1. Let $V$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(-x) = f(x)$ for all $x \in \mathbb{R}$. Show that $V$ is a vector space over $\mathbb{R}$ under the usual addition and scalar multiplication of functions.

**Solution:** Let $W$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. We have seen that $W$ is a vector space with respect to addition and scalar multiplication of functions, so it suffices to show that $V$ is a subspace of $W$.

Certainly $0 \in V$. And if $f, g \in V$ and $c \in \mathbb{R}$, then

$$(cf + g)(-x) = cf(-x) + g(-x) = cf(x) + g(x) = (cf + g)(x)$$

Therefore $cf + g \in W$, so $W$ is a subspace.

2. Let $V$ be a vector space, and $L(V)$ the space of all linear transformations on $V$. Show that $V$ is finite-dimensional if and only if $L(V)$ is finite-dimensional.

**Solution:** Suppose that $V$ is a finite-dimensional vector space, of dimension $n$, and choose a basis $\{v_1, \ldots, v_n\}$ for $V$.

Then $L(V)$ has dimension $n^2$, since it has a basis given by the set $\{E_{i,j} : 1 \leq i, j \leq n\}$, where $E_{i,j}$ is defined by $E_{i,j}(v_k) = 0$ if $j \neq k$, and $E_{i,j}(v_k) = v_i$ if $j = k$. (If we identify $V$ with $\mathbb{F}^n$ via the basis for $V$, then $E_{i,j}$ is the matrix having a 1 in the $i, j$ position and 0 everywhere else). Therefore $L(V)$ is finite-dimensional.

For the other direction, suppose that $L(V)$ is finite-dimensional, and choose some basis $\{e_i\}_{i \in I}$ for $V$. Note that we are not assuming this basis is finite, since after all that is what we are trying to prove.

Fix an element $e_1$ in this basis, and define a linear map $T : V \to L(V)$ as follows: set $T(v) = f_v$, where $f_v(e_i) = v$ if $e_i = e_1$, and $f_v(e_i) = 0$ otherwise. Note that

$$T(cv + w) = f_{cv + w} = cf_v + f_w = cT(v) + T(w)$$

so $T$ is linear.

I claim that $T$ is one-to-one. Indeed, if $T(v) = 0$, then $f_v \in L(V)$ is the zero map. In particular, $f_v(e_1) = 0$. But by definition $f_v(e_1) = v$, so $v = 0$. Therefore $N(T) = \{0\}$, so $T$ is one-to-one.
Thus $T$ is one-to-one, so $T : V \rightarrow R(T)$ is an isomorphism. But $R(T)$ is a subspace of the finite-dimensional vector space $L(V)$, so is also finite-dimensional. Thus $V$ is finite-dimensional because it is isomorphic to $R(T)$.

3. Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional inner product space of dimension $n$. Let $S = \{v_1, \ldots, v_k\}$ be a linearly independent subset of $V$. Define $T : V \rightarrow V$ by

$$ T(x) = \sum_{i=1}^{k} \langle x, v_i \rangle v_i $$

a. Show that $T$ is linear.

*Solution*: Choose two vectors $x, y \in V$ and a scalar $c$. Then

$$ T(cx + y) = \sum_{i=1}^{k} (cx + y, v_i) v_i = \sum_{i=1}^{k} \left( c \langle x, v_i \rangle + \langle y, v_i \rangle \right) v_i = c \sum_{i=1}^{k} \langle x, v_i \rangle v_i + \sum_{i=1}^{k} \langle y, v_i \rangle v_i $$

using the properties of the inner product. Thus $T$ is linear.

b. What is the kernel of $T$?

*Solution*: Suppose that $T(x) = 0$. By definition, this means that

$$ \sum_{i=1}^{k} \langle x, v_i \rangle v_i = 0 $$

and as the set $S$ is linearly independent, it follows that $\langle x, v_i \rangle = 0$ for all $i$. Therefore

$$ \ker(T) = \{ x \in V : \langle x, v_i \rangle = 0, 1 \leq i \leq k \} $$

Recall the definition of the orthogonal complement of the subspace $\text{span}(S)$:

$$ \text{span}(S)^\perp = \{ y \in V : \langle y, z \rangle = 0 \text{ for all } z \in S \} $$

I claim that the subspaces $\text{span}(S)^\perp$ and $\{ x \in V : \langle x, v_i \rangle = 0, 1 \leq i \leq k \}$ are equal.

