1. **Compound interest and present value**

1. Which should you choose: 1000 dollars now or 2000 dollars in 5 years? Assume that the interest rate \( r = 0.2 \).

   **Solution**: Given a rate \( r \), the present value of \( P \) dollars received \( t \) years in the future is
   \[
P e^{-rt}
   \]
   In our case, \( P = 2000 \), \( t = 5 \), and \( r = 0.2 = \frac{1}{5} \), hence the present value is
   \[
   2000 e^{-\frac{2}{5}} = 2000 e^{-1} = \frac{2}{e} \cdot 1000
   \]
   Since \( e > 2 \), this is less than 1000 dollars. So it is better to take 1000 dollars now.

2. **L’Hôpital’s rule**

1. Evaluate \( \lim_{x \to \infty} \left( \frac{x+2}{x+1} \right)^x \).

   **Solution**: Let \( L = \lim_{x \to \infty} \left( \frac{x+2}{x+1} \right)^x \). Then
   \[
   \ln(L) = \lim_{x \to \infty} x \ln \left( \frac{x+2}{x+1} \right) = \lim_{x \to \infty} \frac{\ln \left( \frac{x+2}{x+1} \right)}{\frac{1}{x}}
   \]
   This limit is the indeterminate form \( \frac{0}{0} \), so we can apply L’Hôpital’s rule to get
   \[
   \lim_{x \to \infty} \frac{\ln \left( \frac{x+2}{x+1} \right)}{\frac{1}{x}} = \lim_{x \to \infty} \frac{\ln(x+2) - \ln(x+1)}{-\frac{x^2}{x+1}} = \lim_{x \to \infty} \frac{x+1-(x+2)}{(x+1)(x+2)} = \lim_{x \to \infty} \frac{x^2}{(x+1)(x+2)} = 1
   \]
   Therefore \( L = e^{\ln(L)} = e \).
   There is a faster way to evaluate this limit: if we set \( y = x+1 \), then the limit becomes
   \[
   \lim_{y \to \infty} \left( \frac{y+1}{y} \right)^{y-1} = \lim_{y \to \infty} \left( \frac{y+1}{y} \right)^{y} \left( \frac{y+1}{y} \right)^{-1} = e \cdot 1 = e
   \]
3. Trig Substitutions

1. Evaluate \( \int \sqrt{4x - x^2} \, dx \).

   **Solution:** We want to do a trig substitution, but first we need to complete the square. Recall that
   \[
   (x - 2)^2 = x^2 - 4x + 4
   \]
   so
   \[
   4 - (x - 2)^2 = 4 - (x^2 - 4x + 4) = 4 - x^2
   \]
   Therefore
   \[
   \int \sqrt{4x - x^2} \, dx = \int \sqrt{4 - (x - 2)^2} \, dx = \int \sqrt{4 - u^2} \, du
   \]
   by making the substitution \( u = x - 2 \). Now set \( u = 2 \sin \theta \), so \( du = 2 \cos \theta \, d\theta \). Then the integral becomes
   \[
   \int \sqrt{4 - 4 \sin^2 \theta} \cdot 2 \cos \theta \, d\theta = 4 \int \cos^2 \theta \, d\theta
   \]
   Now recall that \( \cos^2 \theta = \frac{1 + \cos 2\theta}{2} \), so
   \[
   4 \int \cos^2 \theta \, d\theta = 2 \int 1 + \cos 2\theta \, d\theta = 2\theta + \sin 2\theta + C = 2(\theta + \sin \theta \cos \theta) + C
   \]
   Since \( u = 2 \sin \theta \), it follows that
   \[
   \sin \theta = \frac{u}{2} \quad \theta = \sin^{-1} \frac{u}{2} \quad \cos \theta = \frac{\sqrt{4 - u^2}}{2}
   \]
   hence
   \[
   \int \sqrt{4 - u^2} \, du = 2 \sin^{-1} \frac{u}{2} + \frac{u}{2} \sqrt{4 - u^2} + C
   \]
   and then plug in \( u = x - 2 \) to obtain
   \[
   \int \sqrt{4x - x^2} \, dx = 2 \sin^{-1} \frac{x - 2}{2} + \frac{x - 2}{2} \sqrt{4 - (x - 2)^2} + C
   \]

