1. Limits and Continuity

1. Evaluate \( \lim_{x \to 0} \frac{1 - \cos x}{x^2} \).

**Solution:** Multiply the numerator and denominator by \( 1 + \cos x \) to obtain

\[
\frac{1 - \cos x}{x^2} = \frac{1 - \cos^2 x}{x^2 (1 + \cos x)} = \frac{\sin^2 x}{x^2 (1 + \cos x)}
\]

Therefore, using the limit laws shows that

\[
\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin^2 x}{x^2 (1 + \cos x)} = \left[ \lim_{x \to 0} \frac{\sin x}{x} \right]^2 \lim_{x \to 0} \frac{1}{1 + \cos x} = \frac{1}{2}
\]

2. Let \( f(x) = \frac{x^2 - a^2}{x - 2} \) if \( x \neq 2 \), and \( f(2) = 4 \). Find the value(s) of \( a \) that make \( f \) continuous at \( x = 2 \).

**Solution:** In order for \( f \) to be continuous, we need \( \lim_{x \to 2} f(x) = f(2) = 4 \); in particular the limit must exist.

\( f(x) = \frac{(x-a)(x+a)}{x-2} \) will not have a limit at 2 unless one of the factors in the numerator is \( x - 2 \). Therefore we need either \( x - a = x - 2 \) or \( x + a = x - 2 \), so \( a = \pm 2 \). If \( a = 2 \) then

\[
\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2} x + 2 = 4
\]

and if \( a = -2 \) then

\[
\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{(x + 2)(x - 2)}{x - 2} = \lim_{x \to 2} x + 2 = 4
\]

so both \( a = 2 \) and \( a = -2 \) work.

2. Differentiation

1. Find \( \frac{d^{2014}}{dx^{2014}} (\cos x) \).

**Solution:** The first derivative of \( \cos x \) is \( -\sin x \), the second derivative is \( -\cos x \), the third derivative is \( \sin x \), and the fourth derivative is \( \cos x \), which is the original function.
Therefore the \( k \)th derivative of \( \cos x \) is determined by the remainder when \( k \) is divided by 4, i.e. the integer \( 0 \leq r \leq 3 \) such that \( k - r \) is divisible by 4. 2014 has remainder 2 when it is divided by 4, so

\[
\frac{d^{2014}}{dx^{2014}}(\cos x) = \frac{d^2}{dx^2} \cos x = -\cos x
\]

2. Define a function \( f(x) \) by \( f(x) = mx + b \) for \( x < 2 \), and \( f(x) = x^2 + 1 \) for \( x \geq 2 \). Find the values of \( m \) and \( b \) such that \( f \) is differentiable.

Solution: First, \( f \) must be continuous, so the left and right hand limits of \( f(x) \) at \( x = 2 \) must be equal. The left hand limit is

\[
\lim_{x \to 2^-} mx + b = 2m + b
\]

while the right hand limit is

\[
\lim_{x \to 2^+} x^2 + 1 = 5
\]

Therefore \( 2m + b = 5 \).

Next \( f \) must be differentiable at 2, so we need

\[
\lim_{h \to 0^+} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0^-} \frac{f(2 + h) - f(2)}{h}
\]

The right hand limit is

\[
\lim_{h \to 0^+} \frac{(2 + h)^2 + 1 - 5}{h} = \lim_{h \to 0^+} \frac{4 + 4h + h^2 - 4}{h} = \lim_{h \to 0^+} 4 + h = 4
\]

The left hand limit is

\[
\lim_{h \to 0^-} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0^-} \frac{m(2 + h) + b - 5}{h}
\]

and using the fact that \( 2m + b = 5 \), this limit is

\[
\lim_{h \to 0^-} \frac{mh}{h} = m
\]

Since the two one-sided limits must be equal, this implies that \( m = 4 \). Plugging this into \( 2m + b = 5 \) shows that \( b = -3 \).
3. Implicit Differentiation

Find the points \((x_0, y_0)\) for which the tangent line to the curve \(y^2 - x^2 = 1\) is horizontal.

**Solution:** Using implicit differentiation gives

\[
2y \frac{dy}{dx} - 2x = 0
\]

or \(\frac{dy}{dx} = \frac{x}{y}\). Therefore the tangent line is horizontal when \(x = 0\). Plugging \(x = 0\) into the equation of the curve yields

\[y^2 = 1\]

or \(y = \pm 1\). Therefore the tangent line is horizontal at the points \((0, \pm 1)\).

