1. Trigonometric Integrals

1. Evaluate $\int \sin^2 x \, dx$ and $\int \cos^2 x \, dx$.

Solution: Recall the double angle formula
$$\cos(2x) = \cos^2 x - \sin^2 x$$
If we combine this with the Pythagorean identity $\sin^2 x + \cos^2 x = 1$, we obtain the formulas
$$\cos(2x) = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$
which imply that
$$\cos^2 x = \frac{1 + \cos(2x)}{2} \quad \text{and} \quad \sin^2 x = \frac{1 - \cos(2x)}{2}$$
Using these formulas gives
$$\int \sin^2 x \, dx = \int \frac{1 - \cos(2x)}{2} \, dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C = \frac{x}{2} - \frac{\sin x \cos x}{2} + C$$
and similarly
$$\int \cos^2 x \, dx = \int \frac{1 + \cos(2x)}{2} \, dx = \frac{x}{2} + \frac{\sin(2x)}{4} + C = \frac{x}{2} + \frac{\sin x \cos x}{2} + C$$
The last step of both integrals used the formula
$$\sin(2x) = 2\sin x \cos x$$

2. Evaluate $\int \sin^2 x \cos^3 x \, dx$.

Solution: The key idea for evaluating trig integrals with odd powers is to use the identity $\sin^2 x + \cos^2 x = 1$ to try to set up a substitution. In this case, we get
$$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cos x \, dx = \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$$
Now the integral is written in a way that makes a $u$-substitution possible. Let $u = \sin x$, so $du = \cos x \, dx$. Then the integral becomes
$$\int u^2 (1 - u^2) \, du = \int u^2 - u^4 \, du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$
One of the challenges of trig integrals is that the answer can take several different forms, depending on how many trig identities one chooses to use. So as a sanity check, let’s take the derivative of our answer:

\[
\frac{d}{dx} \left[ \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C \right] = \sin^2 x \cos x - \sin^4 x \cos x = \sin^2 x \cos x - \sin^2 x \sin^2 x \cos x
\]

\[
= \sin^2 x \cos x - \sin^2 x (1 - \cos^2 x) \cos x = \sin^2 x \cos x - \sin^2 x \cos x + \sin^2 x \cos^3 x
\]

\[
= \sin^2 x \cos^3 x
\]

which is what we started with.

2. Trig Substitutions

1. Evaluate \( \int \frac{x^2}{\sqrt{9-x^2}} \, dx \).

Solution: Make the substitution \( x = 3 \sin \theta \), so that \( dx = 3 \cos \theta \, d\theta \). Then

\[
\int \frac{x^2}{\sqrt{9-x^2}} \, dx = \int \frac{9 \sin^2 \theta}{3 \cos \theta} \cdot 3 \cos \theta \, d\theta = 9 \int \sin^2 \theta \, d\theta = \frac{9}{2} \left[ \theta + \sin \theta \cos \theta \right] + C
\]

using the result from the previous section. Now \( \theta = \sin^{-1} \frac{x}{3} \), \( \sin \theta = \frac{x}{3} \), and \( \cos \theta = \frac{\sqrt{9-x^2}}{3} \), so

\[
\int \frac{x^2}{\sqrt{9-x^2}} \, dx = \frac{9}{2} \left[ \sin^{-1} \frac{x}{3} + \frac{x}{3} \frac{\sqrt{9-x^2}}{3} \right] + C = \frac{9}{2} \sin^{-1} \frac{x}{3} + \frac{1}{2} x \sqrt{9-x^2} + C
\]

2. Evaluate \( \int \frac{dx}{\sqrt{12x-x^2}} \).

Solution: At first glance, it doesn’t look like we can do a trig substitution. But observe that

\[
(x-6)^2 = x^2 - 12x + 36
\]

hence

\[
12x - x^2 = 36 - (x-6)^2
\]

