MATH 31B: FINAL REVIEW

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1. Inverses

1. Let \( f(x) = \frac{\ln(x)}{\sqrt{x}} \). Write \((0, \infty)\) as the union of two intervals \( I_1 \) and \( I_2 \) such that \( f \) is invertible on each of \( I_1 \) and \( I_2 \). If \( g(x) \) is the inverse of \( f \) on one of these intervals, determine \( g'(\frac{1}{\sqrt{e}}) \).

Solution: Recall that \( f \) is guaranteed to be invertible on an interval if it is increasing or decreasing on that interval. So we should begin by finding the intervals on which \( f \) is increasing or decreasing.

By the quotient rule,
\[
f'(x) = \frac{\frac{\sqrt{x}}{x} - \frac{\ln(x)}{2\sqrt{x}}}{x} = \frac{2 - \ln(x)}{2x^\frac{3}{2}}
\]
Therefore \( f'(x) > 0 \) for \( 0 < x < e^2 \), and \( f'(x) < 0 \) for \( x > e^2 \).

\( f \) is neither increasing nor decreasing at \( x = e^2 \), but this is only a single point, so we can add it to either interval without changing anything. So either \( I_1 = (0, e^2] \) and \( I_2 = (e^2, \infty) \), or \( I_1 = (0, e^2) \) and \( I_2 = [e^2, \infty) \).

For the second part, remember the inverse function theorem:
\[
g'(x) = \frac{1}{f'(g(x))}
\]
We already know \( f'(x) \), and we can guess that \( g\left(\frac{1}{\sqrt{e}}\right) = e \), because \( f(e) = \frac{1}{\sqrt{e}} \). Therefore
\[
g'\left(\frac{1}{\sqrt{e}}\right) = \frac{1}{f'(e)} = \frac{1}{\frac{2 - 1}{2e^2}} = 2e^3
\]

2. Exponential functions and Logarithms

1. Evaluate \( \frac{d}{dx} x^\sin x \).

Solution: Write \( x^\sin x = e^{\ln(x^\sin x)} = e^{\sin(x)\ln(x)} \). Then
\[ \frac{d}{dx} e^{\sin(x) \ln(x)} = e^{\sin(x) \ln(x)} \left[ \cos(x) \ln(x) + \frac{\sin(x)}{x} \right] = x^{\sin(x)} \left[ \cos(x) \ln(x) + \frac{\sin(x)}{x} \right] \]

3. **L’Hôpital’s Rule**

1. Evaluate \( \lim_{x \to \infty} \left( \frac{x+2}{x+1} \right)^x \).

   **Solution:** Let \( L = \lim_{x \to \infty} \left( \frac{x+2}{x+1} \right)^x \). Then
   \[
   \ln(L) = \lim_{x \to \infty} x \ln \left( \frac{x+2}{x+1} \right) = \lim_{x \to \infty} \frac{x+2 - 1}{x+1} = \lim_{x \to \infty} \frac{x+1-(x+2)}{(x+1)(x+2)}
   \]
   \[
   = \lim_{x \to \infty} \frac{x^2}{x(x+1)(x+2)} = \lim_{x \to \infty} \frac{x^2}{x^2 + 3x + 2} = 1
   \]
   Therefore \( L = e^{\ln(L)} = e \).

   There is a faster way to evaluate this limit: if we set \( y = x+1 \), then the limit becomes
   \[
   \lim_{y \to \infty} \left( \frac{y+1}{y} \right)^{y-1} = \lim_{y \to \infty} \left( \frac{y+1}{y} \right)^y \left( \frac{y+1}{y} \right)^{-1} = e \cdot 1 = e
   \]

4. **Integration by Parts**

1. Evaluate \( \int \log_2 x \, dx \).

   **Solution:** Take \( u = \log_2 x \) and \( dv = dx \), so \( du = \frac{1}{\ln(2)x} \, dx \) and \( v = x \). Then
   \[
   \int \log_2 x \, dx = x \log_2 x - \int \frac{1}{\ln(2)} \, dx = x \log_2 x - \frac{x}{\ln(2)} + C
   \]

2. Evaluate \( \int \sin(2x) \cos(3x) \, dx \).

   **Solution:** This is one of the problems in which we integrate by parts twice and end up with a multiple of the original integral. The choice of \( u \) and \( v \) is arbitrary, but you must be consistent.

