MATH 32A: MIDTERM 2 REVIEW

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1. Curvature

1. Consider the curve \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \), where

\[
x(t) = \int_0^t \cos(u) \sin(u) \, du \quad y(t) = \int_0^t \sin^2 u \, du \quad z(t) = \int_0^t \cos u \, du
\]

Find the curvature \( \kappa(t) \).

Solution: The formula for curvature is

\[
\kappa(t) = \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3}
\]

Using the fundamental theorem of calculus, we get

\[
\mathbf{r}'(t) = \langle \cos t \sin t, \sin^2 t, \cos t \rangle
\]

which has length

\[
\left( \cos^2 t \sin^2 t + \sin^2 t \sin^2 t + \cos^2 t \right)^\frac{1}{2} = \left( \sin^2 t + \cos^2 t \right)^\frac{1}{2} = 1
\]

Therefore the denominator of \( \kappa(t) \) is always equal to 1.

Moreover, since \( \mathbf{r}'(t) \) has constant length, it follows that

\[
0 = \frac{d}{dt} ||\mathbf{r}'(t)||^2 = 2\mathbf{r}'(t) \cdot \mathbf{r}''(t)
\]

by the product rule. Therefore the angle between \( \mathbf{r}'(t) \) and \( \mathbf{r}''(t) \) is always \( \frac{\pi}{2} \), so

\[
\kappa(t) = ||\mathbf{r}'(t) \times \mathbf{r}''(t)|| = ||\mathbf{r}'(t)|| \cdot ||\mathbf{r}''(t)|| \sin \frac{\pi}{2} = ||\mathbf{r}''(t)||
\]

Finally,

\[
\mathbf{r}''(t) = \langle \cos^2 t - \sin^2 t, 2 \sin t \cos t, - \sin t \rangle = \langle \cos(2t), \sin(2t), - \sin t \rangle
\]

so

\[
||\mathbf{r}''(t)|| = \left[ \cos^2(2t) + \sin^2(2t) + \sin^2 t \right]^\frac{1}{2} = \sqrt{1 + \sin^2 t}
\]

Therefore \( \kappa(t) = \sqrt{1 + \sin^2 t} \).
2. Find the osculating circle to the curve \( y = \frac{x}{1 + x^2} \) at \( x = 1 \).

**Solution:** Begin by finding the curvature \( \kappa(x) \). Because this curve is the graph of a function in the plane, we can use the formula

\[
\kappa(x) = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}
\]

In this case,

\[
f'(x) = \frac{(1 + x^2) - x(2x)}{(1 + x^2)^2} = \frac{1 - x^2}{(1 + x^2)^2}
\]

and

\[
f''(x) = \frac{-2x(1 + x^2)^2 - (1 - x^2) \cdot 2(1 + x^2) \cdot 2x}{(1 + x^2)^4} = \frac{-2x(1 + x^2) - 4x(1 - x^2)}{(1 + x^2)^3}
\]

\[
= \frac{2x^3 - 6x}{(1 + x^2)^3}
\]

Therefore \( f'(1) = 0 \) and \( f''(1) = -\frac{1}{2} \), so the curvature at \( x = 1 \) is

\[
\kappa(1) = \frac{|f''(1)|}{(1 + f'(1)^2)^{3/2}} = \frac{1}{2}
\]

Thus the radius of the osculating circle is \( \frac{1}{\kappa(1)} = 2 \).

Next, we need to find the normal vector \( \mathbf{N} \) at the point \((1, \frac{1}{2})\). To do so, note that the tangent vector is \( \langle 1, f'(1) \rangle = \langle 1, 0 \rangle \), and that the normal vector is perpendicular to the tangent vector. Therefore either \( \mathbf{N} = \langle 0, 1 \rangle \) or \( \mathbf{N} = \langle 0, -1 \rangle \).

From sketching the curve \( y = f(x) \) and using the fact that \( \mathbf{N} \) is supposed to point “inward”, we see that \( \mathbf{N} = \langle 0, -1 \rangle \). The center of the osculating circle is therefore

\[
\langle 1, \frac{1}{2} \rangle + 2 \langle 0, -1 \rangle = \langle 1, -\frac{3}{2} \rangle
\]

Therefore the osculating circle has center \((1, -\frac{3}{2})\) and radius 2, so has parametrization

\[
\mathbf{c}(t) = \left< 1 + 2 \cos t, -\frac{3}{2} + 2 \sin t \right>
\]

2. **Motion in three-space**

1. Joe and Zane take a break from grading 32A exams by going to a playground. Joe pushes Zane on a swing so that his position is described by the vector-valued function \( \mathbf{c}(t) = \langle -\cos t, 2 - \sin t \rangle \). At time \( t = \frac{\pi}{2} \), Zane decides to jump off the swing. What is the \( x \)-coordinate of his landing point?
Solution: At the instant Zane jumps off the swing, his position is \( c\left(\frac{\pi}{2}\right) = \langle 0, 1 \rangle \) and his velocity is \( c'(\pi) = \langle 1, 0 \rangle \). The easiest way to solve this problem is to start time over at zero again when Zane jumps, and to think of these two vectors as the initial position and initial velocity.

