1. Vectors

1. Let $\mathbf{v} = \langle 3, 2, 3 \rangle$.

   a. Find $e_{\mathbf{v}}$.

   Solution: $||\mathbf{v}|| = \sqrt{9 + 4 + 9} = \sqrt{22}$, so

   
   
   $$e_{\mathbf{v}} = \frac{1}{||\mathbf{v}||} \mathbf{v} = \frac{1}{\sqrt{22}} \langle 3, 2, 3 \rangle$$

   b. Find the vectors parallel to $\mathbf{v}$ which lie on the sphere of radius two centered at the origin.

   Solution: The sphere of radius two centered at the origin is the set of vectors $\mathbf{w}$ such that $||\mathbf{w}|| = 2$. So the two vectors on this sphere which are parallel to $\mathbf{v}$ are

   $$\pm 2e_{\mathbf{v}} = \pm \frac{2}{\sqrt{22}} \langle 3, 2, 3 \rangle$$

2. Find the vector $\mathbf{v}$ such that $||\mathbf{v}|| = 3$ and $\mathbf{v}$ makes an angle $\frac{3\pi}{4}$ with the positive $x$-axis.

   Solution: For any vector $\mathbf{v}$ in the plane, we have

   $$\mathbf{v} = ||\mathbf{v}|| e_{\mathbf{v}} = ||\mathbf{v}|| \langle \cos \theta, \sin \theta \rangle$$

   where $\theta$ is the angle between $\mathbf{v}$ and the positive $x$-axis. So in this case

   $$\mathbf{v} = 3 \langle \cos \frac{3\pi}{4}, \sin \frac{3\pi}{4} \rangle = 3 \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle = \langle -\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2} \rangle$$

2. The dot product and cross product

1. Suppose that $\mathbf{v}, \mathbf{w}$ are vectors such that $||\mathbf{v} + \mathbf{w}|| = 1$, $||\mathbf{v} - \mathbf{w}|| = 2$, and $||\mathbf{v}|| = \sqrt{2}$. Find $||\mathbf{w}||$.

   Solution: Often with lengths it is easier to work with the square of the length. For instance,

   $$||\mathbf{v} + \mathbf{w}||^2 = (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} + 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w}$$
and similarly
\[ ||v - w||^2 = (v - w) \cdot (v - w) = v \cdot v - 2v \cdot w + w \cdot w \]

If we add these two equations together, the result is
\[ ||v + w||^2 + ||v - w||^2 = 2||v||^2 + 2||w||^2 \]

This is called the parallelogram identity, and is important in general. For our purposes, its usefulness lies in the fact that it allows us to solve for \( ||w|| \):

\[ 2||w||^2 = ||v + w||^2 + ||v - w||^2 - 2||v||^2 = 1 + 4 - 2(2) = 1 \]

so \( ||w|| = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \).

2. Consider an inclined plane with angle \( \theta = \frac{\pi}{6} \) above the horizontal and with a pulley attached to the top. A rope through the pulley connects a ten kilogram block on the inclined plane to a bucket of water. If there is no net force on the block, determine the mass of the bucket of water (ignore friction and the mass of the rope and pulley).

Solution: Let \( m \) be the mass of the bucket. Gravity exerts a force of magnitude \( mg \) on the bucket of water, and since the bucket is connected to the block, there is a force of magnitude \( mg \) pulling the block up the plane.

On the other hand, gravity exerts a force of magnitude \( 10g \) on the block. The component of this force parallel to the inclined plane has magnitude \( 10g \sin \frac{\pi}{6} = 5g \), pulling the block down the plane.

Since there is no net force on the block, we must have \( mg = 5g \), so \( m = 5 \) kilograms.