On the one hand, if $x \in \text{span}(S)^\perp$, then $\langle x, v_i \rangle = 0$ for all $i$ since each $v_i$ is in $\text{span}(S)$. On the other hand, every element of $\text{span}(S)$ is a linear combination of the $v_i$, so if $\langle x, v_i \rangle = 0$ for all $i$ then $\langle x, y \rangle = 0$ for all $y \in \text{span}(S)$. Therefore

$$ \ker(T) = \{ x \in V : \langle x, v_i \rangle = 0, 1 \leq i \leq k \} = \text{span}(S)^\perp $$

c. What is the image of $T$?

*Solution*: By the definition of $T$, for any $x \in V$ we have

$$ T(x) = \sum_{i=1}^{k} \langle x, v_i \rangle v_i \in \text{span}(S) $$
Thus $R(T) \subset \text{span}(S)$.

To show that this inclusion is an equality, it suffices to show that the two subspaces have the same dimension. Since $S$ is a linearly independent set, $\text{span}(S)$ has dimension $k$. And by the dimension theorem,

$$\dim R(T) = \dim V - \dim N(T) = n - \dim \text{span}(S)^\perp = n - (n - k) = k$$

Thus $\dim R(T) = k = \dim \text{span}(S)$, so $R(T) = \text{span}(S)$.

4. Let $\alpha = \{v_1, v_2, v_3\}$ be a basis of $\mathbb{R}^3$, with the standard inner product. Find an invertible matrix $A \in M_3(\mathbb{R})$ such that $\beta = \{Av_1, Av_2, Av_3\}$ is an orthogonal basis for $V$.

**Solution:** I think it’s easiest to find the orthogonal basis we want first, then figure out what the matrix should be.

To construct an orthogonal basis out of $\alpha$, use the Gram-Schmidt process: set $u_1 = v_1$, and

$$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} u_1 = v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} v_1$$

$$u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{\|u_1\|^2} u_1 - \frac{\langle v_3, u_2 \rangle}{\|u_2\|^2} u_2 = v_3 - \frac{\langle v_3, u_1 \rangle}{\|u_1\|^2} v_1 - \frac{\langle v_3, u_2 \rangle}{\|u_2\|^2} \left( v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} v_1 \right)$$

$$= v_3 - \frac{\langle v_3, u_2 \rangle}{\|u_2\|^2} v_2 + \left( \frac{\langle v_3, u_2 \rangle}{\|u_2\|^2} \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} - \frac{\langle v_3, u_1 \rangle}{\|u_1\|^2} \right) v_1$$

$\beta = \{u_1, u_2, u_3\}$ is our desired orthogonal basis.

Now let $B$ be the matrix changing $\beta$ coordinates into $\alpha$ coordinates, so

$$B = \begin{bmatrix}
1 & -\frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} & -\frac{\langle v_2, u_2 \rangle}{\|u_2\|^2} \\
0 & 1 & \frac{\langle v_3, u_2 \rangle}{\|u_2\|^2} \\
0 & 0 & \frac{1}{\|u_2\|^2}
\end{bmatrix}$$

and let $Q$ be the matrix changing $\alpha$ coordinates into the standard coordinates. Then $A = QBQ^{-1}$ is the desired matrix.

Let’s consider an example to clarify what’s going on. Let $v_1 = (1, 1, 0), v_2 = (1, 0, 1), v_3 = (0, 0, 1)$. Then $u_1 = v_1$,

$$u_2 = v_2 - \frac{1}{2} v_1 = \left( \frac{1}{2}, -\frac{1}{2}, 1 \right)$$

and

$$u_3 = v_3 - \frac{2}{3} u_2 = v_3 - \frac{2}{3} v_2 + \frac{1}{3} v_1 = \left( -\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)$$

The matrix $B$ is

$$B = \begin{bmatrix}
1 & -\frac{1}{2} & \frac{1}{3} \\
0 & 1 & -\frac{1}{3} \\
0 & 0 & 1
\end{bmatrix}$$
and similarly

\[ Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad Q^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \]

