1. **Inverses**

1. Let $f(x) = \frac{1}{x-3}$. Find the inverse $g(x)$ for $f$.

   **Solution:** Setting $y = (x - 3)^{-1}$ and solving for $x$ gives
   
   $$yx - 3y = 1$$
   
   and
   
   $$x = \frac{1 + 3y}{y}$$
   
   Therefore $g(x) = \frac{1+3x}{x}$.

2. Let $f(x) = x^4 + 32x$. Find a domain on which $f$ is invertible. If $g$ is the inverse of $f$, determine $g'(-31)$.

   **Solution:** $f'(x) = 4x^3 + 32$, which is non-negative when $x \geq -2$. Therefore $f$ is increasing, hence one-to-one, on $[-2, \infty)$.

   After checking $f(x)$ for some small values of $x$, we find that
   
   $$f(-1) = 1 - 32 = -31$$
   
   so $g(-31) = -1$. By the inverse function theorem,
   
   $$g'(-31) = \frac{1}{f'(g(-31))} = \frac{1}{f'(-1)} = \frac{1}{4(-1) + 32} = \frac{1}{32 - 4} = \frac{1}{28}$$

2. **Exponential functions and logarithms**

1. Find the derivative of $x^{x^2+1}$.

   **Solution:** Begin by writing
   
   $$x^{x^2+1} = e^{\ln(x^{x^2+1})} = e^{(x^2+1)\ln(x)}$$
   
   Now use the chain rule:
   
   $$\frac{d}{dx}x^{x^2+1} = \frac{d}{dx}e^{(x^2+1)\ln(x)} = e^{(x^2+1)\ln(x)} \left[ \frac{x^2 + 1}{x} + 2x \ln(x) \right]$$
2. Evaluate \( \int \frac{dx}{x \ln^2(x)} \).

Solution: If \( u = \ln(x) \), then \( du = \frac{1}{x} \, dx \). Therefore the integral is
\[
\int \frac{1}{u^2} \, du = -\frac{1}{u} + C = -\frac{1}{\ln(x)} + C
\]

3. Evaluate \( \int \frac{e^{2x} - 1}{e^{2x} + 1} \, dx \).

Solution: Multiply the top and bottom by \( e^{-x} \) to get
\[
\int \frac{e^{2x} - 1}{e^{2x} + 1} \, dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} \, dx
\]
If \( u = e^x + e^{-x} \), then \( du = (e^x - e^{-x}) \, dx \), so the integral is
\[
\int \frac{du}{u} = \ln(u) + C = \ln(e^x + e^{-x}) + C
\]
Note that the term in the parentheses is always positive, so there is no need for absolute values.

3. Exponential growth and decay

Two types of bacteria in a laboratory experience exponential growth. Bacteria type A has an initial population of 1000 and doubles every minute. Bacteria type B doubles every three minutes. If the two types of bacteria have the same population after 30 minutes, what was the initial population of bacteria B?

Solution: Let \( A(t) \) and \( B(t) \) denote the populations of bacteria types A and B. Both are experiencing exponential growth, so
\[
A(t) = C_1 e^{k_1 t} \quad B(t) = C_2 e^{k_2 t}
\]
for constants \( k_1, k_2, C_1, C_2 \).

To determine \( k_1 \), recall that the doubling time of \( A \) is equal to \( \frac{\ln(2)}{k_1} \). Since the doubling time is one minute, this implies that \( k_1 = \ln(2) \). Similarly, \( k_2 = \frac{\ln(2)}{3} \).

Next, for bacteria A we have
\[
1000 = A(0) = C_1
\]
Therefore
\[
A(30) = 1000 e^{\ln(2):30} = 1000 \cdot 2^{30}
\]
And \( A(30) = B(30) \), so
\[
1000 \cdot 2^{30} = A(30) = B(30) = C_2 e^{30 \frac{\ln(2)}{3}} = C_2 \cdot 2^{10}
\]
Therefore
\[
B(0) = C_2 = 1000 \cdot 2^{10}
\]
4. \( y' = k(y - b) \)

1. Find the solution to \( 2y' + 12 = 4y \) satisfying \( y(0) = 0 \).

   **Solution:** To solve this differential equation, begin by solving for \( y' \):
   \[
   y' = \frac{1}{2}(4y - 12) = 2y - 6 = 2(y - 3)
   \]

   The solution to this differential equation is
   \[
   y(x) = 3 + Ce^{2x}
   \]

   To determine the constant \( C \), plug in the initial condition:
   \[
   0 = y(0) = 3 + C
   \]

   so \( C = -3 \). Therefore
   \[
   y(x) = 3 - 3e^{2x}
   \]

2. A metal rod is heated to 200 degrees Celsius, then submerged in a large bucket of water whose ambient temperature is 20 degrees Celsius. After 5 minutes, the temperature of the rod is 110 degrees Celsius. The rod is then removed from the water. If the ambient temperature of the room is 25 degrees Celsius, what is the temperature of the rod after another 5 minutes?

   **Solution:** Begin by finding the formula for the cooling of the rod in the water. The equation for cooling is
   \[
   y(t) = T_0 + Ce^{-kt} = 20 + Ce^{-kt}
   \]

   and the constant \( C \) can be determined using the initial temperature:
   \[
   200 = y(0) = 20 + C
   \]

   so \( C = 180 \). Now we want to determine the cooling constant \( k \). To do this, use the fact that \( y(5) = 110 \):
   \[
   110 = y(5) = 20 + 180e^{-5k}
   \]

   so \( e^{-5k} = \frac{90}{180} = \frac{1}{2} \). Therefore
   \[
   -5k = \ln\left(\frac{1}{2}\right) = -\ln(2)
   \]

   so \( k = \frac{\ln(2)}{5} \).

