1. Counting

1. Imagine that a sports betting pool is run in the following way: there are 20 teams, 12 weeks, and each week you pick a team to win. However, you can’t pick the same team twice. How many different ways can you make your picks?

*Solution:* Think of your picks as a sequence of length 12. There are 20 choices for the first entry, 19 choices for the second entry, and so on. So the number of ways you can pick the teams is
\[ P(20, 12) = \frac{20!}{(20-12)!} = 20 \cdot 19 \cdot 18 \cdot \ldots \cdot 9 \]

2. How many words can be formed from the word “beginning”?

*Solution:* If all the letters are distinct, then the number of permutations of a 9-letter word is simply 9!. But not all the letters are distinct, so we must divide by the number of ways to permute the repeated letters. So the number of words is
\[ \frac{9!}{2!2!3!} \]

3. You plan to go on a 20-mile walk, which will include 5 northward miles, 5 southward miles, 5 westward miles, and 5 eastward miles. How many such walks are possible?

*Solution:* Think of a walk as a word of length 20, each letter being one of N, S, E, W. There must be 5 of each of the letters, so the number of walks is
\[ \frac{20!}{(5!)^4} \]

2. Probability with equally likely outcomes

1. Suppose that you roll 3 fair dice. What is the probability that exactly two of the numbers are the same?

*Solution:* View the result of rolling the dice as a three element sequence \((r_1, r_2, r_3)\) with \(1 \leq r_i \leq 6\). So the sample space has \(6^3 = 216\) elements.
Let $E$ be the event that exactly two of the dice match. In terms of sequences, $E$ is the set of sequences $(r_1, r_2, r_3)$ for which exactly two of the $r_i$ are equal.

To determine the size of $E$, we must first choose the two entries which are the same. There are $\binom{3}{2} = 3$ ways to do so. Then there are 6 choices for the value of the equal entries, and 5 choices for the remaining entry. So $|E| = 3 \cdot 6 \cdot 5 = 90$. Thus the probability is

$$\frac{|E|}{\Omega} = \frac{90}{216} = \frac{15}{36} = \frac{5}{12}.$$

2. There are 100 students in a class. 50 of them are biology majors, 40 are chemistry majors, and 30 are psychology majors. 10 of the students are double majoring in biology and chemistry, 10 are double majoring in biology and psychology, and 10 are double majoring in chemistry and psychology. Finally, 5 students are majoring in all three subjects. What is the probability that a randomly selected student is not majoring in any of biology, chemistry, or psychology?

Solution: Let $B$ be the set of biology majors, $C$ the set of chemistry majors, and $P$ the set of psychology majors. We want to determine the size of the set $|B \cup C \cup P|$.

Simply adding up the sizes of the sets gives $|B| + |C| + |P| = 50 + 40 + 30 = 120$. But this over-counts all the students majoring in more than one subject. So subtract the sizes of the pairwise intersections:

$$|B| + |C| + |P| - |B \cap C| - |B \cap P| - |C \cap P| = 120 - 30 = 90$$

But now we haven’t counted the triple majors at all: they’ve been counted three times and then subtracted three times. So add in the triple intersection to get

$$|B \cup C \cup P| = 120 - 30 + 5 = 95$$

So the probability that a randomly selected student will not be in $B \cup C \cup P$ is

$$\frac{100 - 95}{100} = \frac{5}{100} = \frac{1}{20}.$$

3. Conditional Probability and Independence

1. Imagine that you roll two fair dice. Find the probability that the first die is a 2, given that the maximum of the two dice is a 4.

Solution: Let $A$ be the event that the first die is a 2, and $B$ the event that the maximum of the two dice is a 4. We want to compute $P(A|B)$.

By definition, $P(A|B) = \frac{P(A \cap B)}{P(B)}$. $A \cap B$ consists of the single event $(2, 4)$, so $P(A \cap B) = \frac{1}{36}$. And $B$ is the event

$$\{(1, 4), (2, 4), (3, 4), (4, 4), (4, 3), (4, 2), (4, 1)\}$$

so $P(B) = \frac{7}{36}$. Therefore $P(A|B) = \frac{1}{7}$. 


2. (Simpson’s paradox) Consider two baseball players. Player \( A \) has a higher batting average than player \( B \) in each of years 1 and 2. If the two years are combined, does it follow that player \( A \) has a higher batting average overall?

Solution: No. Suppose that player \( A \) has 100 at bats in year 1, and gets a hit in 20 of them. And player \( B \) has 10 at bats in year 1, and gets a hit in 1 of them. In year 2, suppose that player \( A \) has 10 at bats, and gets a hit in 5 of them. Player \( B \) has 100 at bats, and gets a hit in 34 of them.

Then player \( A \) has a better batting average than player \( B \) in both years. However, when the years are combined, player \( A \) has 25 hits in 110 at bats, and player \( B \) has 35 hits in 110 at bats.