   So let \( u = \sin(2x) \), so \( du = 2 \cos(2x) \, dx \), and \( dv = \cos(3x) \, dx \), so \( v = \frac{\sin(3x)}{3} \). Then
   \[
   \int \sin(2x) \cos(3x) \, dx = \frac{\sin(2x) \sin(3x)}{3} - \frac{2}{3} \int \cos(2x) \sin(3x) \, dx
   \]
For the second integral, let \( u = \cos(2x) \), so \( du = -2\sin(2x) \, dx \), and \( dv = \sin(3x) \), so \( v = \frac{-\cos(3x)}{3} \). Then

\[
\int \cos(2x) \sin(3x) \, dx = -\frac{\cos(2x) \cos(3x)}{3} - \frac{2}{3} \int \sin(2x) \cos(3x) \, dx
\]

so

\[
\int \sin(2x) \cos(3x) \, dx = \frac{\sin(2x) \sin(3x)}{3} - \frac{2}{3} \left[ -\frac{\cos(2x) \cos(3x)}{3} - \frac{2}{3} \int \sin(2x) \cos(3x) \, dx \right]
\]

\[
= \frac{\sin(2x) \sin(3x)}{3} + \frac{2 \cos(2x) \cos(3x)}{9} + \frac{4}{9} \int \sin(2x) \cos(3x) \, dx
\]

Solving for the original integral, we get

\[
\frac{5}{9} \int \sin(2x) \cos(3x) \, dx = \frac{\sin(2x) \sin(3x)}{3} + \frac{2 \cos(2x) \cos(3x)}{9}
\]

or

\[
\int \sin(2x) \cos(3x) \, dx = \frac{3 \sin(2x) \sin(3x)}{5} + \frac{2 \cos(2x) \cos(3x)}{5} + C
\]

5. Trig Substitutions

1. Evaluate \( \int \sqrt{4x - x^2} \, dx \).

**Solution:** We want to do a trig substitution, but first we need to complete the square. Recall that

\[
(x - 2)^2 = x^2 - 4x + 4
\]

so

\[
4 - (x - 2)^2 = 4 - (x^2 - 4x + 4) = 4 - x^2
\]

Therefore

\[
\int \sqrt{4 - x^2} \, dx = \int \sqrt{4 - (x - 2)^2} \, dx = \int \sqrt{4 - u^2} \, du
\]

by making the substitution \( u = x - 2 \). Now set \( u = 2 \sin \theta \), so \( du = 2 \cos \theta \, d\theta \). Then the integral becomes

\[
\int \sqrt{4 - 4 \sin^2 \theta} \cdot 2 \cos \theta \, d\theta = 4 \int \cos^2 \theta \, d\theta
\]

Now recall that \( \cos^2 \theta = \frac{1 + \cos 2\theta}{2} \), so

\[
4 \int \cos^2 \theta \, d\theta = 2 \int 1 + \cos 2\theta \, d\theta = 2\theta + \sin 2\theta + C = 2(\theta + \sin \theta \cos \theta) + C
\]

Since \( u = 2 \sin \theta \), it follows that

\[
\sin \theta = \frac{u}{2} \quad \theta = \sin^{-1} \left( \frac{u}{2} \right) \quad \cos \theta = \frac{\sqrt{4 - u^2}}{2}
\]

hence

\[
\int \sqrt{4 - u^2} \, du = 2 \sin^{-1} \left( \frac{u}{2} \right) + \frac{u}{2} \sqrt{4 - u^2} + C
\]
and then plug in $u = x - 2$ to obtain 

$$ \int \sqrt{4x - x^2} \, dx = 2 \sin^{-1} \frac{x - 2}{2} + \frac{x - 2}{2} \sqrt{4 - (x - 2)^2} + C $$

