1. Trigonometric Integrals

1. Evaluate \( \int \sin^2 x \, dx \) and \( \int \cos^2 x \, dx \).

Solution: Recall the identities
\[
\cos^2 x = \frac{1 + \cos(2x)}{2} \quad \sin^2 x = \frac{1 - \cos(2x)}{2}
\]
Using these formulas gives
\[
\int \sin^2 x \, dx = \int \frac{1 - \cos(2x)}{2} \, dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C = \frac{x}{2} - \frac{\sin x \cos x}{2} + C
\]
and similarly
\[
\int \cos^2 x \, dx = \int \frac{1 + \cos(2x)}{2} \, dx = \frac{x}{2} + \frac{\sin(2x)}{4} + C = \frac{x}{2} + \frac{\sin x \cos x}{2} + C
\]
The last step of both integrals used the formula \( \sin(2x) = 2 \sin x \cos x \).

2. Evaluate \( \int \sin^2 x \cos^3 x \, dx \).

Solution: The key idea for evaluating trig integrals with odd powers is to use the identity \( \sin^2 x + \cos^2 x = 1 \) to try to set up a substitution. In this case, we get
\[
\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cos x \, dx = \int \sin^2 x (1 - \sin^2 x) \cos x \, dx
\]
Now the integral is written in a way that makes a \( u \)-substitution possible. Let \( u = \sin x \), so \( du = \cos x \, dx \). Then the integral becomes
\[
\int u^2(1 - u^2) \, du = \int u^2 - u^4 \, du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C
\]
One of the challenges of trig integrals is that the answer can take several different forms, depending on how many trig identities one chooses to use. So as a sanity check, let’s take the derivative of our answer:

\[
\frac{d}{dx} \left[ \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C \right] = \sin^2 x \cos x - \sin^4 x \cos x = \sin^2 x \cos x - \sin^2 x \sin^2 x \cos x
\]
\[ \sin^2 x \cos x - \sin^2 x (1 - \cos^2 x) \cos x = \sin^2 x \cos x - \sin^2 x \cos x + \sin^2 x \cos^3 x \]

\[ = \sin^2 x \cos^3 x \]

which is what we started with.

3. Evaluate \( \int \cos^4 x \, dx \).

Solution: To compute this integral, we can repeatedly use the identity from problem 1. Thus
\[ \cos^4 x = (\cos^2 x)^2 = \left( \frac{1 + \cos(2x)}{2} \right)^2 = \frac{1}{4} \left[ 1 + 2 \cos(2x) + \cos^2(2x) \right] \]
\[ = \frac{1}{4} \left[ 1 + 2 \cos(2x) + \frac{1 + \cos(4x)}{2} \right] \]

and
\[ \int \cos^4 x \, dx = \frac{1}{4} \int 1 + 2 \cos(2x) + \frac{1 + \cos(4x)}{2} \, dx \]
\[ = \frac{3}{8} \int dx + \frac{1}{4} \int 2 \cos(2x) \, dx + \frac{1}{8} \int \cos(4x) \, dx \]
\[ = \frac{3}{8} x + \frac{1}{4} \sin(2x) + \frac{1}{32} \sin(4x) + C \]

2. Trig Substitutions

1. Evaluate \( \int \frac{x^2}{\sqrt{9-x^2}} \, dx \).

Solution: Make the substitution \( x = 3 \sin \theta \), so that \( dx = 3 \cos \theta \, d\theta \). Then
\[ \int \frac{x^2}{\sqrt{9-x^2}} \, dx = \int \frac{9 \sin^2 \theta}{3 \cos \theta} 3 \cos \theta \, d\theta = 9 \int \sin^2 \theta \, d\theta \]
\[ = \frac{9}{2} \left[ \theta - \sin \theta \cos \theta \right] + C \]

using the result from the previous section. Now \( \theta = \sin^{-1} \frac{x}{3} \), \( \sin \theta = \frac{x}{3} \), and \( \cos \theta = \frac{\sqrt{9-x^2}}{3} \), so
\[ \int \frac{x^2}{\sqrt{9-x^2}} \, dx = \frac{9}{2} \left[ \frac{\sin^{-1} \frac{x}{3}}{3} - \frac{x \sqrt{9-x^2}}{3} \right] + C = \frac{9}{2} \sin^{-1} \frac{x}{3} - \frac{1}{2} x \sqrt{9-x^2} + C \]