As an aside, this happened to Derek Jeter and David Justice in 1995 and 1996. The thing to remember is that this paradox shows that \( P(A|B ∪ C) \neq P(A|B) + P(A|C) \).

3. i. Find an example of a sample space \( \Omega \) and two events \( A \) and \( B \) which are disjoint but not independent.

Solution: Let \( \Omega = \{H, T\} \) be the sample space corresponding to flipping a fair coin once. Let \( A = \{H\} \) and \( B = \{T\} \). Then \( A \cap B = \emptyset \), so \( A \) and \( B \) are disjoint.

But \( P(A ∩ B) = 0 \neq \frac{1}{4} = P(A)P(B) \), so \( A \) and \( B \) are not independent.

ii. Find an example of a sample space \( \Omega \) and two events \( A, B \) which are independent but not disjoint.

Solution: Let \( \Omega \) be the sample space corresponding to flipping a fair coin twice. Let \( A \) be the event that the first flip is heads, and \( B \) the event that the second flip is heads. Then \( A \) and \( B \) are not disjoint since they both contain \( (H, H) \).

But \( A \) and \( B \) are independent: \( P(A) = \frac{1}{2} = P(B) \), and \( P(A ∩ B) = \frac{1}{4} \).

So the conclusion to draw from this problem is that independence and disjointness are very different notions.

4. Suppose that you flip two fair coins. Let \( A \) be the event that the coins match, and \( B \) the event that the first coin is heads. Are \( A \) and \( B \) independent?

Solution: \( P(A) = \frac{1}{2} \), \( P(B) = \frac{1}{2} \), and \( P(A ∩ B) = \frac{1}{4} \), so yes.

5. Suppose that you have two weighted coins, which are heads with probability \( p \). You play the following game with a friend: before your friend flips the coins, you guess whether they will match (i.e. both be heads or tails) or not. Is it better to guess that the coins will match or that they won’t?

Solution: Using a tree diagram, we see that the two coins are both heads with probability \( p^2 \), and are both tails with probability \( (1-p)^2 \), so they match with probability \( p^2 + (1-p)^2 \).
Let $P$ be the probability that you win by guessing that the coins match, viewed as a function of $p$ for $0 \leq p \leq 1$. Thus

$$P(p) = p^2 + (1 - p)^2 = p^2 + 1 - 2p + p^2 = 2p^2 - 2p + 1$$

and

$$P'(p) = 4p - 2$$

which has a critical point at $p = \frac{1}{2}$. And $P'' = 4$ is positive, so $p = \frac{1}{2}$ is a minimum. Since

$$P\left(\frac{1}{2}\right) = \frac{2}{4} - 1 + 1 = \frac{1}{2}$$

it follows that $P(p) \geq \frac{1}{2}$ for all $p$. So you are better off guessing that the coins will match.

4. Random Variables

1. Consider the functions $F_1(x), F_2(X), F_3(X)$ defined on the real numbers as follows:

$$F_1(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & 0 \leq x < 1 \\ \frac{3}{4} & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

$$F_2(x) = \begin{cases} \frac{1}{2} & x < -10 \\ \frac{3}{4} & -10 \leq x < 0 \\ 1 & x \geq 0 \end{cases}$$

$$F_3(x) = \begin{cases} 0 & x < 2 \\ 1 & x \geq 2 \end{cases}$$

Which (if any) of these functions is the cumulative distribution function of a discrete random variable?

**Solution:** The cumulative distribution function of a discrete random variable $X$ is the function $F(x) = P(X \leq x)$. Such a function must be non-decreasing, which rules out $F_1(x)$. We must also have

$$\lim_{x \to -\infty} F(x) = 0 \quad \lim_{x \to \infty} F(x) = 1$$

which rules out $F_2(x)$. And $F_3(x)$ is the cumulative distribution function of a random variable $X$ with probability mass function $P(X = x) = 0$ for $x \neq 2$ and $P(X = 2) = 1$.

2. Consider the cumulative distribution function defined as follows:

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & 0 \leq x < 1 \\ \frac{3}{4} & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

Find the corresponding probability mass function.

**Solution:** To determine the probability mass function, locate the points of discontinuity of the distribution function $F$ and subtract the left-hand limit from the right-hand limit.
Writing $F(x+)$ for the right-hand limit at $x$ and $F(x−)$ for the left-hand limit, we obtain

$P(X = 0) = F(0+) − F(0−) = \frac{1}{2}$, $P(X = 1) = F(1+) − F(1−) = \frac{3}{4} − \frac{1}{2} = \frac{1}{4}$, and

$P(X = 2) = F(2+) − F(2−) = 1 − \frac{3}{4} = \frac{1}{4}$. Note that

$$\sum_{x} P(X = x) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$$

Intuitively, this means that the sum of the “jumps” of $F$ is 1.