6. Partial Fractions

1. Use partial fractions to evaluate $\int \sec x \, dx$.

Solution: Recall that $\sec x = \frac{1}{\cos x}$, so

$$ \int \sec x \, dx = \int \frac{1}{\cos x} \, dx = \int \frac{\cos x}{\cos^2 x} \, dx = \int \frac{\cos x}{1 - \sin^2 x} \, dx $$

Now take $u = \sin x$, so the integral becomes

$$ \int \frac{du}{1 - u^2} $$

Now we do partial fractions:

$$ \frac{1}{1 - u^2} = \frac{A}{1 - u} + \frac{B}{1 + u} $$

and after clearing denominators we have

$$ 1 = A(1 + u) + B(1 - u) $$

Plugging in $u = 1$ gives $A = \frac{1}{2}$, and plugging in $u = -1$ gives $B = \frac{1}{2}$. Therefore

$$ \int \frac{du}{1 - u^2} = \frac{1}{2} \int \frac{du}{1 - u} + \frac{1}{2} \int \frac{du}{1 + u} = -\frac{1}{2} \ln |1 - u| + \frac{1}{2} \ln |1 + u| + C $$

and plug $u = \sin x$ to get

$$ \int \sec x \, dx = \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + C $$

Now if we use the multiplication trick to evaluate the integral, we get

$$ \int \sec x \, dx = \ln | \sec x + \tan x | + C $$

and using $\sec x = \frac{1}{\cos x}, \tan x = \frac{\sin x}{\cos x}$ gives

$$ \ln | \sec x + \tan x | = \ln \left| \frac{1}{\cos x} + \frac{\sin x}{\cos x} \right| = \ln \left| \frac{1 + \sin x}{\cos x} \right| = \frac{1}{2} \ln \left| \frac{(1 + \sin x)^2}{1 - \sin^2 x} \right| = \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| $$

so these two answers are in fact the same.

2. Evaluate $\int \frac{x^4 + 1}{x^3 + 1} \, dx$.

Solution: We can’t do partial fractions right away because the degree of the numerator is greater than that of the denominator. So start with long division:

$$ x^4 + 1 = x(x^3 + 1) - x + 1 $$
so
\[
\int \frac{x^4 + 1}{x^3 + 1} \, dx = \int x + \frac{1 - x}{x^3 + 1} \, dx = \frac{x^2}{2} + \int \frac{1 - x}{x^3 + 1} \, dx
\]

Now do partial fractions on the second term:
\[
\frac{1 - x}{x^3 + 1} = \frac{1 - x}{(x + 1)(x^2 - x + 1)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 - x + 1}
\]

After clearing denominators, we get
\[
1 - x = A(x^2 - x + 1) + (Bx + C)(x + 1)
\]

First plug in \(x = -1\) to obtain
\[
2 = 3A
\]

so \(A = \frac{2}{3}\). Then set \(x = 0\) to get
\[
1 = A + C
\]

so \(C = 1 - A = \frac{1}{3}\). Finally, from considering the coefficient of \(x^2\) we get
\[
0 = A + B
\]

so \(B = -A = -\frac{2}{3}\). Therefore
\[
\int \frac{1 - x}{x^3 + 1} \, dx = \frac{2}{3} \int \frac{dx}{x + 1} - \frac{1}{3} \int \frac{2x - 1}{x^2 - x + 1} \, dx
\]

Taking \(u = x + 1\) in the first integral and \(u = x^2 - x + 1\) in the second shows that
\[
\frac{2}{3} \int \frac{dx}{x + 1} - \frac{1}{3} \int \frac{2x - 1}{x^2 - x + 1} \, dx = \frac{2}{3} \ln |x + 1| - \frac{1}{3} \ln(x^2 - x + 1) + C
\]

so adding in the first term, we get
\[
\int \frac{x^4 + 1}{x^3 + 1} \, dx = \frac{x^2}{2} + \frac{2}{3} \ln |x + 1| - \frac{1}{3} \ln(x^2 - x + 1) + C
\]

