1. LIMITS AND CONTINUITY

1. Evaluate \( \lim_{x \to 0} \frac{1 - \cos x}{x^2} \).

Solution: Multiply the numerator and denominator by \( 1 + \cos x \) to obtain
\[
\frac{1 - \cos x}{x^2} = \frac{1 - \cos^2 x}{x^2(1 + \cos x)} = \frac{\sin^2 x}{x^2(1 + \cos x)}
\]
Therefore using the limit laws shows that
\[
\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin^2 x}{x^2} \left( \lim_{x \to 0} \frac{1}{1 + \cos x} \right) = \frac{1}{2}
\]

2. Let \( f(x) = \frac{x^2 - a^2}{x - 2} \) if \( x \neq 2 \), and \( f(2) = 4 \). Find the value(s) of \( a \) that make \( f \) continuous at \( x = 2 \).

Solution: In order for \( f \) to be continuous, we need \( \lim_{x \to 2} f(x) = f(2) = 4 \); in particular the limit must exist.
\( f(x) = \frac{(x - a)(x + a)}{x - 2} \) will not have a limit at 2 unless one of the factors in the numerator is \( x - 2 \). Therefore we need either \( x - a = x - 2 \) or \( x + a = x - 2 \), so \( a = \pm 2 \). If \( a = 2 \) then
\[
\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2} x + 2 = 4
\]
and if \( a = -2 \) then
\[
\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{(x + 2)(x - 2)}{x - 2} = \lim_{x \to 2} x + 2 = 4
\]
so both \( a = 2 \) and \( a = -2 \) work.

2. IMPLICIT DIFFERENTIATION

Find the points \((x_0, y_0)\) for which the tangent line to the curve \( y^2 - x^2 = 1 \) is horizontal.
Solution: Using implicit differentiation gives

\[ \frac{dy}{dx} - \frac{2x}{2y} = 0 \]

or \( \frac{dy}{dx} = \frac{x}{y} \). Therefore the tangent line is horizontal at the points \((0, \pm 1)\).

3. Related Rates

Road A runs north-south, and road B runs east-west. The two roads meet at a gas station. At time \( t = 0 \), three cars leave the gas station. Car 1 travels north on road A at a speed of 40 mph, car 2 travels east on road B at 60 mph, and car 3 travels west on road B at 30 mph. Find the rate of change of the area of the triangle formed by the three cars at time \( t = 30 \) minutes.

Solution: Let \( f_1(t), f_2(t), f_3(t) \) denote the distance of cars 1, 2, 3 from the gas station at time \( t \). The area of the triangle is then

\[ A(t) = \frac{1}{2} bh = \frac{1}{2} [f_2(t) + f_3(t)] f_1(t) \]

Differentiating gives

\[ A'(t) = \frac{1}{2} f_1'(t) [f_2(t) + f_3(t)] + \frac{1}{2} f_1(t) [f_2'(t) + f_3'(t)] \]

\( f_1'(t) = 40, \ f_2'(t) = 60, \) and \( f_3'(t) = 30 \). And at time \( t = \frac{1}{2} \) (in units of hours), we have

\[ f_1(t) = \frac{40}{2} = 20, \quad f_2(t) = \frac{60}{2} = 30, \quad f_3(t) = \frac{30}{2} = 15 \]

Therefore

\[ A'\left(\frac{1}{2}\right) = \frac{1}{2} \cdot 40[30 + 15] + \frac{1}{2} \cdot 20[60 + 30] = 20 \cdot 45 + 10 \cdot 90 = 1800 \]

in units of square miles per hour.

4. Local extrema and inflection points

1. Find the local extrema of \( f(x) = \frac{x}{1+x^2} \) and determine whether they are local maxima or local minima.

Solution: By the quotient rule,

\[ f'(x) = \frac{(1 + x^2) - x(2x)}{(1 + x^2)^2} = \frac{1 - x^2}{(1 + x^2)^2} \]

The denominator of \( f' \) is never zero, and the numerator is zero when \( x = \pm 1 \). So the critical points are 1 and \(-1\).
Next, \( f'(x) = \frac{(1-x)(1+x)}{(1+x^2)^2} \) changes sign from negative to positive at \( x = -1 \), and from positive to negative at \( x = 1 \). So \( x = -1 \) is a minimum and \( x = 1 \) is a maximum.

2. Give an example of a function \( f(x) \) and a point \( c \) such that \( f''(c) = 0 \) but \( c \) is not an inflection point of \( f \).

Solution: Let \( f(x) = x^6 \) and \( c = 0 \). Then \( f''(x) = 30x^4 \) and \( f''(0) = 0 \). But \( f'' \) doesn’t change sign at 0, so 0 is not an inflection point.

5. Optimization

Find the point on the graph of \( f(x) = x^2 \) which minimizes the distance to the point \( (3,0) \).