Then \( a(t) = \langle 0, -g \rangle \), so

\[
\mathbf{v}(t) = \mathbf{v}(0) + \int_0^t a(u) \, du = \langle 1, 0 \rangle + \langle 0, -gt \rangle = \langle 1, -gt \rangle
\]

and

\[
\mathbf{r}(t) = \mathbf{r}(0) + \int_0^t \mathbf{v}(u) \, du = \langle 0, 1 \rangle + \left\langle t, -\frac{gt^2}{2} \right\rangle = \left\langle t, 1 - \frac{gt^2}{2} \right\rangle
\]

Zane reaches the ground when

\[
1 - \frac{gt^2}{2} = 0
\]

or

\[
t = \sqrt{\frac{2}{g}}
\]

so the \( x \)-coordinate of his landing point is \( \sqrt{\frac{2}{g}} \).

3. LEVEL CURVES AND GRAPHS

1. Match the functions to the corresponding level curves (see the end for the level curves):
   a. \( f(x, y) = x \cos y + y \cos x \)   b. \( f(x, y) = \sin(x - y) \)   c. \( f(x, y) = x \sin(y) \)   d. \( f(x, y) = xy \).

   Solution: The curve \( xy = c \) is a hyperbola if \( c \neq 0 \), and consists of the coordinate axes if \( x = 0 \). So (d) corresponds to Figure 1.

   If \( \sin(x - y) = c \) with \(-1 \leq c \leq 1\), then \( x - y = \sin^{-1}(c) + 2n\pi \) for some integer \( n \) (or \( +n\pi \) if \( c = 0 \)). These are lines, so (b) corresponds to Figure 2.

   The last two are a bit trickier. The key is to note that \( x \sin y = 0 \) when \( x = 0 \) or when \( y = 0, \pm\pi, \pm 2\pi, \ldots \). So (c) corresponds to Figure 4.

   Therefore (a) corresponds to Figure 3 by process of elimination. We could also see this by noting that the curves in Figure 3 are symmetric about the line \( y = x \), which fits with (a) because \( f(x, y) = f(y, x) \).

2. Match the functions to the corresponding graphs (see the end for the graphs):
   a. \( f(x, y) = (x^2 + y^2) e^{-x^2 + y^2} \)   b. \( f(x, y) = x \sin y \)   c. \( f(x, y) = \tan(x - y) \)   d. \( f(x, y) = y \sin x \).
Solution: The function in (a) is radially symmetric, so corresponds to Figure 8. Because tangent is periodic, we see that (c) corresponds to Figure 7.

The last two are more difficult because the functions are so similar. But note that the function whose graph is Figure 5 increases in magnitude as $y$ does (for $x$ fixed), and oscillates in $x$ for $y$ fixed. Therefore Figure 5 corresponds to (d), and Figure 6 corresponds to (b).

4. Quadric surfaces

1. A surface in $\mathbb{R}^3$ is bounded if there is a positive number $M$ such that $|x|, |y|, |z| \leq M$ for all points $P = (x, y, z)$ on the surface. Which of the following quadric surfaces are bounded?

a. $x^2 + y^2 + z^2 = 100.$

Solution: This surface is bounded because $x, y, z$ can be at most 10 in absolute value.

b. $z = 4x^2 - y^2.$

Solution: This surface is unbounded. For instance, if I set $y = 0$ then I can make $x$ and $z$ as large as I want, since the parabola $z = 4x^2$ in the $xz$ plane is unbounded.

c. $\frac{x^2}{2} + \frac{y^2}{3} = 1.$

Solution: While the $x$ and $y$ components of this surface are bounded ($|x| \leq \sqrt{2}$ and $|y| \leq \sqrt{3}$), there are no restrictions on $z$. So this surface is unbounded.

5. Limits and continuity

1. Compute $\lim_{(x,y)\to(0,0)} \frac{x \sin x}{x^2 + y^2}.$

Solution: If we approach the origin along the line $x = 0$, the result is $\lim_{y \to 0} \frac{0}{y^2} = 0.$ On the other hand, if we approach the origin along the line $y = 0$, the limit is

$$\lim_{x \to 0} \frac{x \sin x}{x^2} = \lim_{x \to 0} \frac{\sin x}{x} = 1$$

These limits are different, so the whole limit does not exist.