7. Improper Integrals

1. Determine the real numbers \(b\) for which \(\int_0^1 x^{-1/2} |\ln(x)|^b \, dx\) converges.

Solution: Since \(\ln(x)\) is negative for \(0 < x < 1\), \(|\ln(x)| = -\ln(x)\) on this interval. Therefore take \(u = -\ln(x)\), so \(\frac{du}{dx} = -\frac{1}{x}\). Then
\[
\int_0^1 x^{-1/2} |\ln(x)|^b \, dx = - \int_{\ln(1)}^{\ln(2)} u^b \, du = \int_{\ln(2)}^{\infty} u^b \, du
\]

which is finite precisely when \(b < -1\).

2. Determine whether \(\int_1^\infty \frac{x}{\sqrt{x^3 + x^2 + 2}} \, dx\) converges or diverges.
Solution: When $x$ is large, the denominator is roughly $x^{\frac{5}{2}}$, so the whole fraction is approximately $\frac{1}{x^2}$. This suggests that the integral should converge. And since

$$\frac{x}{\sqrt{x^5 + x^3 + x}} \leq \frac{x}{\sqrt{x^5}} = \frac{1}{x^{\frac{3}{2}}}$$

and

$$\int_{1}^{\infty} \frac{dx}{x^{\frac{3}{2}}}$$

converges, the comparison test implies that

$$\int_{1}^{\infty} \frac{x}{\sqrt{x^5 + x^3 + x}} \, dx$$

converges.

3. Determine whether $\int_{1}^{\infty} \frac{x^2}{\sqrt{x^5 + x^3 + x}} \, dx$ converges or diverges.

Solution: The intuitive reasoning is essentially the same as the previous problem: when $x$ is large, the denominator is roughly $x^{\frac{5}{2}}$, so the whole fraction is approximately $\frac{1}{x^2}$, which suggests that the integral should diverge.

However, the comparison is slightly more tricky in this case, because it is not true that $\frac{x^2}{\sqrt{x^5 + x^3 + x}} \geq \frac{1}{x^2}$. Instead, observe that

$$x^5 + x^3 + x \leq x^5 + x^5 + x^5 = 3x^5$$

which implies that

$$\frac{1}{\sqrt{x^5 + x^3 + x}} \geq \frac{1}{\sqrt{3x^5}}$$

and therefore that

$$\frac{x^2}{\sqrt{x^5 + x^3 + x}} \geq \frac{x^2}{\sqrt{3x^5}} = \frac{1}{\sqrt{3x^\frac{5}{2}}}$$

Since the integral

$$\int_{1}^{\infty} \frac{1}{\sqrt{3x^\frac{5}{2}}} \, dx = \frac{1}{\sqrt{3}} \int_{1}^{\infty} \frac{1}{x^{\frac{3}{2}}} \, dx$$

diverges, the comparison test implies that

$$\int_{1}^{\infty} \frac{x^2}{\sqrt{x^5 + x^3 + x}} \, dx$$

diverges.
8. Limits

1. a. If \( L = \lim_{n \to \infty} a_n^2 \) exists, does \( \lim_{n \to \infty} a_n \) necessarily exist?

   \textit{Solution:} No. One simple counterexample is \( a_n = (-1)^n \). Then \( a_n^2 = 1 \), so \( \lim_{n \to \infty} a_n^2 = 1 \), but \( \lim_{n \to \infty} a_n \) does not exist.

b. If \( L = \lim_{n \to \infty} a_n^3 \) exists, does \( \lim_{n \to \infty} a_n \) necessarily exist?

   \textit{Solution:} Yes. If \( f(x) = x^{\frac{1}{3}} \), then \( f \) is continuous and \( a_n = f(a_n^3) \), so by limit laws we get
   \[
   \lim_{n \to \infty} a_n = \lim_{n \to \infty} f(a_n^3) = f\left( \lim_{n \to \infty} a_n^3 \right) = f(L) = L^{\frac{1}{3}}
   \]

   The difference between this part and the previous one is that the function \( f(x) = x^{\frac{1}{3}} \) is continuous for all real \( x \), whereas the square-root function is only defined for non-negative \( x \).