4. Related Rates

1. Road A runs north-south, and road B runs east-west. The two roads meet at a gas station. At time \(t = 0\), three cars leave the gas station. Car 1 travels north on road A at a speed of 40 mph, car 2 travels east on road B at 60 mph, and car 3 travels west on road B at 30 mph. Find the rate of change of the area of the triangle formed by the three cars at time \(t = 30\) minutes.

**Solution:** Let \(f_1(t), f_2(t), f_3(t)\) denote the distance of cars 1, 2, 3 from the gas station at time \(t\). The area of the triangle is then

\[A(t) = \frac{1}{2}bh = \frac{1}{2}(f_2(t) + f_3(t))f_1(t)\]

Differentiating both sides with respect to time, we get

\[A'(t) = \frac{1}{2}f_1'(t)(f_2(t) + f_3(t)) + \frac{1}{2}f_1(t)(f_2'(t) + f_3'(t))\]

The problem tells us that \(f_1'(t) = 40, f_2'(t) = 60, \) and \(f_3'(t) = 30\). And at time \(t = \frac{1}{2}\) (in units of hours), we have

\[f_1(t) = \frac{40}{2} = 20, \quad f_2(t) = \frac{60}{2} = 30, \quad f_3(t) = \frac{30}{2} = 15\]

Therefore

\[A'(\frac{1}{2}) = \frac{1}{2} \cdot 40[30 + 15] + \frac{1}{2} \cdot 20[60 + 30] = 20 \cdot 45 + 10 \cdot 90 = 1800\]

in units of square miles per hour.
5. The second derivative and Inflection points

1. Give an example of a function \( f(x) \) and a point \( c \) such that \( f''(c) = 0 \) but \( c \) is not an inflection point of \( f \).

\text{Solution:} \ Let \( f(x) = x^6 \) and \( c = 0 \). Then \( f''(x) = 30x^4 \) and \( f''(0) = 0 \). But \( f'' \) doesn’t change sign at 0, so 0 is not an inflection point.

6. Graph Sketching

1. Sketch the graph of \( f(x) = \frac{x + 1}{x - 1} \). Identify all zeros and horizontal and vertical asymptotes, local extrema, and inflection points.

\text{Solution:} \ f(x) = 0 \ when \ x = -1, \ and \ f \ has \ a \ vertical \ asymptote \ at \ x = 1. \ f \ also \ has \ horizontal \ asymptotes \ at \ 1 \ as \ x \to \pm \infty. \n
Rather than using the quotient rule, we can differentiate \( f \) by first writing

\[ f(x) = \frac{x + 1}{x - 1} = \frac{x - 1 + 2}{x - 1} = 1 + \frac{2}{x - 1} \]

so that

\[ f'(x) = -\frac{2}{(x-1)^2} \]

Therefore the only critical point is \( x = 1 \). But \( f \) has a vertical asymptote at \( x = 1 \), so this can be neither a local min nor a local max. So \( f \) has no local extrema. Also note that \( f'(x) < 0 \) on \( (-\infty, 1) \) and \( (1, \infty) \), so \( f \) is always decreasing.

For concavity,

\[ f''(x) = \frac{4}{(x-1)^3} \]

so the only potential inflection point is \( x = 1 \). Since \( f''(x) < 0 \) for \( x < 1 \) and \( f''(x) > 0 \) for \( x > 1 \), \( x = 1 \) is an inflection point. \( f \) is concave down on \( (-\infty, 1) \) and concave up on \( (1, \infty) \).

7. Optimization

1. Find the point on the graph of \( f(x) = x^2 \) which minimizes the distance to the point \( (3, 0) \).

\text{Solution:} \ It’s important to note that the function we want to minimize is the distance from \( f \) to \( (3, 0) \), not necessarily the function \( f \) itself. By the distance formula, we have

\[ d(x)^2 = x^2 + y^2 = (x - 3)^2 + x^4 = x^2 - 6x + 9 + x^4 \]
Minimizing the distance is equivalent to minimizing the square of the distance, which is easier because there is no square root. The derivative of \( d^2 \) is
\[
4x^3 + 2x - 6 = 2(2x^3 + x - 3) = 2(x - 1)(2x^2 + 2x + 3)
\]
By using the quadratic formula, we see that \( 2x^2 + 2x + 3 \) does not have real roots. Therefore the only critical point is \( x = 1 \). The derivative changes sign from negative to positive at this point, so this is a minimum.

Thus the point on the graph of \( f \) minimizing the distance to \((3, 0)\) is \((1, 1)\).