2. Evaluate \( \int \frac{dx}{\sqrt{12x-x^2}} \).

Solution: At first glance, it doesn’t look like we can do a trig substitution. But observe that
\( (x - 6)^2 = x^2 - 12x + 36 \)
hence
\[12x - x^2 = 36 - (x - 6)^2\]

Therefore
\[\int \frac{dx}{\sqrt{12x - x^2}} = \int \frac{dx}{\sqrt{36 - (x - 6)^2}} = \int \frac{du}{\sqrt{36 - u^2}}\]
by taking \(u = x - 6\).

Now we can do a trig substitution: let \(u = 6 \sin \theta\). Then \(du = 6 \cos \theta \, d\theta\), hence
\[\int \frac{du}{\sqrt{36 - u^2}} = \int \frac{6 \cos \theta}{6 \cos \theta} \, d\theta = \theta + C = \sin^{-1} \frac{u}{6} + C = \sin^{-1} \frac{x - 6}{6} + C\]

3. Partial Fractions

1. Evaluate \(\int \frac{x-1}{(x-2)(x-3)} \, dx\).

**Solution:** The numerator has degree less than the denominator, so we can use partial fractions. Write
\[
\frac{x - 1}{(x - 2)^2(x - 3)} = \frac{A_1}{x - 2} + \frac{A_2}{(x - 2)^2} + \frac{B}{x - 3}
\]

After clearing denominators, we get
\[x - 1 = A_1(x - 2)(x - 3) + A_2(x - 3) + B(x - 2)^2\]

Now plug in \(x = 2\) to obtain \(1 = -A_2\), so \(A_2 = -1\). Similarly, if we plug in \(x = 3\) then we get \(2 = B\).

To determine \(A_1\), let’s look at the coefficient of \(x^2\) on both sides. On the left-hand side, this coefficient is zero, while on the right-hand side the coefficient is \(A_1 + B\). Therefore \(A_1 = -B = -2\). So our integral becomes
\[
\int \frac{x - 1}{(x - 2)^2(x - 3)} \, dx = -2 \int \frac{dx}{x - 2} - \int \frac{dx}{(x - 2)^2} + 2 \int \frac{dx}{x - 3}
\]
\[= 2 \ln |x - 3| - 2 \ln |x - 2| + \frac{1}{x - 2} + C\]

2. Evaluate \(\int \frac{5x}{(x+2)(x^2+1)} \, dx\).

**Solution:** When we do partial fractions with an irreducible quadratic factor like \(x^2 + 1\), the corresponding term in the expansion should have a linear polynomial in the numerator. So write
\[
\frac{5x}{(x+2)(x^2+1)} = \frac{A}{x + 2} + \frac{Bx + C}{x^2 + 1}
\]

Now clear the denominators to get
\[5x = A(x^2 + 1) + (Bx + C)(x + 2)\]
Next, plug in $x = -2$:

$$5(-2) = A(4 + 1)$$

or $A = -2$.

To find $B$, look at the coefficient of $x^2$. On the left-hand side this coefficient is zero, while on the right hand side it is $A + B$. Therefore $B = -A = 2$. Finally, to find $C$ we look at the constant coefficient on both sides to get $0 = A + 2C$, so $C = -\frac{A}{2} = 1$.

Therefore

$$\int \frac{5x}{(x+2)(x^2+1)} \, dx = -2 \int \frac{dx}{x+2} + \int \frac{2x+1}{x^2+1} \, dx = -2 \int \frac{dx}{x+2} + \int \frac{2x}{x^2+1} \, dx + \int \frac{dx}{x^2+1}$$

$$= -2 \ln |x+2| + \ln(x^2+1) + \tan^{-1}(x) + C$$

3. Evaluate $\int \frac{\sec^2 \theta}{\tan^2 \theta - 1} \, d\theta$.

**Solution:** One might be tempted to try to use the trig identity $\sec^2 \theta = 1 + \tan^2 \theta$, but I don’t think that helps here. So instead let’s start with a substitution: let $u = \tan \theta$, so that $du = \sec^2 \theta \, d\theta$. Then

$$\int \frac{\sec^2 \theta}{\tan^2 \theta - 1} \, d\theta = \int \frac{du}{u^2 - 1} = \int \frac{1}{(u-1)(u+1)} \, du$$

