogram $ABCD$ in $\mathbb{R}^2$ may be translated so that vertex A lies on the origin. After translation, denote the coordinate-vectors of the adjacent vertices B and D by v and w. The four vertices therefore have coordinate-vectors $0, v, v + w$, and w.

   By the midpoint rule, segment BD has midpoint at $\frac{1}{2}(v+w)$, and AC has midpoint $\frac{1}{2}(0+(v+w))$. These midpoints are identical, so BD and AC bisect one another.

1.2 - Vector Spaces

2. Write the zero vector of $\mathsf{M}_{3\times 4}(\mathsf{F})$.

Solution: Consider the matrix $\mathbf{0} := \left(\begin{smallmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{smallmatrix}\right)$. For any matrix $A \in \mathsf{M}_{3\times 4}$,

$$\mathbf{0} + A = \left(\begin{smallmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} a_{11}&a_{12}&a_{13}&a_{14} \\ a_{21}&a_{22}&a_{23}&a_{24} \\ a_{31}&a_{32}&a_{33}&a_{34} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0+a_{11}&0+a_{12}&0+a_{13}&0+a_{14} \\ 0+a_{21}&0+a_{22}&0+a_{23}&0+a_{24} \\ 0+a_{31}&0+a_{32}&0+a_{33}&0+a_{34} \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_{11}&a_{12}&a_{13}&a_{14} \\ a_{21}&a_{22}&a_{23}&a_{24} \\ a_{31}&a_{32}&a_{33}&a_{34} \end{smallmatrix}\right) = A$$

Thus $\mathbf{0}$ is the zero vector of $\mathsf{M}_{3\times 4}(\mathsf{F})$.

12. Prove that the set of even functions from $\mathbb{R}$ to $\mathbb{R}$ is a vector space over F (under the usual addition and scalar multiplication).

Solution: Note that (1) $f = g$ means $f(x) = g(x)$ for all $x \in \mathbb{R}$, and (2) the "usual addition and scalar multiplication" means that $(f + g)(x) = f(x) + g(x)$ and $(cf)(x) = cf(x)$ for all $x \in \mathbb{R}$.

The even functions are those $f : \mathbb{R} \to \mathbb{R}$ which satisfy $f(\text{-}x) = f(x)$ for all $x \in \mathbb{R}$. Adding two such functions:

$$(f + g)(\text{-}x) = f(\text{-}x) + g(\text{-}x) = f(x) + g(x) = (f + g)(x),$$

so the sum $f + g$ is an even function. As for scalar multiplication,

$$(cf)(\text{-}x) = cf(\text{-}x) = cf(x) = (cf)(x).$$

The remaining properties hold equally well for any set S, field F, scalars $a, b \in \mathsf{F}$, point $x \in S$ and functions $f, g, h \in \mathcal{F}(S, \mathsf{F})$.

VS1 Commutativity of addition is inherited from F:

$$(f + g)(x) = \underline{f(x) + g(x) = g(x) + f(x)} = (g + f)(x).$$

VS2 Associativity of addition is inherited from F:

$$\begin{aligned}[(f + g) + h](x) &= (f + g)(x) + h(x) \\ &= \underline{(f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x))} \\ &= f(x) + (g + h)(x) = [f + (g + h)](x).\end{aligned}$$

VS3 Define $\mathbf{0}$ by the mapping $\mathbf{0}(x) = 0$. Then

$$(f + \mathbf{0})(x) = f(x) + \mathbf{0}(x) = f(x).$$

VS4 Define -f by the mapping $(\text{-}f)(x) = \text{-}(f(x))$P. Then

$$(f + \text{-}f)(x) = f(x) + (\text{-}f)(x) = f(x) + \text{-}f(x) = 0 = \mathbf{0}(x).$$

VS5 Briefly, $(1f)(x) = 1f(x) = f(x)$.