3. Find the area of the triangle with vertices \( P = (1, 1, 5), Q = (3, 4, 3), \) and \( R = (1, 5, 7) \).

Solution: The area of a triangle determined by two vectors \( \mathbf{v}, \mathbf{w} \) is half the area of the parallelogram defined by the vectors, so is equal to \( \frac{1}{2}||\mathbf{v} \times \mathbf{w}|| \). In this case, take \( \mathbf{v} = PQ = (2, 3, -2) \) and \( \mathbf{w} = PR = (0, 4, 2) \). Then
\[
\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -2 \\ 0 & 4 & 2 \end{vmatrix} = \mathbf{i}(6 + 8) - \mathbf{j}(4 - 0) + \mathbf{k}(8 - 0) = \langle 14, -4, 8 \rangle
\]

so the area is
\[
\frac{1}{2}||\mathbf{v} \times \mathbf{w}|| = \frac{1}{2} \sqrt{196 + 16 + 64} = \frac{1}{2} \sqrt{276} = \sqrt{69}
\]
3. Equations of lines and planes

1. Find the equation of the line \( \ell \) given by the intersection of the two planes \( x + y + z = 0 \) and \( x = 1 \).

Solution: The direction vector \( \mathbf{v} \) of \( \ell \) should be perpendicular to the normal vectors of both planes, so can be taken to be \( \mathbf{n}_1 \times \mathbf{n}_2 \), where \( \mathbf{n}_1 = \langle 1, 1, 1 \rangle \) and \( \mathbf{n}_2 = \langle 1, 0, 0 \rangle \). Therefore

\[
\mathbf{v} = \begin{vmatrix}
i & j & k \\
1 & 1 & 1 \\
1 & 0 & 0 \\
\end{vmatrix} = i(0) - j(-1) + k(-1) = \langle 0, 1, -1 \rangle
\]

To find a point on the line, it is enough to find a point lying in both planes. The \( x \)-coordinate must be one, and setting \( z = 0 \), we get \( y = -1 \) from the equation of the first plane. Therefore \( \ell \) has equation

\[
\langle 1, -1, 0 \rangle + t \langle 0, 1, -1 \rangle = \langle 1, t - 1, -t \rangle
\]

2. Find the equation of the plane containing the line \( x = y = z \) and the point \( P = (-2, 3, 1) \).

Solution: Three points determine the plane containing them, except in the rare case that they happen to be collinear. In this problem \( P \) is not on the line \( x = y = z \), so we’re free to pick two points on the line. For simplicity, take \( Q = (0, 0, 0) \) and \( R = (1, 1, 1) \) (of course any other two points on the line would work as well). The points \( P, Q, R \) then determine two vectors in the plane, namely \( \mathbf{v} = QP = \langle -2, 3, 1 \rangle \) and \( \mathbf{w} = QR = \langle 1, 1, 1 \rangle \).

The normal vector to the plane can then be taken to be

\[
\mathbf{n} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix}
i & j & k \\
-2 & 3 & 1 \\
1 & 1 & 1 \\
\end{vmatrix} = i(3 - 1) - j(-2 - 1) + k(-2 - 3) = \langle 2, 3, -5 \rangle
\]

For the point on the plane, it’s easiest to take \( Q \). Therefore the equation of the plane is

\[
2x + 3y - 5z = 0
\]

3. Find the equation of the plane consisting of all points which are equidistant from \( P = (0, 1, 0) \) and \( Q = (0, 0, 1) \).

Solution: If \( R = (x, y, z) \) is such a point, then the vectors \( \mathbf{v} = RP = \langle x, y - 1, z \rangle \) and \( \mathbf{w} = RQ = \langle x, y, z - 1 \rangle \) must have the same length. Therefore

\[
x^2 + (y - 1)^2 + z^2 = ||\mathbf{v}||^2 = ||\mathbf{w}||^2 = x^2 + y^2 + (z - 1)^2
\]

which implies that

\[
y^2 - 2y + 1 + z^2 = y^2 + z^2 - 2z + 1
\]
or $y = z$. Therefore the equation of the plane is $y - z = 0$.

4. Parametrizations and vector-valued functions

1. Find a parametrization of the intersection of the plane $x + y + z = 0$ with the surface $\frac{x^2}{4} + y^2 = 1$.
   
   Solution: Start with the second equation. The projection of this surface into the $xy$ plane is the ellipse $\frac{x^2}{4} + y^2 = 1$, which has parametrization
   
   $$x = 2 \cos t \quad y = \sin t \quad 0 \leq t \leq 2\pi$$
   
   The second equation tells us nothing about $z$, but from the first equation we get
   
   $$z = -x - y = -2 \cos t - \sin t$$

2. a. Find a parametrization of the curve $x^3 + y^3 = 3xy$ (Hint: set $y = tx$ and solve for $x, y$ in terms of $t$).
   