Solution: It’s important to note that the function we want to minimize is the distance from \( f \) to \( (3,0) \), not necessarily the function \( f \) itself. By the distance formula, we have

\[
d(x)^2 = (x-3)^2 + x^4 = x^2 - 6x + 9 + x^4
\]

Minimizing the distance is equivalent to minimizing the square of the distance. The derivative of \( d^2 \) is

\[
4x^3 + 2x - 6 = 2(2x^3 + x - 3) = 2(x-1)(2x^2 + 2x + 3)
\]

By using the quadratic formula, we see that \( 2x^2 + 2x + 3 \) does not have real roots. Therefore the only critical point is \( x = 1 \). The derivative changes sign from negative to positive at this point, so this is a minimum.

Thus the point on the graph of \( f \) minimizing the distance to \( (3,0) \) is \( (1,1) \).

6. Integration and the Fundamental Theorem of Calculus

1. Evaluate \( \int_{-1}^{1} |x|^3 \, dx \).

Solution: \( |x| = x \) if \( x \geq 0 \), and \( |x| = -x \) if \( x \leq 0 \). Therefore

\[
\int_{-1}^{1} |x|^3 \, dx = \int_{-1}^{0} |x|^3 \, dx + \int_{0}^{1} |x|^3 \, dx = \int_{-1}^{0} -x^3 \, dx + \int_{0}^{1} x^3 \, dx
\]

\[
= \left[ -\frac{x^4}{4} \right]_{-1}^{0} + \left[ \frac{x^4}{4} \right]_{0}^{1} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}
\]

2. Evaluate \( \frac{d}{dx} \int_{x+1}^{x^3} \sin(t^2) \, dt \).

Solution: Let \( F(x) \) be an antiderivative of \( \sin(x^2) \). Then

\[
\int_{x+1}^{x^3} \sin(t^2) \, dt = F(x^3) - F(x+1)
\]
so by the Fundamental Theorem of Calculus and the chain rule,
\[
\frac{d}{dx} \int_{x+1}^{x^3} \sin(t^2) \, dt = F'(x^3) \cdot 3x^2 - F'(x + 1) = 3x^2 \sin(x^6) - \sin((x + 1)^2)
\]

7. Substitution

Evaluate \( \int_0^1 x^3(x^4 + 1)^2 \, dx \).

**Solution:** Set \( u(x) = x^4 + 1 \). Then \( \frac{du}{dx} = 4x^3 \), so \( x^3 \, dx = \frac{1}{4} \, du \). The new limits of integration are \( u(0) = 1 \) and \( u(1) = 2 \). Therefore
\[
\int_0^1 x^3(x^4 + 1)^2 \, dx = \frac{1}{4} \int_1^2 u^2 \, du = \frac{u^3}{12} \bigg|_1^2 = \frac{7}{12}
\]

8. Volume of solids

1. Consider the region \( R \) bounded by the graphs of \( f(x) = 2x - x^2 \) and \( g(x) = x^2 \).

   a. Find the area of this region.

   **Solution:** To find the area, we must first determine the points of intersection of the two functions. Setting \( f(x) = g(x) \) gives \( 2x - x^2 = x^2 \), or \( 2x^2 - 2x = 0 \), which has roots \( x = 0 \) and \( x = 1 \).

   Checking \( f \) and \( g \) at the point \( x = \frac{1}{2} \) shows that \( f(x) \geq g(x) \) on \([0, 1]\), hence the area is given by
   \[
   \int_0^1 f(x) - g(x) \, dx = \int_0^1 2x - 2x^2 \, dx = \left[ \frac{x^2}{3} - \frac{2x^3}{3} \right]_0^1 = \frac{1}{3}
   \]

   b. Find the volume of the solid obtained by rotating \( R \) about the \( x \)-axis.

   **Solution:** Since we’re rotating about the \( x \)-axis, it’s easier to use the washer method. The volume is
   \[
   \pi \int_0^1 (f(x))^2 - (g(x))^2 \, dx = \pi \int_0^1 4x^2 - 4x^3 + x^4 - x^4 \, dx = \pi \int_0^1 4x^2 - 4x^3 \, dx
   \]
   \[
   = \pi \left[ \frac{4}{3}x^3 - x^4 \right]_0^1 = \frac{\pi}{3}
   \]

   c. Find the volume of the solid obtained by revolving the region \( R \) about the \( y \) axis.

   **Solution:** Since we’re rotating about the \( y \)-axis, the shell method is probably the best bet. The volume is given by
   \[
   2\pi \int_0^1 x(f(x) - g(x)) \, dx = 2\pi \int_0^1 x(2x - 2x^2) \, dx = \pi \int_0^1 4x^2 - 4x^3 \, dx
   \]
\[ = \pi \left[ \frac{4x^3}{3} - x^4 \right]_0^1 = \frac{\pi}{3} \]