VS6 Associativity of scalars is inherited from F:

$$((ab)f)(x) = \underline{(ab)f(x) = a(bf(x))} = a((bf)(x)) = (a(bf))(x).$$

VS7 Distributivity(1) is inherited from F:

$$\begin{aligned}(a(f + g))(x) &= a(f + g)(x) \\ &= \underline{a(f(x) + g(x)) = af(x) + ag(x)} \\ &= (af)(x) + (ag)(x) = (af + ag)(x).\end{aligned}$$

VS8 Distributivity(2) is inherited from F:

$$((a+b)f)(x) = \underline{(a+b)f(x) = af(x) + bf(x)}$$
$$= (af)(x) + (bf)(x) = (af+bf)(x).$$

16. Let $\mathsf{V} = \mathsf{M}_{m\times n}(\mathbb{R})$. Let $F = \mathbb{Q}$, the field of rational numbers. Is V a vector space over F (under the usual addition and scalar multiplication)?

Solution: Adding two matrices in V, $(A+B)_{ij} = A_{ij} + B_{ij} \in \mathbb{R}$, so the matrix sum is in V. Likewise, $(cA)_{ij} = cA_{ij} \in \mathbb{R}$, for any $c \in \mathbb{Q} \subseteq \mathbb{R}$.

VS1 Commutativity of addition is inherited from $\mathbb{R}$:

$$(A+B)_{ij} = \underline{A_{ij} + B_{ij} = B_{ij} + A_{ij}} = (B+A)_{ij}$$

VS2 Associativity of addition is inherited from $\mathbb{R}$:

$$((A+B)+C)_{ij} = (A+B)_{ij} + C_{ij}$$
$$= \underline{(A_{ij}+B_{ij}) + C_{ij} = A_{ij} + (B_{ij} + C_{ij})}$$
$$= A_{ij} + (B+C)_{ij} = (A+(B+C))_{ij}$$

VS3 Define $\mathbf{0}$ by $\mathbf{0}_{ij} = 0$ for all i, j. Then

$$(A+0)_{ij} = A_{ij} + \mathbf{0}_{ij} = A_{ij} + 0 = A_{ij}$$

VS4 Define -A by (-$A)_{ij}$ = -A_{ij}. Then

$$(A + \text{-}A)_{ij} = A_{ij} + (\text{-}A)_{ij} = A_{ij} + \text{-}A_{ij} = 0 = \mathbf{0}_{ij}.$$

VS5 Briefly, $(1A)_{ij} = 1A_{ij} = A_{ij}$.

VS6 Associativity of multiplication is inherited from $\mathbb{R}$:

$$((ab)A)_{ij} = \underline{(ab)A_{ij} = a(bA_{ij})} = a((bA)_{ij}) = (a(bA))_{ij}.$$

VS7 Distributivity(1) is inherited from $\mathbb{R}$:

$$(a(A+B))_{ij} = a(A+B)_{ij} = \underline{a(A_{ij}+B_{ij}) = aA_{ij} + aBij} = (aA)_{ij} + (aB)_{ij} = (aA+aB)_{ij}.$$

VS8 Distributivity(2) is inherited from $\mathbb{R}$:

$$((a+b)A)_{ij} = \underline{(a+b)A_{ij} = aA_{ij} + bA_{ij}} = (aA)_{ij} + (bA)_{ij} = (aA+bA)_{ij}.$$

18. Let $\mathsf{V} = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in \mathsf{V}$ and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2)$$

Is V a vector space over $\mathbb{R}$ with these operations?

Solution: No. Consider the points $x = (0,0)$ and $y = (1,1)$. Then

$$x + y = (0,0) + (1,1) = (0 + 2(1), 0 + 3(1)) = (2,3)$$

whereas

$$y + x = (1,1) + (0,0) = (1 + 2(0), 1 + 3(0)) = (1,1).$$

This additive relation is not commutative, so V is not a vector space.

19. Let $\mathsf{V} = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. Is V a vector space under the usual addition and with scalar multiplication defined as

$$c(a_1, a_2) = \begin{cases} (0,0) & \text{if } c = 0 \\ (ca_1, \frac{a_2}{c}) & \text{if } c \neq 0 \end{cases}$$

Solution: No. Let $a = b = 1$, and let $x = (1,1)$. Then

$$(a + b)x = 2x = (2(1), \tfrac{1}{2}) = (2, \tfrac{1}{2}),$$

whereas

$$ax + bx = 1x + 1x = (1(1), \tfrac{1}{1}) + (1(1), \tfrac{1}{1}) = (2,2).$$

These relations do not satisfy distributivity(1).