   Solution: Plugging $y = tx$ into the equation of the curve, we get
   
   $$x^3 + t^3x^3 = 3tx^2$$
   
   Therefore either $x = 0$ or
   
   $$(1 + t^3)x = 3t$$
   
   so $x = \frac{3t}{1 + t^3}$. Then $y = tx = \frac{3t^2}{1 + t^3}$.

   Where did the idea of setting $y = tx$ come from? If you were to graph $x^3 + y^3 = 3xy$, the result is the following curve:
Every line \( y = tx \) through the origin intersections the curve in another point, except when \( t = 0 \) or \( t = -1 \) (the line \( y = -x \) is an asymptote to the curve). So this is why setting \( y = tx \) allows us to parametrize the curve.

When you parametrize a curve, it can be helpful to work out how an imaginary object travels along the curve as a function of the “time” \( t \). This parametrization is a little different from the other ones we’ve done because \( x(t) \) and \( y(t) \) are discontinuous at \( t = -1 \).

The parametrization “starts” at the origin at time \( t = -\infty \), then travels along the curve in the fourth quadrant as \( t \) increases to \(-1\). Then the object travels toward the origin along the curve in the second quadrant as \( t \) increases from \(-1\) to \( 0 \), reaching the origin at time \( t = 0 \). Finally, the parametrization traces out the loop in the first quadrant as \( t \to \infty \).

b. Find the points on the curve where the tangent line is horizontal or vertical.

Solution: Using the result of part (a), we get

\[
\frac{dx}{dt} = \frac{3(1 + t^3) - 3t(3t^2)}{(1 + t^3)^2} = \frac{3 - 6t^3}{(1 + t^3)^2}
\]

and similarly

\[
\frac{dy}{dt} = \frac{6t(1 + t^3) - 3t^2(3t^2)}{(1 + t)^2} = \frac{6t - 3t^4}{(1 + t)^2}
\]

Therefore \( \frac{dx}{dt} = 0 \) when \( 6t^3 = 3 \), or \( t = \frac{1}{\sqrt[3]{2}} \). So the tangent line is vertical at this time. Similarly, \( \frac{dy}{dt} = 0 \) when \( 6t = 3t^4 \), or \( t = 0, \frac{\sqrt{2}}{2} \). Thus the tangent line is horizontal at these times.

3. The velocity of an object at time \( t \) is given by the vector-valued function \( \mathbf{v}(t) = \langle 2t, \cos t, \frac{2t}{1+t^2} \rangle \). If the object starts at the origin, find the position of the object at \( t = 2\pi \).

Solution: Let \( \mathbf{r}(t) \) be the position vector of the object. Then \( \mathbf{r}(t) \) is the antiderivative of \( \mathbf{v}(t) \), and since \( \mathbf{r}(0) = \mathbf{0} \), the fundamental theorem of calculus implies that

\[
\mathbf{r}(2\pi) = \mathbf{r}(2\pi) - \mathbf{r}(0) = \int_0^{2\pi} \mathbf{v}(t) \, dt = \int_0^{2\pi} \left\langle 2t, \cos t, \frac{2t}{1+t^2} \right\rangle \, dt
\]

\[
= \left[ t^2, \sin t, \ln(1+t^2) \right]_0^{2\pi} = \left( 4\pi^2, 0, \ln(1 + 4\pi^2) \right)
\]