A computation gives

\[ A = QBQ^{-1} = \begin{bmatrix} \frac{5}{2} & \frac{1}{2} & -\frac{1}{3} \\ \frac{5}{2} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \]

and to check that this works, note that

\[ Av_1 = \begin{bmatrix} \frac{5}{2} & \frac{1}{2} & -\frac{1}{3} \\ \frac{5}{2} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = u_1 \]

\[ Av_2 = \begin{bmatrix} \frac{5}{2} & \frac{1}{2} & -\frac{1}{3} \\ \frac{5}{2} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = u_2 \]

and finally

\[ Av_3 = \begin{bmatrix} \frac{5}{2} & \frac{1}{2} & -\frac{1}{3} \\ \frac{5}{2} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} = u_3 \]

5. Let \( T \) be a linear transformation of a vector space \( V \) over \( \mathbb{R} \).

a. Let \( \lambda \) be an eigenvalue of \( T \). Show that \( \lambda^2 \) is an eigenvalue of \( T^2 \).

*Solution:* Since \( \lambda \) is an eigenvalue of \( T \), there is a non-zero vector \( v \in V \) such that \( Tv = \lambda v \). Then by linearity,

\[ T^2 v = T(Tv) = T(\lambda v) = \lambda T(v) = \lambda^2 v \]

which shows that \( \lambda^2 \) is an eigenvalue of \( T^2 \).

b. If \( T^2 = I \), then show that the only possible eigenvalues of \( T \) are \( \pm 1 \).

*Solution:* Suppose that \( \lambda \) is an eigenvalue of \( T \), and choose a non-zero vector \( v \) such that \( Tv = \lambda v \). Then by part (a) we know that \( T^2 v = \lambda^2 v \). On the other hand, \( T^2 = I \), hence

\[ v = Iv = T^2 v = \lambda^2 v \]

Therefore \( (1 - \lambda^2)v = 0 \), and since \( v \neq 0 \), this implies that \( \lambda^2 = 1 \), or \( \lambda = \pm 1 \).

6. Let \( A = \begin{bmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{bmatrix} \in M_3(\mathbb{R}) \).

a. Find all eigenvalues of \( A \).
Solution: The eigenvalues of $A$ are the roots of the polynomial $\det(xI - A)$. To compute this determinant, we use expansion by minors in the third column:

$$\begin{vmatrix} x + 1 & -3 & 1 \\ 3 & x - 5 & 1 \\ 3 & -3 & x - 1 \end{vmatrix} = \begin{vmatrix} x - 5 & -3 & 1 \\ 3 & x - 5 & 1 \\ 3 & -3 & x - 5 \end{vmatrix} - \begin{vmatrix} x + 1 & -3 & 1 \\ 3 & x - 5 & 1 \\ 3 & -3 & x - 5 \end{vmatrix} + (x - 1) \begin{vmatrix} x + 1 & -3 \\ 3 & x - 5 \end{vmatrix}$$

$$= -9 - 3x + 15 - (-3x - 3 + 9) + (x - 1)(x^2 - 4x - 5 + 9) = (x - 1)(x^2 - 4x + 4) = (x - 1)(x - 2)^2$$

Therefore the eigenvalues of $A$ are 1 and 2.

b. Find bases for the eigenspaces of $A$, and show that $A$ is diagonalizable.

Solution: The eigenspaces of $A$ are the kernels of $A - I$ and $A - 2I$. For $\lambda = 1$, we have

$$A - I = \begin{bmatrix} -2 & 3 & -1 \\ -3 & 4 & -1 \\ -3 & 3 & 0 \end{bmatrix}$$

so the vector $(1,1,1)$ is in the kernel. Moreover, the kernel has dimension one since the algebraic multiplicity of $\lambda = 1$ is 1. Therefore $(1,1,1)$ is a basis for the kernel.

For $\lambda = 2$,

$$A - 2I = \begin{bmatrix} -3 & 3 & -1 \\ -3 & 3 & -1 \\ -3 & 3 & -1 \end{bmatrix}$$

and the vectors $(1,1,0), (0,1,3)$ are a basis for the kernel.

The set $\{(1,1,1), (1,1,0), (0,1,3)\}$ is a basis of $\mathbb{R}^3$ consisting of eigenvectors for $A$, so $A$ is diagonalizable.

c. Find an invertible matrix $Q$ and a diagonal matrix $D$ such that $D = Q^{-1}AQ$.