*2. When a light beam travels from a point \( A \) above a swimming pool to a point \( B \) below the water, it chooses the path that takes the least time. Let \( v_1 \) be the velocity of light in air and \( v_2 \) the velocity of light in water. Prove Snell’s law of refraction: \( \frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2} \), where \( \theta_1 \) and \( \theta_2 \) are the vertical angles between the path the light takes and the water.

**Solution:** A good picture is essential in this problem, and would most likely be provided on an exam. This is problem 44 on p. 224 of the textbook if you want to see the book’s picture. Following the book’s picture, let \( h_1 \) be the height of the point \( A \) above the water and \( h_2 \) the vertical distance of the point \( B \) below the water.

This problem is very difficult to get started on, because while we know that we want to minimize time, it’s not at all clear what variable time should be a function of. I think the key to the problem is realizing that the horizontal distance \( L \) between the points \( A \) and \( B \) is fixed, but the point \( C \) at which the light crosses the water can vary.

Let \( x \) be the horizontal distance of this point from \( A \), define \( L_1 \) to be the distance between \( A \) and \( C \), and similarly let \( L_2 \) be the distance from \( C \) to \( B \). Then by the Pythagorean theorem
\[
L_1^2 = h_1^2 + x^2 \quad L_2^2 = h_2^2 + (L - x)^2
\]
Since light travels at a constant velocity \( v_1 \) in air and a constant velocity \( v_2 \) in water, the time it takes to travel from point \( A \) to point \( B \) is
\[
t = \frac{L_1}{v_1} + \frac{L_2}{v_2}
\]
Then write \( t \) as a function of \( x \):
\[
t(x) = \frac{\sqrt{x^2 + h_1^2}}{v_1} + \frac{\sqrt{(L - x)^2 + h_2^2}}{v_2}
\]
Since \( h_1, h_2, v_1, v_2, L \) are all constants, it follows that
\[
\frac{dt}{dx} = \frac{x}{v_1 \sqrt{x^2 + h_1^2}} - \frac{L - x}{v_2 \sqrt{(L - x)^2 + h_2^2}}
\]
Setting \( \frac{dt}{dx} = 0 \), we obtain

\[
x = \frac{L - x}{v_1 \sqrt{x^2 + h_1^2}} = \frac{L - x}{v_2 \sqrt{(L - x)^2 + h_2^2}}
\]

or

\[
x = \frac{L - x}{v_1 L_1} = \frac{L - x}{v_2 L_2}
\]

Finally, since

\[
\sin(\theta_1) = \frac{x}{L_1} \quad \sin(\theta_2) = \frac{L - x}{L_2}
\]

it follows that

\[
\sin \theta_1 = \frac{\sin \theta_2}{v_1} = \frac{\sin \theta_2}{v_2}
\]

which proves Snell’s law.

(You might find this solution slightly unsatisfying. After all, we didn’t find the value of \( x \) for which \( t(x) \) is minimized. But this is unnecessary: \( t(x) \) is a continuous function of \( x \) on the interval \( [0, L] \), so it has a maximum and a minimum. The endpoints can safely be ruled out as possible minima for physical reasons, because \( v_1 \) and \( v_2 \) are not too different from each other. Thus the minimum must come at a critical point, and what we showed is that Snell’s law must hold at any critical point).

8. Integration and the Fundamental Theorem of Calculus

1. Evaluate \( \int_{-1}^1 |x|^3 \, dx \).

Solution: \( |x| = x \) if \( x \geq 0 \), and \( |x| = -x \) if \( x \leq 0 \). Therefore

\[
\int_{-1}^1 |x|^3 \, dx = \int_{-1}^0 |x|^3 \, dx + \int_{0}^1 |x|^3 \, dx = \int_{-1}^0 -x^3 \, dx + \int_{0}^1 x^3 \, dx
\]

\[
= \left[ -\frac{x^4}{4} \right]_{-1}^0 + \left[ \frac{x^4}{4} \right]_0^1 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}
\]

2. Evaluate \( \frac{d}{dx} \int_{x+1}^{x^3} \sin(t^2) \, dt \).

Solution: Let \( F(x) \) be an antiderivative of \( \sin(x^2) \). Then

\[
\int_{x+1}^{x^3} \sin(t^2) \, dt = F(x^3) - F(x + 1)
\]

so by the Fundamental Theorem of Calculus and the chain rule,

\[
\frac{d}{dx} \int_{x+1}^{x^3} \sin(t^2) \, dt = F'(x^3) \cdot 3x^2 - F'(x + 1) = 3x^2 \sin(x^6) - \sin((x + 1)^2)
\]
9. Substitution