Therefore

\[
\int \frac{dx}{\sqrt{12x-x^2}} = \int \frac{dx}{\sqrt{36 - (x-6)^2}} = \int \frac{du}{\sqrt{36 - u^2}}
\]

by taking \( u = x - 6 \).
Now we can do a trig substitution: let \( u = 6 \sin \theta \). Then \( du = 6 \cos \theta \ d\theta \), hence

\[
\int \frac{du}{\sqrt{36-u^2}} = \int \frac{6 \cos \theta}{6 \cos \theta} \ d\theta = \theta + C = \sin^{-1} \frac{u}{6} + C = \sin^{-1} \frac{x-6}{6} + C
\]

3. Partial Fractions

1. Evaluate \( \int \frac{x-1}{(x-2)^2(x-3)} \ dx \).

**Solution:** The numerator has degree less than the denominator, so we can use partial fractions. Write

\[
\frac{x-1}{(x-2)^2(x-3)} = \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} + \frac{B}{x-3}
\]

After clearing denominators, we get

\[
x - 1 = A_1(x-2)(x-3) + A_2(x-3) + B(x-2)^2
\]

Now plug in \( x = 2 \) to obtain \( 1 = -A_2 \), so \( A_2 = -1 \). Similarly, if we plug in \( x = 3 \) then we get \( 2 = B \).

To determine \( A_1 \), let’s look at the coefficient of \( x^2 \) on both sides. On the left-hand side, this coefficient is zero, while on the right-hand side the coefficient is \( A_1 + B \). Therefore \( A_1 = -B = -2 \). So our integral becomes

\[
\int \frac{x-1}{(x-2)^2(x-3)} \ dx = -2 \int \frac{dx}{x-2} - \int \frac{dx}{(x-2)^2} + 2 \int \frac{dx}{x-3}
\]

\[
= 2 \ln |x-3| - 2 \ln |x-2| + \frac{1}{x-2} + C
\]

2. Evaluate \( \int \frac{5x}{(x+2)(x^2+1)} \ dx \).

**Solution:** When we do partial fractions with an irreducible quadratic factor like \( x^2 + 1 \), the corresponding term in the expansion should have a linear polynomial in the numerator. So write

\[
\frac{5x}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}
\]

Now clear the denominators to get

\[
5x = A(x^2 + 1) + (Bx+C)(x+2)
\]

Next, plug in \( x = -2 \):

\[
5(-2) = A(4+1)
\]

or \( A = -2 \).

To find \( B \), look at the coefficient of \( x^2 \). On the left-hand side this coefficient is zero, while on the right hand side it is \( A + B \). Therefore \( B = -A = 2 \). Finally, to find \( C \) we look at the constant coefficient on both sides to get \( 0 = A + 2C \), so \( C = -\frac{A}{2} = 1 \).
Therefore
\[
\int \frac{5x}{(x+2)(x^2+1)} \, dx = -2 \int \frac{dx}{x+2} + \int \frac{2x+1}{x^2+1} \, dx = -2 \int \frac{dx}{x+2} + \int \frac{2x}{x^2+1} \, dx + \int \frac{dx}{x^2+1} \\
= -2 \ln |x+2| + \ln(x^2+1) + \tan^{-1}(x) + C
\]

3. Evaluate \( \int \frac{\sec^2 \theta}{\tan^2 \theta - 1} \, d\theta \).

*Solution:* One might be tempted to try to use the trig identity \( \sec^2 \theta = 1 + \tan^2 \theta \), but I don’t think that helps here. So instead let’s start with a substitution: let \( u = \tan \theta \), so that \( du = \sec^2 \theta \, d\theta \). Then
\[
\int \frac{\sec^2 \theta}{\tan^2 \theta - 1} \, d\theta = \int \frac{du}{u^2 - 1} = \int \frac{du}{(u-1)(u+1)}
\]
Now we can proceed using partial fractions. Write
\[
\frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}
\]
and clear denominators to obtain
\[
1 = A(u+1) + B(u-1)
\]
Plugging in \( u = 1 \) gives \( 1 = 2A \), so \( A = \frac{1}{2} \). Similarly, plugging in \( u = -1 \) gives \( B = -\frac{1}{2} \). Therefore
\[
\int \frac{du}{(u-1)(u+1)} = \frac{1}{2} \int \frac{du}{u-1} - \frac{1}{2} \int \frac{du}{u+1} = \frac{1}{2} \ln |u-1| - \frac{1}{2} \ln |u+1| + C
\]
and plugging in \( u = \tan \theta \) gives
\[
\int \frac{\sec^2 \theta}{\tan^2 \theta - 1} \, d\theta = \frac{1}{2} \ln |\tan \theta - 1| - \frac{1}{2} \ln |\tan \theta + 1| + C
\]