Now we can proceed using partial fractions. Write

$$\frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}$$

and clear denominators to obtain

$$1 = A(u+1) + B(u-1)$$

Plugging in $u = 1$ gives $1 = 2A$, so $A = \frac{1}{2}$. Similarly, plugging in $u = -1$ gives $B = -\frac{1}{2}$. Therefore

$$\int \frac{du}{(u-1)(u+1)} = \frac{1}{2} \int \frac{du}{u-1} - \frac{1}{2} \int \frac{du}{u+1} = \frac{1}{2} \ln |u-1| - \frac{1}{2} \ln |u+1| + C$$

and plugging in $u = \tan \theta$ gives

$$\int \frac{\sec^2 \theta}{\tan^2 \theta - 1} \, d\theta = \frac{1}{2} \ln |\tan \theta - 1| - \frac{1}{2} \ln |\tan \theta + 1| + C$$

4. **Improper Integrals**

1. Determine whether $\int_0^1 x^{-\frac{7}{9}} \, dx$ converges, and if so, evaluate it:

**Solution:** The function $x^{-\frac{7}{9}}$ has an infinite discontinuity at $x = 0$, so this is an improper integral. We can evaluate it using the limit definition:

$$\int_0^1 x^{-\frac{7}{9}} \, dx = \lim_{a \to 0^+} \int_a^1 x^{-\frac{7}{9}} \, dx = \lim_{a \to 0^+} \left[ 8x^{\frac{2}{9}} \right]_a^1 = \lim_{a \to 0^+} 8 - 8a^{\frac{2}{9}} = 8$$
Therefore the integral converges, and the value of the integral is 8.

2. Determine whether \( \int_{1}^{\infty} \frac{dx}{2x^{\frac{3}{2}} + x} \) converges.

**Solution:** \( 2x^{\frac{1}{2}} \leq x \) for \( x \geq 4 \), hence \( 2x^{\frac{3}{2}} + x \leq 2x + \frac{1}{2x} \geq \frac{1}{2x} \) for \( x \geq 4 \).

We can break our integral up into two pieces:

\[
\int_{1}^{\infty} \frac{dx}{2x^{\frac{3}{2}} + x} = \int_{1}^{4} \frac{dx}{2x^{\frac{3}{2}} + x} + \int_{4}^{\infty} \frac{dx}{2x^{\frac{3}{2}} + x}
\]

The first integral is just some finite number, so it doesn’t affect the convergence or divergence of the original integral. For the second integral, the comparison test gives

\[
\int_{4}^{\infty} \frac{dx}{2x^{\frac{3}{2}} + x} \geq \int_{4}^{\infty} \frac{dx}{2x} = \frac{1}{2} \lim_{R \to \infty} \left[ \ln(x) \right]_{4}^{R} = \frac{1}{2} \lim_{R \to \infty} (\ln(R) - \ln(4)) = \infty
\]

Therefore

\[
\int_{4}^{\infty} \frac{dx}{2x^{\frac{3}{2}} + x}
\]

diverges, which shows that

\[
\int_{1}^{\infty} \frac{dx}{2x^{\frac{3}{2}} + x}
\]

also diverges.

5. **Numerical Integration**

1. Find \( N \) such that

\[
\left| \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos x \, dx - T_{N} \right| \leq 10^{-5}
\]

**Solution:** The formula for the error bound for the trapezoid rule is given by

\[
\text{Error}(T_{N}) \leq \frac{K_{2}(b-a)^{3}}{12N^{2}}
\]

where \( K_{2} = \max_{x \in [a,b]} |f''(x)| \).