9. Summing a series

1. Determine whether the series \( \sum_{n=2}^{\infty} \frac{4}{3n} \) converges, and if so, find its sum.

   \textit{Solution:} \( \sum_{n=2}^{\infty} \frac{4}{3n} = 4 \sum_{n=2}^{\infty} \frac{1}{3n} \) is a geometric series with \( r = \frac{1}{3} \), so converges. The sum is
   \[
   \sum_{n=2}^{\infty} \frac{4}{3n} = \frac{4}{1 - \frac{1}{3}} = \frac{4}{\frac{2}{3}} = \frac{4 \cdot 3}{2} = 2
   \]

2. Find the sum of the series \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 2n} \).

   \textit{Solution:} First of all, note that the series converges by comparison to \( \frac{1}{n^2} \). But we want to actually find the sum.

   To do so, use partial fractions to write
   \[
   \frac{1}{n^2 + 2n} = \frac{1}{2n} - \frac{1}{2(n + 2)} = \frac{1}{2} \left[ \frac{1}{n} - \frac{1}{n + 2} \right]
   \]

   Now look at the partial sums \( S_N \) of \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 2n} \). Writing out the first few, we get
   \[
   S_1 = 1 - \frac{1}{3}, \quad S_2 = \left( 1 - \frac{1}{3} \right) + \left( \frac{1}{2} - \frac{1}{4} \right) = 1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4}
   \]
   \[
   S_3 = \left( 1 - \frac{1}{3} \right) + \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}
   \]
   and
   \[
   S_4 = \left( 1 - \frac{1}{3} \right) + \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) + \left( \frac{1}{4} - \frac{1}{6} \right) = 1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}
   \]
Therefore in general
\[ S_N = 1 + \frac{1}{2} - \frac{1}{N + 1} - \frac{1}{N + 2} \]
so the sum of the infinite series is
\[
\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n} = \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{n} - \frac{1}{n + 2} \right] = \frac{1}{2} \lim_{N \to \infty} \left[ 1 + \frac{1}{2} - \frac{1}{N + 1} - \frac{1}{N + 2} \right] = \frac{3}{4}
\]

10. Series with Positive Terms

1. Determine whether the series \( \sum_{n=1}^{\infty} \frac{3 + 2 \cos(n\pi)}{n^2} \) converges.

*Solution:* Let \( a_n = \frac{3 + 2 \cos(n\pi)}{n^2} \). Since \( \cos(n\pi) = 1 \) if \( n \) is even and \(-1\) if \( n \) is odd, we see that \( a_n = \frac{5}{n^2} \) if \( n \) is even and \( a_n = \frac{1}{n^2} \) if \( n \) is odd.

Therefore \( 0 \leq a_n \leq \frac{5}{n^2} \) for all \( n \), and
\[
\sum_{n=1}^{\infty} \frac{5}{n^2} = 5 \sum_{n=1}^{\infty} \frac{1}{n^2}
\]
converges by the \( p \)-series test, so the original series converges by the comparison test.

2. Determine whether the series \( \sum_{n=1}^{\infty} ne^{-n} \) converges.

*Solution:* There are a few ways to solve this problem, but one good method is the integral test. The function \( f(x) = xe^{-x} \) is positive and continuous, and
\[ f'(x) = e^{-x} - xe^{-x} = (1 - x)e^{-x} \]
is negative for \( x > 1 \). Thus \( f \) is decreasing on \([1, \infty)\), so the integral test can be applied.

Integrating by parts gives
\[
\int xe^{-x} \, dx = -xe^{-x} + \int e^{-x} \, dx = -xe^{-x} - e^{-x} + C
\]
hence
\[
\int_1^{\infty} xe^{-x} \, dx = \lim_{R \to \infty} \left[ -xe^{-x} - e^{-x} \right]_1^R = \frac{2}{e} - \lim_{R \to \infty} Re^{-R} + e^{-R} = \frac{2}{e}
\]
using L’Hopital’s rule. Since the integral converges, the sum also converges.
11. Absolute and conditional convergence

1. For which of the following series does the alternating series test imply convergence?

a. \( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+1}{n} \).