4. Partial Fractions

1. Use partial fractions to evaluate \( \int \sec x \, dx \).

   **Solution:** Recall that \( \sec x = \frac{1}{\cos x} \), so
   \[
   \int \sec x \, dx = \int \frac{1}{\cos x} \, dx = \int \frac{\cos x}{\cos^2 x} \, dx = \int \frac{\cos x}{1 - \sin^2 x} \, dx
   \]
   Now take \( u = \sin x \), so the integral becomes
   \[
   \int \frac{du}{1 - u^2}
   \]
Now we do partial fractions:
\[
\frac{1}{1 - u^2} = \frac{A}{1 - u} + \frac{B}{1 + u}
\]
and after clearing denominators we have
\[
1 = A(1 + u) + B(1 - u)
\]
Plugging in \(u = 1\) gives \(A = \frac{1}{2}\), and plugging in \(u = -1\) gives \(B = \frac{1}{2}\). Therefore
\[
\int \frac{du}{1 - u^2} = \frac{1}{2} \int \frac{du}{1 - u} + \frac{1}{2} \int \frac{du}{1 + u} = -\frac{1}{2} \ln |1 - u| + \frac{1}{2} \ln |1 + u| + C
\]
and plug \(u = \sin x\) to get
\[
\int \sec x \, dx = \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + C
\]
Now if we use the multiplication trick to evaluate the integral, we get
\[
\int \sec x \, dx = \ln |\sec x + \tan x| + C
\]
and using \(\sec x = \frac{1}{\cos x}, \tan x = \frac{\sin x}{\cos x}\) gives
\[
\ln |\sec x + \tan x| = \ln \left| \frac{1}{\cos x} + \frac{\sin x}{\cos x} \right| = \ln \left| \frac{1 + \sin x}{\cos x} \right| = \frac{1}{2} \ln \left| \frac{(1 + \sin x)^2}{1 - \sin^2 x} \right| = \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right|
\]
so these two answers are in fact the same.

5. Improper Integrals

1. Determine the real numbers \(b\) for which \(\int_{0}^{\frac{1}{2}} x^{-1/2} |\ln(x)|^b \, dx\) converges.

\textit{Solution:} Since \(\ln(x)\) is negative for \(0 < x < 1\), \(|\ln(x)| = -\ln(x)\) on this interval. Therefore take \(u = -\ln(x)\), so \(\frac{du}{dx} = -\frac{1}{x}\). Then
\[
\int_{0}^{\frac{1}{2}} x^{-1/2} |\ln(x)|^b \, dx = -\int_{\ln(2)}^{0} u^b \, du = \int_{0}^{\ln(2)} u^b \, du
\]
which is finite precisely when \(b < -1\).

6. Numerical Integration

1. Find a value of \(N\) for which \(\text{Error}(S_N) \leq 10^{-3}\) for \(\int_{1}^{2} \frac{x+1}{x} \, dx\).

\textit{Solution:} Recall the error bound for Simpson’s rule:
\[
\text{Error}(S_N) \leq \frac{K_4(2 - 1)^5}{180N^4}
\]
where \(K_4\) is chosen so that \(|f^{(4)}(x)| \leq K_4\) for all \(x \in [1, 2]\).
If we write $f(x) = \frac{x+1}{x} = 1 + \frac{1}{x}$, then $f^{(4)}(x) = 24x^{-5}$. This is a decreasing function, so its maximum on $[1, 2]$ is at $x = 1$. Therefore we may take $K_4 = 24$.

The desired inequality is

$$\frac{24}{180N^4} \leq 10^{-3}$$

or

$$N^4 \geq \frac{24 \cdot 10^3}{180} = \frac{4}{3} \cdot 10^2$$

Therefore

$$N \geq \left(\frac{4}{3} \cdot 10^2\right)^{\frac{1}{4}} \approx 3.398$$

so $N \geq 4$ works. Without the benefit of a calculator, we could take

$$N^4 \geq \frac{3}{2} \cdot 100 = 150$$

because $\frac{3}{2} > \frac{4}{3}$. Since $4^4 = 256$, we see again that $N \geq 4$ suffices.