1. Evaluate \( \int_{0}^{1} x^3(x^4 + 1)^2 \, dx \).

   **Solution:** Set \( u(x) = x^4 + 1 \). Then \( \frac{du}{dx} = 4x^3 \), so \( x^3 \, dx = \frac{1}{4} \, du \). The new limits of integration are \( u(0) = 1 \) and \( u(1) = 2 \). Therefore
   \[
   \int_{0}^{1} x^3(x^4 + 1)^2 \, dx = \frac{1}{4} \int_{1}^{2} u^2 \, du = \frac{u^3}{12}\bigg|_{1}^{2} = \frac{7}{12}
   \]

2. Evaluate the indefinite integral \( \int x^3(x^2 + 1)^{10} \, dx \).

   **Solution:** In principle it’s possible to expand out \((x^2 + 1)^{10}\), but I wouldn’t recommend trying this. Instead use a \( u \)-substitution.

   Take \( u = x^2 + 1 \). We can split up the \( x^3 \) into \( x \) and \( x^2 = u - 1 \), and since \( du = 2x \, dx \) the integral becomes
   \[
   \frac{1}{2} \int (u - 1)u^{10} \, du = \frac{1}{2} \int u^{11} - u^{10} = 2 \left[ \frac{u^{12}}{12} - \frac{u^{11}}{11} \right] + C
   \]
   \[
   = \frac{(x^2 + 1)^{12}}{24} - \frac{(x^2 + 1)^{11}}{22} + C
   \]

   In contrast to definite integrals, with indefinite integrals it’s important to substitute in \( u = x^2 + 1 \).

10. Volume of solids

1. Find the volume of the solid whose base is given by the region under the graph of \( y = 1 - x^2 \) (and above the \( x \)-axis) and whose cross sections perpendicular to the \( y \)-axis are squares.

   **Solution:** Since the cross sections are perpendicular to the \( y \)-axis, the integral will be with respect to \( y \). The volume is given by
   \[
   V = \int_{0}^{1} A(y) \, dy
   \]
   where \( A(y) \) is the cross-sectional area at height \( y \). The cross sections are squares, so \( A(y) = b(y)^2 \), where \( b(y) \) is the length of the base at height \( y \).

   Since the graph of the parabola is symmetric about the \( y \)-axis, the length of the base is equal to twice the positive \( x \)-coordinate at height \( y \). Solving \( y = 1 - x^2 \) for \( x \), we get
   \[
   b(y) = 2\sqrt{1 - y}
   \]
   so that
   \[
   V = \int_{0}^{1} \left[2\sqrt{1 - y}\right]^2 \, dy = 4 \int_{0}^{1} (1 - y) \, dy = \left[4y - 2y^2\right]_{0}^{1} = 2
   \]
2. Consider the region $R$ bounded by the graphs of $f(x) = 2x - x^2$ and $g(x) = x^2$.

a. Find the area of this region.

Solution: To find the area, we must first determine the points of intersection of the two functions. Setting $f(x) = g(x)$ gives $2x - x^2 = x^2$, or $2x^2 - 2x = 0$, which has roots $x = 0$ and $x = 1$.

Checking $f$ and $g$ at the point $x = \frac{1}{2}$ shows that $f(x) \geq g(x)$ on $[0, 1]$, hence the area is given by

$$\int_0^1 f(x) - g(x) \, dx = \int_0^1 2x - 2x^2 \, dx = \left[ x^2 - \frac{2x^3}{3} \right]_0^1 = \frac{1}{3}$$

b. Find the volume of the solid obtained by rotating $R$ about the $x$-axis.

Solution: Since we’re rotating about the $x$-axis, it’s easier to use the washer method. The volume is

$$\pi \int_0^1 f(x)^2 - g(x)^2 \, dx = \pi \int_0^1 4x^2 - 4x^3 + x^4 - x^4 \, dx = \pi \int_0^1 4x^2 - 4x^3 \, dx$$

$$= \pi \left[ \frac{4}{3} x^3 - x^4 \right]_0^1 = \frac{\pi}{3}$$

c. Find the volume of the solid obtained by revolving the region $R$ about the $y$-axis.

Solution: Since we’re rotating about the $y$-axis, the shell method is probably the best bet. The volume is given by

$$2\pi \int_0^1 x(f(x) - g(x)) \, dx = 2\pi \int_0^1 x(2x - 2x^2) \, dx = \pi \int_0^1 4x^2 - 4x^3 \, dx$$

$$= \pi \left[ \frac{4x^3}{3} - x^4 \right]_0^1 = \frac{\pi}{3}$$