If \( f(x) = \cos x \), then \( f'(x) = -\sin x \) and \( f''(x) = -\cos x \). Therefore

\[
K_{2} = \max_{x \in [-\frac{1}{2}, \frac{1}{2}]} |\cos x| = 1
\]

hence

\[
\text{Error}(T_{N}) \leq \frac{(\frac{1}{2} - (-\frac{1}{2}))^{3}}{12N^{2}} = \frac{1}{12N^{2}}
\]
We want the error to be at most $10^{-5}$. Thus

$$\frac{1}{12N^2} \leq 10^{-5}$$

and solving for $N$ gives

$$N \geq \sqrt{\frac{10^5}{12}} \approx 91.2$$

so any $N \geq 92$ will do.

How could we handle the last part without using a calculator? Using the fact that

$$\frac{10^5}{12} \leq \frac{10^5}{10} = 10^4$$

we see that any $N \geq \sqrt{10^4} = 10^2 = 100$ suffices.

6. ARC LENGTH

For $t \in [0, \frac{1}{2}]$, let $f(t)$ be the arclength of the curve

$$y = \frac{x^{1-t}}{2(1-t)} - \frac{x^{t+1}}{2(t+1)}$$
on $[0, 1]$. Find the maximum of $f(t)$ on $[0, \frac{1}{2}]$ and the point at which it is attained.

**Solution:** Recall that the arclength formula is

$$\int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

By the power rule,

$$\frac{dy}{dx} = \frac{x^{-t}}{2} - \frac{x^t}{2}$$

hence

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^{-2t}}{4} - \frac{1}{2} + \frac{x^{2t}}{4} = \frac{x^{-2t}}{4} + \frac{1}{2} + \frac{x^{2t}}{4} = \left(\frac{x^{-t}}{2} + \frac{x^t}{2}\right)^2$$

Plugging this into the arclength formula gives

$$f(t) = \int_0^1 \frac{x^t}{2} + \frac{x^{-t}}{2} \, dx = \frac{1}{2(1+t)} + \frac{1}{2(1-t)} = \frac{1}{1-t^2}$$

Therefore $f$ is increasing on $[0, \frac{1}{2}]$, so has a maximum of $\frac{4}{3}$ at $t = \frac{1}{2}$. 
7. Fluid Force

1. Find the fluid force on a metal plate bounded by the parabola \( y = -x^2 \) and the line \( y = -1 \). The surface of the liquid is the line \( y = 0 \), and the liquid has pressure \( \rho \).

Solution: The formula for fluid force is

\[
F = \rho g \int_0^1 yf(y) \, dy
\]

where \( f(y) \) is the width at depth \( y \). In this case, \( f(y) = 2\sqrt{y} \). Therefore

\[
F = 2\rho g \int_0^1 y\sqrt{y} \, dy = 2\rho g \int_0^1 y^{3/2} \, dy = 2\rho g \left[ \frac{2}{5}y^{5/2} \right]_0^1 = \frac{4}{5}\rho g
\]

8. Taylor Polynomials

1. Let \( f(x) = e^{-x} \). Find the third Taylor polynomial \( T_3(x) \) for \( f \) centered at \( x = 0 \), and use the error bound to find the maximum value for \( |f(0.1) - T_3(0.1)| \).

Solution: Recall that

\[
T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3
\]

Therefore we need to take 3 derivatives:

\[
f'(x) = -e^{-x} \quad f''(x) = e^{-x} \quad f^{(3)}(x) = -e^{-x}
\]

Plugging in \( x = 0 \), we get

\[
T_3(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6}
\]

Next, recall that the error bound formula is

\[
|f(0.1) - T_3(0.1)| \leq \frac{K|0.1 - 0|^4}{4!} = \frac{K}{24 \cdot 10^4}
\]

where \( K \) is an upper bound for \( |f^{(4)}(x)| \) on \([0, 0.1]\). To get the best possible error bound, we take

\[
K = \max_{x \in [0,0.1]} |f^{(4)}(x)|
\]

Now \( f^{(4)}(x) = e^{-x} \), which is a decreasing function. Therefore

\[
K = \max_{x \in [0,0.1]} |f^{(4)}(x)| = e^0 = 1
\]

hence

\[
|f(0.1) - T_3(0.1)| \leq \frac{1}{24 \cdot 10^4} \approx 4.16 \times 10^{-6}
\]

On the midterm, I think that it would suffice to leave the answer as \( \frac{1}{24 \cdot 10^4} \).