7. **Arc length**

1. Which is larger: the arc length of the graph of $f(x) = \frac{1}{2}x^2 + 1$ on $[0, 1]$ or the arc length of the graph of $g(x) = \frac{1}{3}x^3$ on $[0, 1]$?

   **Solution:** $f'(x) = x$, so the arc length of the first graph is

   $$\int_{0}^{1} \sqrt{1 + x^2} \, dx$$

   Similarly, $g'(x) = x^2$, so the arc length of the second graph is

   $$\int_{0}^{1} \sqrt{1 + x^4} \, dx$$

   And $x^2 \geq x^4$ on $[0, 1]$ (with equality only at the endpoints), so the first integral is larger.

   We can check this using a computer:

   $$\int_{0}^{1} \sqrt{1 + x^2} \, dx \approx 1.15$$

   and

   $$\int_{0}^{1} \sqrt{1 + x^4} \, dx \approx 1.09$$
8. Fluid Force

1. Determine the fluid force on one side of a metal plate bounded by the x-axis and the graph of the function \( g(x) = \sqrt{|x|} - 1 \). The surface of the liquid is the line \( y = 0 \), and the liquid has pressure \( \rho \).

Solution: The formula for fluid force is

\[
F = \rho g \int_a^b y f(y) \, dy
\]

where \( f(y) \) is the width of the plate at depth \( y \).

The function \( g(x) \) is symmetric about the y-axis, so we can find the distance between the y-axis and the graph of \( g \) (for \( x \geq 0 \)), then multiply by 2.

At depth \( y \), the point on the graph of \( g \) is given by \((x, -y) = (x, g(x))\). Therefore

\[-y = g(x) = \sqrt{x} - 1\]

and solving for \( x \) gives \( x = (1 - y)^2 \). Therefore \( f(y) = 2(1 - y)^2 \), so

\[
F = 2\rho g \int_0^1 y(1-y)^2 \, dy = 2\rho g \int_0^1 y - 2y^2 + y^3 \, dy = 2\rho g \left[ \frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4} \right]_0^1 = 2\rho g \left[ \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right]
\]

\[
= \frac{\rho g}{6}
\]

9. The formal definition of convergence

1. Use the formal definition of the limit of a sequence to show that \( \lim_{n \to \infty} \frac{\sin(n)}{\sqrt{n}} = 0 \).

Solution: We need to show the following: for all \( \varepsilon > 0 \), there exists \( N \) (depending on \( \varepsilon \)) such that if \( n \geq N \) then \( \left| \frac{\sin(n)}{\sqrt{n}} \right| < \varepsilon \).

Fix \( \varepsilon > 0 \). Then

\[
\left| \frac{\sin(n)}{\sqrt{n}} \right| \leq \frac{1}{\sqrt{n}}
\]

Choose \( N \) such that \( N > \frac{1}{\varepsilon^2} \). If \( n \geq N \), then \( \sqrt{n} \geq \sqrt{N} > \frac{1}{\varepsilon} \), hence

\[
\left| \frac{\sin(n)}{\sqrt{n}} \right| \leq \frac{1}{\sqrt{n}} < \frac{1}{\varepsilon} = \varepsilon
\]

As \( \varepsilon \) was arbitrary, this shows that the sequence converges to 0.

*2. Let \( \{a_n\}, \{b_n\} \) be two sequences with \( \lim_{n \to \infty} a_n = a \) and \( \lim_{n \to \infty} b_n = a \). Define a new sequence by \( c_n = b_n^2 \) if \( n \) is even, and \( c_n = \frac{a_{n+1}}{2} \) if \( n \) is odd. So \( c_n \) is the sequence

\[ a_1, b_1, a_2, b_2, a_3, b_3, \ldots \]

Use the formal definition of the limit to show that \( \lim_{n \to \infty} c_n = a \).
Solution: Fix $\varepsilon > 0$. Since $\lim_{n \to \infty} a_n = a$, there exists $N_1$ such that if $n \geq N_1$ then $|a_n - a| < \varepsilon$. Similarly, there exists $N_2$ such that if $n \geq N_2$ then $|b_n - a| < \varepsilon$.