4. A student is trying to throw a tennis ball into his friend’s dorm room. The student is standing 10 meters from the dorm, his friend’s window is 10 meters above the ground, and the student throws the ball at an angle of 60 degrees. What must the initial velocity of the ball be in order for the student to succeed in his goal?
Solution: Let \( v \) be the unknown velocity. The acceleration vector of the ball is \( \mathbf{a}(t) = (0, -g) \), so the velocity vector is
\[
\mathbf{v}(t) = \mathbf{v}(0) + \int_0^t \mathbf{a}(u) \, du = v \left\langle \cos \frac{\pi}{3}, \sin \frac{\pi}{3} \right\rangle + (0, -gt) = \left\langle \frac{v}{2}, \frac{v\sqrt{3}}{2} - gt \right\rangle
\]
Think of the student’s position as the origin. Then the ball’s position at time \( t \) is given by
\[
\mathbf{r}(t) = \int_0^t \mathbf{v}(u) \, du = \left\langle \frac{vt}{2}, \frac{v\sqrt{3}t - gt^2}{2} \right\rangle
\]
To go through the window, the ball must satisfy \( x(t) = 10 \) and \( y(t) = 10 \) for some time \( t \). Therefore we must have
\[
\frac{vt}{2} = 10 \quad \frac{v\sqrt{3}t - gt^2}{2} = 10
\]
The first equation tells us that \( t = \frac{20}{v} \), and substituting this into the second equation, we get
\[
10 = \frac{v\sqrt{3}t - gt^2}{2} = \frac{20\sqrt{3} - g\left(\frac{20}{v}\right)^2}{2}
\]
Therefore
\[
20\sqrt{3} - 20 = \frac{400g}{v^2}
\]
or
\[
v = \sqrt{\frac{400g}{20\sqrt{3} - 20}} \approx 16.36
\]
Naturally, you would not be expected to simplify the square root on the exam.

5. Arc length

1. Find the arc length of \( \mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle \) for \( 0 \leq t \leq 2\pi \).

Solution: \( \mathbf{r}'(t) = \langle 1 - \cos t, \sin t \rangle \), so
\[
||\mathbf{r}'(t)||^2 = (1 - \cos t)^2 + \sin^2 t = 1 - 2\cos t + \cos^2 t + \sin^2 t = 2 - 2\cos t
\]
In order to integrate \( ||\mathbf{r}'(t)|| \), we need the trig identity
\[
\sin^2 \left( \frac{t}{2} \right) = \frac{1 - \cos t}{2}
\]
which comes from the double angle formula for cosine. Therefore
\[
||\mathbf{r}'(t)||^2 = 2(1 - \cos t) = 4\sin^2 \left( \frac{t}{2} \right)
\]
so the arc length is
\[
\int_0^{2\pi} ||\mathbf{r}'(t)|| \, dt = \int_0^{2\pi} 2 \left| \sin \left( \frac{t}{2} \right) \right| \, dt = 4 \int_0^{\pi} |\sin u| \, du
\]
by setting \( u = \frac{t}{2} \). Since \( \sin u \) is non-negative for \( 0 \leq u \leq \pi \), the absolute values can be removed, and the result is
\[
4 \int_0^\pi \sin u \, du = -4 \cos u |_0^\pi = 8
\]

2. Find an arc length parametrization of the line \( y = mx + b \).

Solution: The line can be parametrized by \( \mathbf{r}(t) = \langle t, mt + b \rangle \) for \( t \) in \( (-\infty, \infty) \), so the arc length function is
\[
s(t) = \int_0^t ||\mathbf{r}'(u)|| \, du = \int_0^t \sqrt{1 + m^2} \, du = \sqrt{1 + m^2}t
\]
Thus \( t = \frac{s}{\sqrt{1 + m^2}} \), so the arc length parametrization is
\[
\mathbf{r}_1(s) = \left\langle \frac{s}{\sqrt{1 + m^2}}, \frac{ms}{\sqrt{1 + m^2}} + b \right\rangle
\]

3. Find an arc length parametrization of the helix \( \mathbf{r}(t) = \langle \cos t, \sin t, t \rangle \) for \( 0 \leq t \leq 2\pi \).

Solution: The derivative is
\[
\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle
\]
which has length
\[
||\mathbf{r}'(t)|| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}
\]
Therefore
\[
s(t) = \int_0^t \sqrt{2} \, du = \sqrt{2}t
\]
so the arc length parametrization is
\[
\mathbf{r}_1(s) = \left\langle \cos \frac{s}{\sqrt{2}}, \sin \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right\rangle
\]