20. Let V be the set of sequences $\{a_n\}$ of real numbers. For $\{a_n\}, \{b_n\} \in \mathsf{V}$ and $t \in \mathbb{R}$, define

$$\{a_n\} + \{b_n\} = \{a_n + b_n\} \quad \text{and} \quad t\{a_n\} = \{ta_n\}$$

Prove that, under these operations, V is a vector space over $\mathbb{R}$.

Solution: For $\{a_n\}, \{b_n\} \in \mathsf{V}$ and $s, t \in \mathbb{R}$, $\{a_n\} + \{b_n\} = \{a_n + b_n\}$, and $c\{a_n\} = \{ca_n\}$, both of which are sequences of real numbers.

VS1 Commutativity of addition is inherited from $\mathbb{R}$:

$$\{a_n\} + \{b_n\} = \underline{\{a_n + b_n\} = \{b_n + a_n\}} = \{a_n\} + \{b_n\}$$

VS2 Associativity of addition is inherited from $\mathbb{R}$:

$$\begin{aligned} (\{a_n\} + \{b_n\}) + \{c_n\} &= \{a_n + b_n\} + \{c_n\} \\ &= \underline{\{(a_n + b_n) + c_n\} = \{a_n + (b_n + c_n)\}} \\ &= \{a_n\} + \{b_n + c_n\} = \{a_n\} + (\{b_n\} + \{c_n\}). \end{aligned}$$

VS3 Define $\mathbf{0} := \{\mathbf{0}_n\}$ by $\mathbf{0}_n = 0$ for all n. Then

$$\{a_n\} + \mathbf{0} = \{a_n + \mathbf{0}_n\} = \{a_n + 0\} = \{a_n\}$$

VS4 Define $\text{-}\{a_n\} := \{z_n\}$ by $z_n = \text{-}a_n$ for all n. Then

$$\{a_n\} + \text{-}\{a_n\} = \{a_n + z_n\} = \{a_n + \text{-}a_n\} = \{0\} = \mathbf{0}.$$

VS5 Briefly, $1\{a_n\} = \{1a_n\} = \{a_n\}$.

VS6 Associativity of multiplication is inherited from $\mathbb{R}$:

$$(cd)\{a_n\} = \underline{\{(cd)a_n\} = \{c(da_n)\}} = c\{da_n\} = c(d\{a_n\}).$$

VS7 Distributivity(1) is inherited from $\mathbb{R}$:

$$c(\{a_n\} + \{b_n\}) = c\{a_n + b_n\} = \underline{\{c(a_n + b_n)\} = \{ca_n + cb_n\}} = \{ca_n\} + \{cb_n\} = c\{a_n\} + c\{b_n\}.$$

VS8 Distributivity(2) is inherited from $\mathbb{R}$:

$$(c + d)\{a_n\} = \underline{\{(c + d)a_n\} = \{ca_n + da_n\}} = \{ca_n\} + \{da_n\} = c\{a_n\} + d\{a_n\}.$$

□

We may also prove this using the method in the note below. Given $f \in \mathcal{F}(\mathbb{N}, \mathbb{R})$, a real-valued function on the natural numbers, define $T(f) = \{f(n)\}$. T is linear:

$$T(f + cg) = \{(f + cg)(n)\} = \{f(n) + cg(n)\} = \{f(n)\} + \{cg(n)\} = \{f(n)\} + c\{g(n)\} = T(f) + cT(g).$$

Given a sequence $\{a_n\}$, define $f = T^{-1}(\{a_n\})$ by the mapping $f(n) = a_n$. Thus

$$T(T^{-1}(\{a_n\})) = T(f) = \{f(n)\} = \{a_n\},$$

so T is onto. Thus the set of real sequences is a vector space.

Note on section 1.1

You might have noticed that the proofs for problems 12, 16, and 20 were basically the same. We can save a lot of time by relating these spaces to a known vector space, $\mathcal{F}(\mathcal{I}, \mathsf{F})$, the space of F-valued functions on some indexing set $\mathcal{I}$.