4. **Improper Integrals**

1. Determine whether \( \int_0^1 x^{-\frac{7}{5}} \, dx \) converges, and if so, evaluate it:

*Solution:* The function \( x^{-\frac{7}{5}} \) has an infinite discontinuity at \( x = 0 \), so this is an improper integral. We can evaluate it using the limit definition:
\[
\int_0^1 x^{-\frac{7}{5}} \, dx = \lim_{a \to 0^+} \int_a^1 x^{-\frac{7}{5}} \, dx = \lim_{a \to 0^+} \left[ x^{-\frac{2}{5}} \right]_a^1 = \lim_{a \to 0^+} \left[ 8x^{-\frac{2}{5}} \right]_a^1 = \lim_{a \to 0^+} 8 - 8a^{\frac{2}{5}} = 8
\]
Therefore the integral converges, and the value of the integral is 8.

2. Determine whether \( \int_1^\infty \frac{dx}{2x^{\frac{1}{2}} + x} \) converges.

*Solution:* \( 2x^{\frac{1}{2}} \leq x \) for \( x \geq 4 \), hence \( 2x^{\frac{1}{2}} + x \leq 2x \) and \( \frac{1}{2x^{\frac{1}{2}} + x} \leq \frac{1}{2x} \) for \( x \geq 4 \).
We can break our integral up into two pieces:
\[
\int_1^\infty \frac{dx}{2x^{1/2} + 1} = \int_1^4 \frac{dx}{2x^{1/2} + x} + \int_4^\infty \frac{dx}{2x^{1/2} + x}
\]
The first integral is just some finite number, so it doesn’t affect the convergence or divergence of the original integral. For the second integral, the comparison test gives
\[
\int_4^\infty \frac{dx}{2x^{1/2} + x} \geq \int_4^\infty \frac{dx}{2x} = \frac{1}{2} \lim_{R \to \infty} \left[ \ln(x) \right]_4^R = \frac{1}{2} \lim_{R \to \infty} \ln(R) - \ln(4) = \infty
\]
Therefore
\[
\int_4^\infty \frac{dx}{2x^{1/2} + x}
\]
diverges, which shows that
\[
\int_1^\infty \frac{dx}{2x^{1/2} + x}
\]
also diverges.

5. **Numerical Integration**

1. Find \( N \) such that
\[
\left| \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos x \, dx - T_N \right| \leq 10^{-5}
\]

*Solution:* The formula for the error bound for the trapezoid rule is given by
\[
\text{Error}(T_N) \leq \frac{K_2(b - a)^3}{12N^2}
\]
where \( K_2 = \max_{x \in [a,b]} |f''(x)|. \)

If \( f(x) = \cos x \), then \( f'(x) = -\sin x \) and \( f''(x) = -\cos x \). Therefore
\[
K_2 = \max_{x \in [-\frac{1}{2}, \frac{1}{2}]} |\cos x| = 1
\]

hence
\[
\text{Error}(T_N) \leq \frac{\left(\frac{1}{2} - (-\frac{1}{2})\right)^3}{12N^2} = \frac{1}{12N^2}
\]
We want the error to be at most \( 10^{-5} \). Thus
\[
\frac{1}{12N^2} \leq 10^{-5}
\]
and solving for \( N \) gives
\[
N \geq \sqrt{\frac{10^5}{12}} \approx 91.2
\]
so any \( N \geq 92 \) will do.
How could we handle the last part without using a calculator? Using the fact that
\[
\frac{10^5}{12} \leq \frac{10^5}{10} = 10^4
\]
we see that any \( N \geq \sqrt{10^4} = 10^2 = 100 \) suffices.

