1. Limits and Continuity

1. Evaluate \( \lim_{x \to 1} \frac{x^3 - 1}{x - 1} \).

\textbf{Solution:} Both the numerator and denominator are zero at \( c = 1 \), so we cannot use our limit laws to determine the limit. But some algebraic manipulation does the trick:

\[
\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{x - 1} = \lim_{x \to 1} x^2 + x + 1 = 3
\]

2. Evaluate \( \lim_{x \to \infty} \frac{x^3 + 1}{2x^3 + 3x} \).

\textbf{Solution:} Divide the numerator and denominator by \( x^3 \) to obtain

\[
\lim_{x \to \infty} \frac{x^3 + 1}{2x^3 + 3x} = \lim_{x \to \infty} \frac{1 + \frac{1}{x^3}}{2 + \frac{3}{x^3}}
\]

Now \( \lim_{x \to \infty} \frac{1}{x^3} = \lim_{x \to 0} \frac{3}{x^2} = 0 \), hence

\[
\lim_{x \to \infty} \frac{1 + \frac{1}{x^3}}{2 + \frac{3}{x^2}} = \frac{1 + 0}{2 + 0} = \frac{1}{2}
\]

This problem illustrates a useful principle: when evaluating \( \lim_{x \to \infty} \frac{p(x)}{q(x)} \) for polynomials \( p(x) \) and \( q(x) \), we need only consider the terms of highest degree in \( p(x) \) and \( q(x) \). In this problem \( p(x) = x^3 + 1, q(x) = 2x^3 + 3x \), and the highest degree terms are \( x^3 \) and \( 2x^3 \).

3. Evaluate \( \lim_{x \to \infty} \sqrt{x^2 + x - x} \).

\textbf{Solution:} The trick to solving this sort of problem is to multiply by the conjugate:

\[
\lim_{x \to \infty} \sqrt{x^2 + x - x} = \lim_{x \to \infty} \frac{\sqrt{x^2 + x} - x}{x} \cdot \frac{\sqrt{x^2 + x} + x}{\sqrt{x^2 + x} + x} = \lim_{x \to \infty} \frac{x^2 + x - x^2}{\sqrt{x^2 + x} + x} = \lim_{x \to \infty} \frac{x}{\sqrt{x^2 + x} + x}
\]

Now divide the numerator and denominator by \( x \):

\[
\lim_{x \to \infty} \frac{x}{\sqrt{x^2 + x} + x} = \lim_{x \to \infty} \frac{1}{\frac{1}{x} + 1} = \frac{1}{1 + 1} = \frac{1}{2}
\]
4. If \( f(x) + g(x) \) is continuous at a point \( c \), are \( f \) and \( g \) necessarily continuous at \( c \)? Similarly, if \( fg \) is continuous at \( c \), are \( f \) and \( g \) necessarily continuous at \( c \)?

**Solution:** No for both. Let \( f(x) = 1 \) for \( x \geq 0 \), and \( f(x) = -1 \) for \( x < 0 \), and let \( g(x) = -f(x) \). Both \( f \) and \( g \) are discontinuous at \( c = 0 \), but \( f(x) + g(x) = 0 \) is continuous, as is \( f(x)g(x) = -f(x)^2 = -1 \).

*5. Let \( g(x) = \sin \frac{1}{x} \). Could we apply the squeeze theorem to the functions \( f(x) = -1 \) and \( h(x) = 1 \) to determine \( \lim_{x \to 0} g(x) \)? What about \( f(x) = -x^2 \) and \( h(x) = x^2 \)?

**Solution:** No for both. In the first case, we have the inequality \(-1 \leq \sin \frac{1}{x} \leq 1 \) for all \( x \neq 0 \), but this doesn’t help us since \( f \) and \( h \) have different limits at 0.

For the second case, \( \lim_{x \to 0} f(x) = 0 = \lim_{x \to 0} h(x) \), but the functions \( f \) and \( h \) don’t squeeze \( g \) on any interval containing 0: if \( x_n = \frac{2\pi}{2n+1} \) for \( n = 1, 2, 3, \ldots \), then

\[
g(x_n) = \sin \frac{(2n+1)\pi}{2} = \pm 1
\]

since \( 2n+1 \) is always odd, but

\[
h(x_n) = \frac{4}{(2n+1)^2} < 1
\]

and

\[
f(x_n) = -\frac{4}{(2n+1)^2} > -1
\]

Thus one of the inequalities \( g(x_n) \leq h(x_n) \) or \( f(x_n) \leq g(x_n) \) does not hold.

6. Use the formal definition of the limit to show that \( \lim_{x \to 0} x^2 = 0 \).

**Solution:** Let \( f(x) = x^2 \) and \( L = 0 \). We have to show that for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( 0 < |x - 0| < \delta \) then \( |f(x) - L| < \varepsilon \).

Fix \( \varepsilon > 0 \). Then \( |f(x) - L| = |x^2 - 0| = |x|^2 \). So let \( \delta = \sqrt{\varepsilon} \). If \( 0 < |x| < \delta \), then

\[
|f(x) - L| = |x|^2 < \delta^2 = (\sqrt{\varepsilon})^2 = \varepsilon
\]

As \( \varepsilon \) was arbitrary, this shows that \( \lim_{x \to 0} x^2 = 0 \).