For instance, we can think of matrices as functions on the indexing set $\mathcal{I}_{m\times n} = \{(1, 1), (1, 2), \ldots, (m, n)\}$. Each set of indices has a corresponding F-value, so we can treat the indexing set like the domain of an F-valued function. Now, define $T : \mathcal{F}(\mathcal{I}_{m\times n}, \mathsf{F}) \to \mathsf{M}_{m\times n}$ by the mapping

$$(T(f))_{ij} = f((i, j))$$

That is, the ijth element of the matrix $T(f)$ is equal to the value of the function f at the point (i, j). This is linear, since

$$(T(f + cg))_{ij} = (f + cg)((i, j)) = f((i, j)) + cg((i, j)) = (T(f))_{ij} + c(T(g))_{ij}.$$

This is onto, since we can easily reverse the mapping, sending a given matrix to a corresponding function.

$$(T^{-1}M)((i, j)) = M_{ij}.$$

The point of this is, now $\mathsf{M}_{m\times n}$ is the range of a linear function applied to a vector space. It therefore must also be a vector space. Instead of verifying all those axioms, we defined a single mapping.

This very useful fact has a rather onerous proof, since section 1.1 is so sparse on theorems. However, it only needs to be proved once, and an be used with all manner of spaces.

Theorem: Suppose V is a vector space over F, S is a set with a binary addition and $\mathsf{F}-$multiplication relations (not necessarily closed), and $T : \mathsf{V} \to S$ is an surjection (onto function) such that $T(u+cv) = T(u) + cT(v)$ for all $u, v \in \mathsf{V}$ and $c \in \mathsf{F}$. Then S is a vector space under the given addition and multiplication.

Proof: Given $x, y \in S$, there are u, v in V such that $T(u) = x$ and $T(v) = y$ (since T is onto). By closure of V,

$$x + y = T(u) + T(v) = T(u + v) \in S$$

and for any $c \in \mathsf{F}$,

$$cx = cT(u) = T(cu) \in S.$$

VS1 Commutativity of addition is inherited from V:

$$x + y = T(u) + T(v) = \underline{T(u + v) = T(v + u)} = T(v) + T(u) = y + x.$$

VS2 Associativity of addition is inherited from V:

$$\begin{aligned}(x + y) + z &= (T(u) + T(v)) + T(w) = T(u + v) + T(w) \\ &= \underline{T((u + v) + w) = T(u + (v + w))} \\ &= T(u) + T(v + w) = T(u) + (T(v) + T(w)) = x + (y + z).\end{aligned}$$

VS3 Define $\mathbf{0} = T(\mathbf{0})$. Then

$$x + \mathbf{0} = T(u) + T(\mathbf{0}) = T(u + \mathbf{0}) = T(u) = x.$$

VS4 Define $\text{-}x = T(\text{-}u)$. Then

$$x + \text{-}x = T(u) + T(\text{-}u) = T(u + \text{-}u) = T(\mathbf{0}) = \mathbf{0}.$$

VS5 Briefly, $1x = 1T(u) = T(1u) = T(u) = x$.

VS6 Associativity of scalar multiplication is inherited from V:

$$(ab)x = (ab)T(u) = \underline{T((ab)u) = T(a(bu))} = aT(bu) = a(bT(u)) = a(bx).$$

VS7 Distributivity(1) is inherited from V:

$$a(x+y) = a(T(u)+T(v)) = aT(u+v) = \underline{T(a(u + v)) = T(au + av)} = aT(u)+aT(v) = ax+ay.$$

VS8 Distributivity(2) is inherited from V:

$$(a + b)x = (a + b)T(u) = \underline{T((a + b)u) = T(au + bu)} = aT(u) + bT(u) = ax + bx.$$

This is related to the proof that "the range of a linear transform is a subspace of the codomain", but that proof assumes that the codomain of the linear function is already a vector space.