Choose $N$ such that $N \geq \max\{2N_1 - 1, 2N_2\}$, and suppose that $n \geq N$.

If $n$ is odd, then $c_n = a_{n+1}$, and $n \geq N \geq 2N_1 - 1$ implies that

$$\frac{n + 1}{2} \geq \frac{2N_1 - 1 + 1}{2} = N_1$$

so $|c_n - a| = |a_{n+1} - a| < \varepsilon$.

Similarly, if $n$ is even, then $c_n = b_{n/2}$. Since $n \geq N \geq 2N_2$, it follows that

$$\frac{n}{2} \geq \frac{2N_2}{2} = N_2$$

so $|c_n| - a = |b_{n/2} - a| < \varepsilon$.

Therefore $|c_n - a| < \varepsilon$ for all $n \geq N$, so $\lim_{n \to \infty} c_n = a$.

3. Let $\{a_n\}$ be a sequence such that $a_n$ is an integer for all $n$, and suppose that $\lim_{n \to \infty} a_n = a$, where $a$ is also an integer. Show that there exists $N$ such that $a_n = a$ for all $n \geq N$.

Solution: Since $\lim_{n \to \infty} a_n = a$, we know that for all $\varepsilon > 0$ there exists $N$ such that if $n \geq N$ then $|a_n - a| < \varepsilon$.

In particular, taking $\varepsilon = \frac{1}{2}$ implies that there exists $N$ such that $|a_n - a| < \frac{1}{2}$ for all $n \geq N$.

But $a_n$ and $a$ are integers, so if $|a_n - a| < \frac{1}{2}$ then $a_n = a$. Therefore $a_n = a$ for all $n \geq N$.

10. Summing a series

1. Determine whether the series $\sum_{n=2}^{\infty} \frac{4}{3^n}$ converges, and if so, find its sum.

Solution: $\sum_{n=2}^{\infty} \frac{4}{3^n} = 4 \sum_{n=2}^{\infty} \frac{1}{3^n}$ is a geometric series with $r = \frac{1}{3}$, so converges. The sum is

$$\sum_{n=2}^{\infty} \frac{4}{3^n} = \frac{4 \cdot \frac{1}{3}}{1 - \frac{1}{3}} = \frac{4}{9 - 3} = \frac{2}{3}$$

2. Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n}$.

Solution: First of all, note that the series converges by comparison to $\frac{1}{n^2}$. But we want to actually find the sum.

To do so, use partial fractions to write

$$\frac{1}{n^2 + 2n} = \frac{1}{2n} - \frac{1}{2(n+2)} = \frac{1}{2} \left[ \frac{1}{n} - \frac{1}{n + 2} \right]$$
Now look at the partial sums $S_N$ of $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2}$. Writing out the first few, we get

\[
S_1 = 1 - \frac{1}{3} \quad S_2 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) = 1 + \frac{1}{2} - \frac{3}{4}
\]

\[
S_3 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}
\]

and

\[
S_4 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) = 1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}
\]

Therefore in general

\[
S_N = 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}
\]

so the sum of the infinite series is

\[
\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n} = \frac{1}{2} \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+2}\right] = \frac{1}{2} \lim_{N \to \infty} \left[1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right] = \frac{3}{4}
\]

11. Series with positive terms

1. Determine whether the series $\sum_{n=1}^{\infty} \frac{3+2\cos(n\pi)}{n^2}$ converges.

Solution: Let $a_n = \frac{3+2\cos(n\pi)}{n^2}$. Since $\cos(n\pi) = 1$ if $n$ is even and $-1$ if $n$ is odd, we see that $a_n = \frac{5}{n^2}$ if $n$ is even and $a_n = \frac{1}{n^2}$ if $n$ is odd.