2. Compute $\lim_{(x,y)\to(0,1)} \frac{x \ln(x^2)}{y}.$

Solution: The key to solving this problem is using limit laws. $\lim_{y \to 1} \frac{1}{y} = 1$, while using L’Hôpital’s rule shows that

$$\lim_{x \to 0} x \ln(x^2) = \lim_{x \to 0} \frac{\ln(x^2)}{\frac{1}{x}} = \lim_{x \to 0} \frac{2}{x} = -2 \lim_{x \to 0} x = 0$$
Therefore by the limit law for a product,
\[
\lim_{(x,y) \to (0,1)} \frac{x \ln(x^2)}{y} = 1 \cdot 0 = 0
\]

3. Compute \( \lim_{(x,y) \to (0,0)} \frac{x^4y^4}{(x^2+y^2)^7} \).

**Solution:** The limit along the \( x \) or \( y \) axes is zero, while the limit along the parabola \( x = y^2 \) is
\[
\lim_{y \to 0} \frac{y^8 y^4}{(y^4 + y^4)^3} = \lim_{y \to 0} \frac{y^{12}}{8y^{12}} = \frac{1}{8}
\]
Therefore the limit does not exist.

4. Compute \( \lim_{(x,y) \to (-1,1)} \frac{\sin(xy - x + y - 1)}{xy - x + y - 1} \).

**Solution:** Both the numerator and denominator tend to zero as \( (x,y) \to (-1,1) \), so we can’t just plug the point in. The key to the problem is writing
\[
xy - x + y - 1 = (x + 1)(y - 1)
\]
Therefore if \( u = x + 1 \) and \( v = y - 1 \), then
\[
\lim_{(x,y) \to (-1,1)} \frac{\sin(xy - x + y - 1)}{xy - x + y - 1} = \lim_{(u,v) \to (0,0)} \frac{\sin(uv)}{uv} = \lim_{t \to 0} \frac{\sin t}{t} = 1
\]

6. **Partial derivatives**

1. For \( f(x, y, z, w) = x^2yzw + y^2xw\sin z + y^2ze^w\cos x + \frac{zxw}{1+y^2} \), find \( f_{xyxzwyz} \).

**Solution:** All of the functions (and their derivatives) involved in the definition of \( f \) are continuous, so we can take the derivatives in any order we please. And since differentiation is linear, we can differentiate each term in the sum separately, using the order which is most convenient.

For the first term \( x^2yzw \), use the two \( y \) derivatives (or two \( z \) derivatives) to get zero. For the second term, use the two \( x \) derivatives to get zero. For the third term, use the two \( z \) derivatives to get zero, and for the last term, use the two \( x \) derivatives to get zero.

So adding the four terms together, we get \( f_{xyxzwyz} = 0 \) (which is almost always the answer to these types of problems).

2. Two functions \( u(x, y) \) and \( v(x, y) \) are said to be harmonic conjugates if \( \Delta u = 0, \Delta v = 0 \), \( u_x = v_y \), and \( u_y = -v_x \). (Recall that \( \Delta f = f_{xx} + f_{yy} \).)

a. Let \( u(x, y) = e^x \sin y \). Verify that \( \Delta u = 0 \).
Solution: $u_x = e^x \sin y$ and $u_{xx} = e^x \sin y$, while $u_y = e^x \cos y$ and $u_{yy} = -e^x \sin y$. Therefore
\[
\Delta u = u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0
\]
b. Find a harmonic conjugate $v(x, y)$ for $u$.

Solution: Start with the equation $v_y = u_x = e^x \sin y$. Taking an antiderivative with respect to $y$, we get
\[
v(x, y) = -e^x \cos y + w(x)
\]
where $w(x)$ is a function only of $x$. Then using the equation
\[-e^x \cos y + w'(x) = v_x = -u_y = -e^x \cos y
\]
we see that $w'(x) = 0$, so $w$ is a constant. We can set $w = 0$ since we only need one harmonic conjugate.

Finally, we should verify that $v$ is actually harmonic. But this works the same way as part (a): $v_{xx} = -e^x \cos y$ and $v_{yy} = e^x \cos y$, so $\Delta v = 0$.

3. A function $u(t, x)$ is said to satisfy the (one-dimensional) wave equation with initial conditions $f(x)$ and $g(x)$ if $u_{tt} = u_{xx}$ and $u(0, x) = f(x), u_t(0, x) = g(x)$.

a. Verify that $u(t, x) = \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} g(s) \, ds$ satisfies the wave equation with initial conditions $f(x)$ and $g(x)$.