   Now write out the equation for cooling once the rod is removed from the water:
   \[
   y(t) = T_0 + Ce^{-kt} = 25 + Ce^{-\ln(2)\frac{t}{5}}
   \]

   It’s easiest to reset time to zero once the rod is removed from the water. The new initial temperature is the final temperature of the rod in the water, namely 110 degrees. Therefore
   \[
   110 = y(0) = 25 + C
   \]
so \( C = 85 \). So the equation is
\[
y(t) = 25 + 85e^{-\frac{\ln(2)}{5}t}
\]
To find the final temperature, plug in \( t = 5 \):
\[
y(5) = 25 + 85e^{-\ln(2)} = 25 + \frac{85}{e^{\ln(2)}}
\]
This is probably a fine answer, but you can also use the fact that \( e^{\ln(2)} = 2 \) to simplify:
\[
y(5) = 25 + \frac{85}{2} = \frac{135}{2}
\]
in degrees Celsius.

5. **L’Hospital’s Rule**

1. Evaluate \( \lim_{x \to 0} \frac{e^{\sin x}}{\cos(\ln(x^2 + 1))} \).

**Solution:** Before using L’Hospital’s rule, always check that the limit is in fact an indeterminate form. In this case, taking \( f(x) = e^{\sin x} \) and \( g(x) = \cos(\ln(x^2 + 1)) \), we see that
\[
\lim_{x \to 0} f(x) = e^0 = 1
\]
and
\[
\lim_{x \to 0} g(x) = \cos(\ln(1)) = \cos(0) = 1
\]
Therefore L’Hospital’s rule does not apply, but we can simply plug in 0 to obtain
\[
\lim_{x \to 0} \frac{e^{\sin x}}{\cos(\ln(x^2 + 1))} = 1
\]

2. Evaluate \( \lim_{x \to \frac{\pi}{2}} (x - \frac{\pi}{2})\tan x \).

**Solution:** Writing \( \tan x = \frac{\sin x}{\cos x} \) gives
\[
\lim_{x \to \frac{\pi}{2}} (x - \frac{\pi}{2})\tan x = \lim_{x \to \frac{\pi}{2}} \frac{(x - \frac{\pi}{2})\sin x}{\cos x}
\]
Let \( f(x) = (x - \frac{\pi}{2})\sin x \) and \( g(x) = \cos x \). Then \( f(\frac{\pi}{2}) = 0 = g(\frac{\pi}{2}) \) and
\[
f'(x) = \sin x + (x - \frac{\pi}{2})\cos x \quad g'(x) = -\sin x
\]
which are both non-zero near \( \frac{\pi}{2} \). Therefore L’Hospital’s rule can be applied, yielding
\[
\lim_{x \to \frac{\pi}{2}} \frac{(x - \frac{\pi}{2})\sin x}{\cos x} = \lim_{x \to \frac{\pi}{2}} \frac{\sin x + (x - \frac{\pi}{2})\cos x}{-\sin x} = -1
\]

3. Show that \( \lim_{x \to \infty} \frac{\ln(x)}{x^a} = 0 \) for all \( a > 0 \).
**Solution:** Take $f(x) = \ln(x)$ and $g(x) = x^a$. Then \( \lim_{x \to \infty} f(x) = \infty = \lim_{x \to \infty} g(x) \), hence we can apply L’Hopital’s rule to obtain
\[
\lim_{x \to \infty} \frac{\ln(x)}{x^a} = \lim_{x \to \infty} \frac{\frac{1}{x}}{ax^{a-1}} = \lim_{x \to \infty} \frac{1}{ax^a} = 0
\]
since \( a > 0 \).

4. Evaluate \( \lim_{x \to \infty} \left(1 + \frac{1}{x^2}\right)^x \).

**Solution:** Define \( L = \lim_{x \to \infty} \left(1 + \frac{1}{x^2}\right)^x \). Then
\[
\ln(L) = \lim_{x \to \infty} x \ln \left(1 + \frac{1}{x^2}\right) = \lim_{x \to \infty} \frac{x \cdot \frac{1}{x^2}}{1 + \frac{1}{x^2}}
\]
This is an indeterminate form of the form \( \frac{0}{0} \), so by L’Hopital’s rule,
\[
\ln(L) = \lim_{x \to \infty} \frac{1 + \frac{1}{x^2}}{x^2} \cdot \frac{-2}{x^3} = \lim_{x \to \infty} \frac{1}{x^2 + 1} = 1
\]
Therefore \( L = e^{\ln(L)} = e^1 = e \).

5. Evaluate \( \lim_{x \to \infty} \frac{\sqrt{x^2 + 1}}{x} \).

**Solution:** This is an indeterminate form of the form \( \frac{\infty}{\infty} \), so we can try to apply L’Hopital’s rule. The result is
\[
\lim_{x \to \infty} \frac{\sqrt{x^2 + 1}}{x} = \lim_{x \to \infty} \frac{\frac{2x}{2\sqrt{x^2 + 1}}}{1} = \lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}}
\]
All we’ve done is flipped the terms! In this problem, L’Hopital’s rule is valid but unhelpful. Instead, divide the top and bottom by \( x \), using \( x^{-1} = \sqrt{x^{-2}} \):
\[
\lim_{x \to \infty} \frac{\sqrt{x^2 + 1}}{x} = \lim_{x \to \infty} \sqrt{\frac{x^2 + 1}{x^2}} = \lim_{x \to \infty} \sqrt{1 + \frac{1}{x^2}} = 1
\]