Therefore $0 \leq a_n \leq \frac{5}{n^2}$ for all $n$, and

\[
\sum_{n=1}^{\infty} \frac{5}{n^2} = 5 \sum_{n=1}^{\infty} \frac{1}{n^2}
\]

converges by the $p$-series test, so the original series converges by the comparison test.

2. Determine whether the series $\sum_{n=1}^{\infty} ne^{-n}$ converges.

Solution: There are a few ways to solve this problem, but one good method is the integral test. The function $f(x) = xe^{-x}$ is positive and continuous, and

\[
f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}
\]

is negative for $x > 1$. Thus $f$ is decreasing on $[1, \infty)$, so the integral test can be applied.

Integrating by parts gives

\[
\int xe^{-x} \, dx = -xe^{-x} + \int e^{-x} \, dx = -xe^{-x} - e^{-x} + C
\]
hence
\[
\int_{1}^{\infty} xe^{-x} \, dx = \lim_{R \to \infty} \left[ -xe^{-x} - e^{-x} \right]_{1}^{R} = \frac{2}{e} - \lim_{R \to \infty} Re^{-R} + e^{-R} = \frac{2}{e}
\]
using L’Hopital’s rule. Since the integral converges, the sum also converges.

12. Absolute and conditional convergence

1. For which of the following series does the alternating series test imply convergence?

a. \( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+1}{n} \).

Solution: This series is alternating with \( a_n = \frac{n+1}{n} \). While \( a_1 > a_2 > a_3 > \cdots > 0 \), note that
\[
\lim_{n \to \infty} a_n = 1
\]
so the alternating series test does not hold. In fact, the series diverges by the divergence test.

b. \( \sum_{n=1}^{\infty} \frac{\sin \frac{\pi n}{2}}{n} \).

Solution: We can apply the alternating series test in this case, but we have to be careful. Observe that \( \sin \frac{\pi n}{2} = 0 \) when \( n \) is even, and that if \( n \) is odd then \( \sin \frac{\pi n}{2} = \pm 1 \) depending on whether \( n \) has remainder 1 or 3 when it is divided by 4.

Therefore summing over all odd \( n \), we get
\[
\sum_{n=1}^{\infty} \frac{\sin \frac{\pi n}{2}}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1}
\]
and this series converges by the alternating series test.

c. \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \).

Solution: This series is alternating, with \( a_n = \frac{1}{\sqrt{n}} \). Since \( a_n \) is decreasing and \( \lim_{n \to \infty} a_n = 0 \), the alternating series test implies that the series converges.

2. Determine whether the series
\[
\sum_{n=1}^{\infty} \frac{\sin(e^{\cos n})}{n^3 + n}
\]
converges absolutely, conditionally, or not at all.

Solution: Don’t be fooled by the complicated expression in the numerator; just remember that \( |\sin(x)| \leq 1 \) for all real \( x \). Therefore
\[
\left| \frac{\sin(e^{\cos n})}{n^3 + n} \right| \leq \frac{1}{n^3 + n}
\]
Next, \( n^3 + n \geq n^3 \) for all \( n \), so
\[
\frac{1}{n^3 + n} \leq \frac{1}{n^3}
\]
and
\[
\sum_{n=1}^{\infty} \frac{1}{n^3}
\]
is finite by the p-series test. Therefore the series
\[
\sum_{n=1}^{\infty} \left| \frac{\sin(e^{\cos n})}{n^3 + n} \right|
\]
converges by the comparison test, which means that our original series converges absolutely.

13. Ratio and root tests

1. Determine whether the series \( \sum_{n=1}^{\infty} \frac{n!}{n^n} \) converges or diverges.

**Solution:** Use the ratio test:
\[
\lim_{n \to \infty} \frac{(n+1)! \cdot n^n}{n! \cdot (n+1)^{n+1}} \cdot \frac{1}{n+1} = \lim_{n \to \infty} \frac{(n+1)n^n}{(n+1)^{n+1}} = \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^n = \frac{1}{e} < 1
\]
so the series converges.

*2. Define a sequence \( \{a_n\} \) by \( a_n = 2^{-n} \) if \( n \) is odd, and \( a_n = 2^{2-n} \) if \( n \) is even. Determine whether \( \sum_{n=1}^{\infty} a_n \) converges or diverges.