Solution: First, note that
\[
u(0, x) = \frac{f(x) + f(x)}{2} + \frac{1}{2} \int_{x}^{x} g(s) \, ds = f(x)
\]
and that
\[
u_t(t, x) = \frac{f'(x+t) - f'(x-t)}{2} + \frac{g(x+t) + g(x-t)}{2}
\]
by the fundamental theorem of calculus, so that
\[
u_t(0, x) = g(x)
\]
We have already found $u_t(t, x)$, so differentiating with respect to $t$ again, we get
\[
u_{tt}(t, x) = \frac{f''(x+t) + f''(x-t)}{2} + \frac{g'(x+t) - g'(x-t)}{2}
\]
Similarly,
\[
u_x(t, x) = \frac{f'(x+t) + f'(x-t)}{2} + \frac{g(x+t) - g(x-t)}{2}
\]
and
\[
u_{xx}(t, x) = \frac{f''(x+t) + f''(x-t)}{2} + \frac{g'(x+t) - g'(x-t)}{2}
\]
Therefore $u_{tt} = u_{xx}$, so $u$ is a solution to the wave equation.
b. Determine \( u(t, x) \) explicitly for \( f(x) = e^{-x} = g(x) \).

**Solution:** Using part (a), we have

\[
u(x, t) = \frac{e^{-x-t} + e^{-x+t}}{2} + \frac{1}{2} \int_{x-t}^{x+t} e^{-s} \, ds = \frac{e^{-x-t} + e^{-x+t}}{2} - \frac{1}{2} e^{-x} \bigg|_{x-t}^{x+t} = e^{-x} + e^{-x} - e^{-x+t} - e^{-x-t} = e^{-x}.
\]

C. The energy of the wave is defined to be

\[E(t) = \int_{-\infty}^{\infty} [u_t(t, x)^2 + u_x(t, x)^2] \, dx.\]

For the function \( u(t, x) \) from part (b), verify that \( E'(t) = 0 \) (i.e., energy is conserved).

**Solution:** \( u_t(t, x) = e^{t-x} \) and \( u_x(t, x) = -e^{t-x} \). Therefore

\[E(t) = 2 \int_{-\infty}^{\infty} e^{2(t-x)} \, dx = 2 \int_{-\infty}^{\infty} e^{-u} \, du\]

by making the substitution \( u = 2(x - t) \), so \( du = 2dx \). The integral is now independent of \( t \), so \( E'(t) = 0 \).

7. **Tangent planes and linear approximation**

1. Find the points \((a, b, f(a, b))\) on the graph of \( f(x, y) = x^2 + 2x + y^2 - 2y + 1\) such that the tangent plane at the point is parallel to the \( xy \)-plane.

**Solution:** The tangent plane has equation

\[z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)\]

with normal vector \((f_x(a, b), f_y(a, b), -1)\). In order for the plane to be parallel to the \( xy \)-plane, the normal vector must be parallel to the \( z \)-axis, so \( f_x(a, b) = 0 = f_y(a, b) \).

And \( f_x = 2x + 2, f_y = 2y - 2 \), so \( f_x(a, b) = 0 = f_y(a, b) \) when \( a = -1, b = 1 \). Therefore the point on the graph is \((-1, 1, -1)\).

2. The surface area of a right circular cone of radius \( r \) and height \( h \) is \( S(r, h) = \pi r(r + \sqrt{r^2 + h^2}) \). Use linear approximation to estimate \( S(2, 4) \).

**Solution:** The formula for linear approximation is:

\[S(r, h) \approx S(r_0, h_0) + S_r(r_0, h_0)(r - r_0) + S_h(r_0, h_0)(h - h_0)\]

The two partial derivatives are

\[S_r(r_0, h_0) = \pi(r_0 + \sqrt{r_0^2 + h_0^2}) + \pi r_0 \left( 1 + \frac{r_0}{\sqrt{r_0^2 + h_0^2}} \right)\]

and

\[S_h(r_0, h_0) = \frac{\pi r_0 h_0}{\sqrt{r_0^2 + h_0^2}}\]
To finish the problem, we must choose a point \((r_0, h_0)\) which is reasonably close to \((2, 4)\), and such that \(S(r_0, h_0), S_r(r_0, h_0), \text{ and } S_h(r_0, h_0)\) can be easily evaluated. The most natural point is probably \((3, 4)\), because then \(\sqrt{3^2 + 4^2} = 5\).

With \((r_0, h_0) = (3, 4)\), we have

\[
S(3, 4) = 24\pi \quad S_r(3, 4) = 8\pi + \frac{24\pi}{5} = \frac{64\pi}{5} \quad S_h(3, 4) = \frac{12\pi}{5}
\]

so

\[
S(2, 4) \approx 24\pi + \frac{64\pi}{5}(2 - 3) + \frac{12\pi}{5}(4 - 4) = \frac{120\pi}{5} - \frac{64\pi}{5} = \frac{56\pi}{5}
\]

Therefore \(\frac{S(2, 4)}{\pi} \approx \frac{56}{5}\).

8. Figures
Figure 8