2. **Differentiation**

1. Use the limit definition to show that the function \( f(x) = x|x| \) is differentiable at \( x = 0 \).

**Solution:** Recall that \( |x| = x \) if \( x \geq 0 \), and \( |x| = -x \) if \( x < 0 \). So we need to calculate the two one-sided limits at 0 and show that they are equal.

For the right-handed limit, we have

\[
\lim_{h \to 0^+} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0^+} \frac{h^2 - 0^2}{h} = \lim_{h \to 0^+} \frac{h^2}{h} = \lim_{h \to 0^+} h = 0
\]
while the limit from the left is

\[ \lim_{h \to 0^-} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0^-} \frac{h(-h) - 0^2}{h} = \lim_{h \to 0^-} \frac{-h^2}{h} = \lim_{h \to 0^-} -h = 0 \]

Thus the two limits agree, so \( f \) is differentiable at 0, with \( f'(0) = 0 \).

2. Let \( f(x) = \begin{cases} x \quad x < -1 \\ x^2 \quad x \geq -1 \end{cases} \). Find the points at which \( f \) is not differentiable.

**Solution:** If a function is differentiable at a point, then it is necessarily continuous at that point, so we can begin by looking for points where \( f \) is not continuous. The functions \( x \) and \( x^2 \) are everywhere continuous, so the only point we need to consider is the point where \( f \) switches definitions, namely \( x = -1 \).

The two one-sided limits at \( x = -1 \) are

\[ \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} x = -1 \quad \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} x^2 = 1 \]

Thus \( f \) is not continuous (hence not differentiable) at \( x = -1 \).

Next we should look for points where \( f(x) \) is continuous but not differentiable. The function \( x \) is everywhere differentiable, but \( x^2 \) is not differentiable at 0:

\[ \lim_{h \to 0} \frac{h^2}{h} = \lim_{h \to 0} \frac{1}{h^2} \]

which does not exist.

Therefore the two points at which \( f \) is not differentiable are \( x = -1 \) and \( x = 0 \).

3. Find the following derivatives:

a. \( \frac{d}{dx} \sin(\sqrt{x}) \).

**Solution:** This is a composite function of the form \( f(g(x)) \). The outside function is \( f(x) = \sin x \), and the inside function is \( g(x) = \sqrt{x} \). \( f'(x) = \cos x \) and \( g'(x) = \frac{1}{2\sqrt{x}} \), so by the chain rule

\[ \frac{d}{dx} \sin(\sqrt{x}) = \cos(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} = \frac{\cos(\sqrt{x})}{2\sqrt{x}} \]

b. \( \frac{d}{dx} \sqrt{\sin x} \).

**Solution:** \( \sqrt{\sin x} \) is a composite of the same two functions as part (a), but the inside and outside functions are reversed. So in this case

\[ \frac{d}{dx} = \frac{1}{2\sqrt{\sin x}} \cdot \cos x = \frac{\cos x}{2\sqrt{\sin x}} \]

c. \( \frac{d}{dx} \tan(2x) \).

**Solution:**
Let \( f(x) = x + \sin x \) and \( g(x) = \tan(2x) \). Then \( f'(x) = 1 + \cos x \), and \( g'(x) = 2\sec^2(2x) \) by the chain rule. Therefore by the quotient rule,

\[
\frac{d}{dx} \frac{x + \sin x}{\tan(2x)} = \frac{(1 + \cos x)(\tan(2x)) - (x + \sin x)(2\sec^2(2x))}{\tan^2(2x)}
\]

We can simplify this expression by recalling that \( \tan \theta = \frac{\sin \theta}{\cos \theta} \) and \( \sec \theta = \frac{1}{\cos \theta} \). So multiplying the numerator and denominator by \( \cos^2(2x) \) yields

\[
\frac{(1 + \cos x)(\tan(2x)) - (x + \sin x)(2\sec^2(2x))}{\tan^2(2x)} = \frac{(1 + \cos x)\sin(2x)\cos(2x) - 2(x + \sin x)}{\sin^2(2x)}
\]

4. Let \( f(x) = 2x^3 - 9x^2 + 12x + 1 \). Find all points \( x \) such that the tangent line to the graph of \( f \) is horizontal.

**Solution:** The slope of the tangent line to the graph of \( f \) at the point \( x \) is \( f'(x) \), so we need to find all \( x \) such that \( f'(x) = 0 \).

By the power rule and the linearity of the derivative,

\[
f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2)
\]

Therefore \( f'(x) = 0 \) when \( x = 1 \) and \( x = 2 \).

*5. Find a function \( f(x) \) such that \( f \) is differentiable at every point, but such that \( f'(x) \) fails to be differentiable at some point.

**Solution:** To rephrase the question, we want a function \( f(x) \) such that \( f'(x) \) exists everywhere, but such that \( f''(x) \) does not exist at some point.

I think the best way to approach this problem is to find a function \( g(x) \) such that \( g'(x) \) fails to exist at some point, then find a function \( f \) such that \( f'(x) = g(x) \). There are many possible choices for \( g \), but to make our life easy, we should choose \( g \) in a way that makes it easy to guess \( f \).

Set \( g(x) = x^{\frac{2}{3}} \). Then \( g \) is not differentiable at \( x = 0 \), as we saw in problem 2. If \( f(x) = \frac{3}{5}x^{\frac{5}{3}} \), then \( f'(x) = x^{\frac{2}{3}} = g(x) \) by the power rule. Therefore \( f'(x) \) exists for all \( x \), but \( f''(x) = g'(x) \) does not exist at \( x = 0 \).