1.3 - Subspaces

2. Determine the transpose (and trace, if applicable) of each of the matrices that follow.

d. $A = \begin{pmatrix} 10 & 0 & -8 \\ 2 & -4 & 3 \\ -5 & 7 & 6 \end{pmatrix}$

Solution: $A^t = \begin{pmatrix} 10 & 2 & -5 \\ 0 & -4 & 7 \\ -8 & 3 & 6 \end{pmatrix}$, $\operatorname{tr}(A) = 10 + -4 + 6 = 12$.

f. $\begin{pmatrix} -2 & 5 & 1 & 4 \\ 7 & 0 & 1 & -6 \end{pmatrix}$

Solution: $A^t = \begin{pmatrix} -2 & 7 \\ 5 & 0 \\ 1 & 1 \\ 4 & -6 \end{pmatrix}$. No trace - matrix is not square.

5. Prove that $A + A^t$ is symmetric for any square matrix A.

Solution: Observe that

$$(A + A^t)^t = A^t + (A^t)^t = A^t + A = A + A^t.$$

The matrix $A + A^t$ is equal to its transpose, and is therefore symmetric.

8. Determine whether the following sets are subspaces of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$

b. $\mathsf{W}_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + a_3 = 2\}$

Solution: No. Observe that $v = (1, 0, 1)$ lies in W_2, but $2v = (2, 0, 2)$ does not.

d. $\mathsf{W}_4 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - a_3 = 0\}$

Solution: Yes. Given $u = (u_1, u_2, u_3)$ and (v_1, v_2, v_3) in W_4, and $r \in \mathbb{R}$, consider $u + rv$:

$$\begin{aligned}(u_1 + rv_1) - 4(u_2 + rv_2) - (u_3 + rv_3) &= (u_1 - 4u_2 - u_3) + (rv_1 - 4rv_2 - rv_3) \\ &= (u_1 - 4u_2 - u_3) + r(v_1 - 4v_2 - v_3) = 0 + r0 = 0\end{aligned}$$

Thus $u + rv$ lies in W_4.

f. $\mathsf{W}_6 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$

Solution: No. Let $v = (0, \sqrt{2}, 1)$ and let $w = (0, \sqrt{2}, -1)$. These both lie in W_6, but $w + v = (0, 2\sqrt{2}, 0)$ does not.

11. Is the set $\mathsf{W} = \{f(x) \in \mathsf{P}(F) : f(x) = 0 \text{ or } f(x) \text{ has degree } n\}$ a subspace of $\mathsf{P}(F)$ if $n \geq 1$?

Solution: No. Consider the polynomials $p = x^n + x^{n-1}$ and $q = -x^n$, both of degree n, so they lie in W. Their sum is x^{n-1}, which does not lie in W.

13. Let S be a nonempty set and F a field. Prove that for any $s_0 \in S$, $\mathsf{V}_{s_0} = \{f \in \mathcal{F}(S, F) : f(s_0) = 0\}$ is a subspace of $\mathcal{F}(S, F)$.

Solution: Fix $s_0 \in S$. Given $f, g \in \mathsf{V}_{s_0}$ and $c \in \mathsf{F}$, observe that

$$(f + cg)(s_0) = f(s_0) + (cg)(s_0) = f(s_0) + cg(s_0) = 0.$$

Thus $f + cg$ lies in V_{s_0}.

15. Is the set of all differentiable real-valued functions on $\mathbb{R}$ a subspace of $\mathsf{C}(\mathbb{R})$?

Solution: Every differentiable function must be continuous, so the differentiable functions compose a subset of the $\mathsf{C}(\mathbb{R})$. The linearity property of derivatives (you can take this for granted, but it must be mentioned) states that, given f, g differentiable real-valued functions on $\mathbb{R}$, the derivative $(f + cg)'$ exists and is equal to $f' + cg'$. Thus $f + cg$ is differentiable, and so the differentiable functions compose a subspace of $\mathsf{C}(\mathbb{R})$.

20. Prove that if W is a subspace of a vector space V and $w_1, w_2, \ldots, w_n$ are in W, then $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in \mathsf{W}$ for any scalars $a_1, \ldots, a_n$.

Solution: Of course this is true for a single vector. We will prove, by induction, that it is true for any finite collection:

Suppose this fact is known for collections of $n-1$ vectors in W. Then

$$a_1w_1 + a_2w_2 + \cdots + a_nw_n = (a_1w_1 + \cdots + a_{n-1}w_{n-1}) + a_nw_n = w + v.$$

This is a sum of two vectors in W, so it must lie in W as well. Thus the fact is known for collections of n vectors in W.