6. Arc length

1. Find the arc length of the plane curve given by the equation \( x^\frac{2}{3} + y^\frac{2}{3} = 1 \).

\textit{Solution:} Observe that this curve is symmetric about both the \( x \)-axis and the \( y \)-axis. That is, if a point \((x, y)\) is on the curve, then so are the points \((-x, y), (x, -y),\) and \((-x, -y)\).

Therefore to find the arc length of the plane curve, it is enough to find the arc length in the first quadrant, and then multiply by 4.

To do this, first solve for \( y \):
\[
y = (1 - x^\frac{2}{3})^\frac{3}{2}
\]
Taking a derivative gives
\[
\frac{dy}{dx} = \frac{3}{2}(1 - x^\frac{2}{3})^{\frac{1}{2}} \cdot (-\frac{2}{3}x^{-\frac{4}{3}}) = -x^{-\frac{1}{3}}(1 - x^\frac{2}{3})^{\frac{1}{2}}
\]
so that
\[
1 + \left(\frac{dy}{dx}\right)^2 = 1 + x^{-\frac{2}{3}}(1 - x^\frac{2}{3}) = 1 + x^{-\frac{2}{3}} - 1 = x^{-\frac{2}{3}}
\]
so the arc length is
\[
\int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_0^1 x^{-\frac{1}{3}} \, dx = \frac{3}{2}x^{\frac{2}{3}}\bigg|_0^1 = \frac{3}{2}
\]
Finally, we multiply by 4 to see that the arc length of the curve is 6.

7. Fluid Force

1. Find the fluid force on a metal plate bounded by the parabola \( y = -x^2 \) and the line \( y = -1 \). The surface of the liquid is the line \( y = 0 \), and the liquid has pressure \( \rho \).

\textit{Solution:} The formula for fluid force is
\[
F = \rho g \int_0^1 yf(y) \, dy
\]
where \( f(y) \) is the width at depth \( y \). In this case, \( f(y) = 2\sqrt{y} \). Therefore
\[
F = 2\rho g \int_0^1 y\sqrt{y} \, dy = 2\rho g \int_0^1 y^{\frac{3}{2}} \, dy = 2\rho g \left[ \frac{2}{5}y^{\frac{5}{2}} \right]_0^1 = \frac{4}{5}\rho g
\]
8. Taylor Polynomials

1. Let \( f(x) = e^{-x} \). Find the third Taylor polynomial \( T_3(x) \) for \( f \) centered at \( x = 0 \), and use the error bound to find the maximum value for \( |f(0.1) - T_3(0.1)| \).

Solution: Recall that

\[
T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3
\]

Therefore we need to take 3 derivatives:

\[
f'(x) = -e^{-x} \quad f''(x) = e^{-x} \quad f^{(3)}(x) = -e^{-x}
\]

Plugging in \( x = 0 \), we get

\[
T_3(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6}
\]

Next, recall that the error bound formula is

\[
|f(0.1) - T_3(0.1)| \leq \frac{K|0.1 - 0|^4}{4!} = \frac{K}{24 \cdot 10^4}
\]

where \( K \) is an upper bound for \( |f^{(4)}(x)| \) on \([0, 0.1]\). To get the best possible error bound, we take

\[
K = \max_{x \in [0,0.1]} |f^{(4)}(x)|
\]

Now \( f^{(4)}(x) = e^{-x} \), which is a decreasing function. Therefore

\[
K = \max_{x \in [0,0.1]} |f^{(4)}(x)| = e^0 = 1
\]

hence

\[
|f(0.1) - T_3(0.1)| \leq \frac{1}{24 \cdot 10^4} \approx 4.16 \times 10^{-6}
\]

On the midterm, I think that it would suffice to leave the answer as \( \frac{1}{24 \cdot 10^4} \).