**Solution:** The sequence is \( \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \).
If we try to apply the ratio test, we run into a problem: \( \frac{a_{n+1}}{a_n} = 2 \) if \( n \) is odd, and \( \frac{a_{n+1}}{a_n} = \frac{1}{2} \) if \( n \) is even. So \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \) doesn’t exist.

However, the root test still works:
\[
\lim_{n \to \infty} (2^{-n})^{1/n} = \frac{1}{2}
\]
and
\[
\lim_{n \to \infty} (2^{2-n})^{1/n} = \frac{1}{2} \lim_{n \to \infty} 4^{\frac{1}{n}} = \frac{1}{2}
\]
so arguing as in problem 2 of section 9 shows that
\[
\lim_{n \to \infty} a_n^{1/n} = \frac{1}{2} < 1
\]
Therefore the series converges.
14. Power Series and Taylor series

1. Find a series representation of \( \ln(2) \).

*Solution*: Begin with the geometric series:

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n
\]

which is valid for \(|x| < 1\). Plugging in \(-x\) for \(x\), we get

\[
\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n
\]

still valid for \(|x| < 1\). Now integrate term-by-term to obtain

\[
\ln(1 + x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}
\]

This series still has radius of convergence 1, and if we plug in \(x = 1\) we get

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}
\]

which converges by the alternating series test. Therefore

\[
\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}
\]

2. Determine the value of \( \sum_{n=2}^{\infty} \frac{n(n-1)}{3^n} \).

*Solution*: Once again, start with the geometric series

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n
\]

and differentiate term-by-term to obtain

\[
\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}
\]

Differentiate a second time to get

\[
\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1) x^{n-2}
\]
Now the original series was valid for $|x| < 1$, so this series is also valid for $|x| < 1$. Therefore we can plug in $x = \frac{1}{3}$:

$$\frac{2}{(1 - \frac{1}{3})^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{3^{n-2}}$$

To make this look like the original series, multiply both sides by $3^{-2}$:

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{3^n} = \frac{2}{3^2(1 - \frac{1}{3})^3} = \frac{6}{3^3(1 - \frac{1}{3})^3} = \frac{6}{(3-1)^3} = \frac{3}{4}$$

3. Find the interval of convergence of the power series $\sum_{n=0}^{\infty} 2^{2n} \frac{n}{n^2+1} x^n$.

**Solution:** To find the radius of convergence, use the ratio test:

$$\lim_{n \to \infty} \frac{2^{2(n+1)}|x|^{n+1}(n+1)}{(n+1)^2 + 1} \cdot \frac{n^2 + 1}{2^{2n} \cdot n \cdot |x|^n} = \lim_{n \to \infty} 4|x| \frac{(n+1)(n^2 + 1)}{n(n^2 + 2n + 2)}$$

$$= \lim_{n \to \infty} 4|x| \frac{n^3 + n^2 + n + 1}{n^3 + 2n^2 + 2n} = 4|x|$$

and this limit is less than 1 precisely when $|x| < \frac{1}{4}$. Thus the radius of convergence $R$ is $\frac{1}{4}$.

For the endpoints, plugging in $x = \frac{1}{4}$ yields

$$\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$$

For this series, note that $n^2 + 1 \leq 2n^2$ for all $n \geq 1$, hence $\frac{1}{n^2+1} \geq \frac{1}{2n^2}$ and $\frac{n}{n^2+1} \geq \frac{n}{2n^2} = \frac{1}{2n}$ for all $n$. But

$$\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges by the p-series test, so the original series diverges by the comparison test.

Next, plugging in $x = -\frac{1}{4}$ gives

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 1}$$

This is an alternating series with $a_n = \frac{n}{n^2+1}$, and since $a_1 > a_2 > a_3 > \cdots > 0$ and $\lim_{n \to \infty} a_n = 0$, this series converges by the alternating series test.

Therefore the interval of convergence is $[-\frac{1}{4}, \frac{1}{4})$. 