Solution: This series is alternating with \( a_n = \frac{n+1}{n} \). While \( a_1 > a_2 > a_3 > \cdots > 0 \), note that
\[
\lim_{n \to \infty} a_n = 1
\]
so the alternating series test does not hold. In fact, the series diverges by the divergence test.

b. \( \sum_{n=1}^{\infty} \frac{\sin \frac{\pi n}{2}}{n} \).

Solution: We can apply the alternating series test in this case, but we have to be careful. Observe that \( \sin \frac{\pi n}{2} = 0 \) when \( n \) is even, and that if \( n \) is odd then \( \sin \frac{\pi n}{2} = \pm 1 \) depending on whether \( n \) has remainder 1 or 3 when it is divided by 4.
Therefore summing over all odd \( n \), we get
\[
\sum_{n=1}^{\infty} \frac{\sin \frac{\pi n}{2}}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}
\]
and this series converges by the alternating series test.

c. \( \sum_{n=1}^{\infty} \frac{(-1)\sqrt{n}}{n^2} \).

Solution: This series is alternating, with \( a_n = \frac{1}{\sqrt{n}} \). Since \( a_n \) is decreasing and \( \lim_{n \to \infty} a_n = 0 \), the alternating series test implies that the series converges.

2. Determine whether the series
\[
\sum_{n=1}^{\infty} \frac{\sin(e^{\cos n})}{n^3 + n}
\]
converges absolutely, conditionally, or not at all.

Solution: Don’t be fooled by the complicated expression in the numerator; just remember that \( |\sin(x)| \leq 1 \) for all real \( x \). Therefore
\[
\left| \frac{\sin(e^{\cos n})}{n^3 + n} \right| \leq \frac{1}{n^3 + n}
\]
Next, \( n^3 + n \geq n^3 \) for all \( n \), so
\[
\frac{1}{n^3 + n} \leq \frac{1}{n^3}
\]
and \( \sum_{n=1}^{\infty} \frac{1}{n^3} \) is finite by the p-series test. Therefore the series
\[
\sum_{n=1}^{\infty} \left| \frac{\sin(e^{\cos n})}{n^3 + n} \right|
\]
converges by the comparison test, which means that our original series converges absolutely.

12. Ratio and root tests

1. Determine whether the series \( \sum_{n=1}^{\infty} \frac{n!}{n^n} \) converges or diverges.

Solution: Use the ratio test:
\[
\lim_{n \to \infty} \frac{(n+1)! \cdot n^n}{n! \cdot (n+1)^{n+1}} = \lim_{n \to \infty} \frac{(n+1)n^n}{(n+1)(n+1)^{n+1}} = \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^n = \frac{1}{e} < 1
\]
so the series converges.

2. Define a sequence \( \{a_n\} \) by \( a_n = 2^{-n} \) if \( n \) is odd, and \( a_n = 2^{2-n} \) if \( n \) is even. Determine whether \( \sum_{n=1}^{\infty} a_n \) converges or diverges.

Solution: The sequence is \( \frac{1}{2}, 1, \frac{1}{8}, \frac{1}{4}, \frac{1}{32}, \frac{1}{16}, \ldots \).

If we try to apply the ratio test, we run into a problem: \( \frac{a_{n+1}}{a_n} = 2 \) if \( n \) is odd, and \( \frac{a_{n+1}}{a_n} = \frac{1}{8} \) if \( n \) is even. So \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \) doesn’t exist.

However, the root test still works:
\[
\lim_{n \to \infty} \left( 2^{-n} \right)^{\frac{1}{n}} = \frac{1}{2}
\]
and
\[
\lim_{n \to \infty} \left( 2^{2-n} \right)^{\frac{1}{n}} = \frac{1}{2} \lim_{n \to \infty} 4^{\frac{1}{n}} = \frac{1}{2}
\]
so
\[
\lim_{n \to \infty} a_n^{\frac{1}{n}} = \frac{1}{2} < 1
\]
Therefore the series converges.