21. Show that the set of convergent sequences $\{a_n\}$ is a subspace of the vector space of sequences of real numbers.

Solution: The linearity property of limits (again, you can quote this without proof) states that if the sequence $\{a_n\}$ converges to a, and $\{b_n\}$ converges b, then $\{a_n\} + r\{b_n\} = \{a_n + rb_n\}$ is convergent, and has limit $a + rb$, for all real numbers r. Thus the convergent sequences form a linear subspace of the set of real sequences.

24. Show that F^n is the direct sum of the subspaces

$$\mathsf{W}_1 = \{(a_1, a_2, \ldots, a_n) \in \mathsf{F}^n : a_n = 0\}$$

and

$$\mathsf{W}_2 = \{(a_1, a_2, \ldots, a_n) \in \mathsf{F}^n : a_1 = a_2 = \cdots = a_{n-1} = 0\}$$

Solution: Given any $v \in \mathsf{F}^n, v = (v_1, v_2, \ldots, v_n)$ one may express v as the sum of $w_1 = (v_1, v_2, \ldots, v_{n-1}, 0)$, which lies in W_1, and $w_2 = (0, 0, \ldots, 0, v_n)$, which lies in W_2. Thus $\mathsf{F}^n = \mathsf{W}_1 + \mathsf{W}_2$.

Now, suppose $v = (v_1, v_2, \ldots, v_n)$ lies in both W_1 and W_2. To fit the first condition, $v_n = 0$. To fit the second, $v_1 = v_2 = \cdots = v_{n-1} = 0$. Thus $v_1 = v_2 = \cdots = v_{n-1} = v_n = 0$, so $v = 0$. Thus $\mathsf{W}_1 \cap \mathsf{W}_2$ contains only the zero vector.

Together, $\mathsf{F}^n = \mathsf{W}_1 \oplus \mathsf{W}_2$.

28. Prove that the set W_1 of all skew-symmetric $n \times n$ matrices with entries from F is a subspace of $\mathsf{M}_{\times n \times n}(F)$. Let W_2 be the subspace of symmetric $n \times n$ matrices. If $\operatorname{char}(F) \neq 2$, prove that

$$\mathsf{M}_{\times n \times n}(F) = \mathsf{W}_1 \oplus \mathsf{W}_2.$$

Solution: Given a matrix $A \in \mathsf{M}_{n\times n}(\mathsf{F})$, define $w_1 = \frac{1}{2}(A - A^t)$ and $w_2 = \frac{1}{2}(A + A^t)$. Observe that

$$w_1^t = \tfrac{1}{2}(A - A^t)^t = \tfrac{1}{2}(A^t - (A^t)^t) = \tfrac{1}{2}(A^t - A) = \text{-}\tfrac{1}{2}(A - A^t) = -\text{-}w_1,$$

so $w_1 \in \mathsf{W}_1$, and

$$w_2^t = \tfrac{1}{2}(A + A^t)^t = \tfrac{1}{2}(A^t + (A^t)^t) = \tfrac{1}{2}(A^t + A) = \tfrac{1}{2}(A + A^t) = w_2,$$

so $w_2 \in \mathsf{W}_2$. Finally observe that

$$w_1 + w_2 = \tfrac{1}{2}(A - A^t) + \tfrac{1}{2}(A + A^t) = A.$$

Thus $\mathsf{M}_{n\times n}(\mathsf{F}) = \mathsf{W}_1 + \mathsf{W}_2$.

Now suppose $A \in \mathsf{W}_1 \cap \mathsf{W}_2$, so $A = A^t = \text{-}A$. If $\operatorname{char}(F) \neq 2$, then $A = \text{-}A \iff A_{ij} = \text{-}A_{ij}$ implies $A_{ij} = 0$ for all ij. Thus $\mathsf{W}_1 \cap \mathsf{W}_2$ contains only the zero vector.

Together, $\mathsf{M}_{n\times n}(\mathsf{F}) = \mathsf{W}_1 \oplus \mathsf{W}_2$.