13. Power series and Taylor series

1. Find a series representation of \( \ln(2) \).

Solution: Begin with the geometric series:
\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n
\]
which is valid for \( |x| < 1 \). Plugging in \(-x\) for \( x \), we get
\[
\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n
\]
still valid for $|x| < 1$. Now integrate term-by-term to obtain

$$\ln(1 + x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

This series still has radius of convergence 1, and if we plug in $x = 1$ we get

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

which converges by the alternating series test. Therefore

$$\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

2. Determine the value of $\sum_{n=2}^{\infty} \frac{n(n-1)}{3^n}$.

Solution: Once again, start with the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

and differentiate term-by-term to obtain

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$$

Differentiate a second time to get

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$$

Now the original series was valid for $|x| < 1$, so this series is also valid for $|x| < 1$. Therefore we can plug in $x = \frac{1}{3}$:

$$\frac{2}{(1-\frac{1}{3})^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{3^{n-2}}$$

To make this look like the original series, multiply both sides by $3^{-2}$:

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{3^n} = \frac{2}{3^2(1-\frac{1}{3})^3} = \frac{6}{3^3(1-\frac{1}{3})^3} = \frac{6}{(3-1)^3} = \frac{6}{8} = \frac{3}{4}$$

3. Find the interval of convergence of the power series $\sum_{n=0}^{\infty} 2^{2n} \frac{n}{n^2+1} x^n$.

Solution: To find the radius of convergence, use the ratio test:

$$\lim_{n \to \infty} \frac{2^{2(n+1)} |x|^{n+1} (n+1)}{(n+1)^2 + 1} \cdot \frac{n^2 + 1}{2^{2n} \cdot n \cdot |x|^n} = \lim_{n \to \infty} 4|x| (n+1)(n^2 + 1) / n(n^2 + 2n + 2)$$
\[
\lim_{{n \to \infty}} 4|x|^{n^3 + n^2 + n + 1 \over n^3 + 2n^2 + 2n} = 4|x|
\]
and this limit is less than 1 precisely when \(|x| < {1 \over 4}\). Thus the radius of convergence \(R\) is \(1 / 4\).

For the endpoints, plugging in \(x = \frac{1}{4}\) yields

\[
\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}
\]

For this series, note that \(n^2 + 1 \leq 2n^2\) for all \(n \geq 1\), hence \(1 / (n^2 + 1) \geq 1 / 2n^2\) and \(n / (n^2 + 1) \geq n / 2n^2 = 1 / 2n\)
for all \(n \geq 1\). But

\[
\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}
\]
diverges by the p-series test, so the original series diverges by the comparison test.

Next, plugging in \(x = -\frac{1}{4}\) gives

\[
\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 1}
\]

This is an alternating series with \(a_n = n / (n^2 + 1)\), and since \(a_1 > a_2 > a_3 > \cdots > 0\) and \(\lim_{n \to \infty} a_n = 0\), this series converges by the alternating series test.

Therefore the interval of convergence is \([-\frac{1}{4}, \frac{1}{4})\).

4. Determine the Taylor series of the function \(f(x) = xe^x\) centered at \(c = 0\).

\textbf{Solution:} The general approach to Taylor series problems is to look for a formula for \(f^{(n)}(0)\). In this case the pattern is relatively simple: \(f^{(n)}(x) = xe^x + ne^x\), so \(f^{(n)}(0) = n\).

The formula for the Taylor series centered at \(c = 0\) is

\[
T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n
\]

Substituting in \(f^{(n)}(0) = n\), we get

\[
T(x) = \sum_{n=1}^{\infty} \frac{n}{n!} x^n = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}
\]

Note that the last sum starts at \(n = 1\). This is because \(\frac{n}{n!} = 0\) when \(n = 0\), and because \((-1)!\) is undefined.

The above method is relatively quick and easy, but if you already know the Taylor series for \(e^x\), then you can also simply multiply it by \(x\). So the Taylor series for \(xe^x\) would then be

\[
x \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}
\]
This is the same series as

\[ \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} \]

but with the index of summation shifted by one.