Solution: Take $Q$ to be the matrix

$$Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Then a computation gives

$$Q^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -2 & 3 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

and

$$Q^{-1}AQ = \begin{bmatrix} 3 & -3 & 1 \\ -2 & 3 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 1 \\ -2 & 3 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 2 \\ 1 & 0 & 6 \end{bmatrix}$$
which is the desired diagonal matrix $D$.

7. Let $V = P_3(\mathbb{R})$ be the space of all polynomials with coefficients in $\mathbb{R}$ having degree at most 3. Define an inner product on $V$ by

$$\langle f, g \rangle = \int_0^1 f(x)g(x) \, dx$$

a. Show that $W = P_2(\mathbb{R})$ is a subspace of $V$.

Solution: $W$ contains the zero polynomial, and if $f, g$ are two polynomials of degree $\leq 2$ then the degree of $cf + g$ is also at most 2. Therefore $cf + g \in W$, so $W$ is a subspace.

b. Let $\alpha = \{1, X, X^2\}$. Use the Gram-Schmidt process on $\alpha'$ to find an orthogonal basis $\alpha$ of $W$.

Solution: Let $u_1 = 1$. Next, define

$$u_2 = X - \frac{\langle X, u_1 \rangle}{||u_1||^2} u_1 = X - \frac{1}{2}$$

Finally, set

$$u_3 = X^2 - \frac{\langle X^2, u_1 \rangle}{||u_1||^2} u_1 - \frac{\langle X^2, u_2 \rangle}{||u_2||^2} u_2 = X^2 - X + \frac{1}{6}$$

Then $\alpha = \{u_1, u_2, u_3\}$ is an orthogonal basis for $W$.

c. Extend $\alpha$ to an orthogonal basis $\beta$ of $V$.

Solution: $\beta' = \{1, X, X^2, X^3\}$ is a basis for $V$, so we continue the Gram-Schmidt process one more step to get an orthogonal basis $\beta$. So take

$$u_4 = X^3 - \frac{\langle X^3, u_1 \rangle}{||u_1||^2} u_1 - \frac{\langle X^3, u_2 \rangle}{||u_2||^2} u_2 - \frac{\langle X^3, u_3 \rangle}{||u_3||^2} u_3$$

$$= X^3 - \frac{3}{2}X^2 + \frac{3}{5}X - \frac{1}{20}$$

Then $\beta = \{u_1, u_2, u_3, u_4\}$ is an orthogonal basis for $V$.

8. Let $V$ be a three-dimensional vector space and $T$ a linear operator on $V$. Show that there is a basis $\beta$ for $V$ such that $[T]_\beta$ is upper triangular if and only if there are subspaces $V_1, V_2 \subset V$ such that

(i). $V_1$ and $V_2$ are stable under $T$ (i.e. $T(V_i) \subset V_i$);

(ii). $\{0\} \subset V_1 \subset V_2 \subset V$. 

Solution: Suppose that there is a basis $\beta = \{v_1, v_2, v_3\}$ for which $[T]_\beta$ is upper triangular, say
\[
[T]_\beta = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}
\]
Let $V_1 = \text{span}\{v_1\}$ and $V_2 = \text{span}\{v_1, v_2\}$. Then $\{0\} \subsetneq V_1 \subsetneq V_2 \subsetneq V$ since $V_1$ is a one-dimensional vector space and $V_2$ is a two-dimensional vector space.

Moreover, we know that
\[
[v_1]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [v_2]_\beta = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\]
Therefore by using the matrix representation of $T$, we see that $Tv_1 = a_{11}v_1$ and $Tv_2 = a_{12}v_1 + a_{22}v_2$. Thus $V_1$ and $V_2$ are stable under $T$.

For the converse, suppose that there are subspaces $V_1, V_2$ of $V$ such that $V_1$ and $V_2$ are stable under $T$ and $\{0\} \subsetneq V_1 \subsetneq V_2 \subsetneq V$. By counting dimensions, we see that $V_1$ must have dimension 1 and $V_2$ must have dimension 2.

Choose a non-zero vector $v_1 \in V_1$. Then $V_1 = \text{span}\{v_1\}$ since $V_1$ has dimension 1, and since $V_1$ is stable under $T$, we know that $Tv_1 \in V_1 = \text{span}\{v_1\}$, i.e. $Tv_1 = a_{11}v_1$ for some scalar $a_{11}$.

Next, since $V_2$ has dimension 2 we may choose $v_2 \in V$ such that $v_2 \not\in \text{span}\{v_1\}$. Then $\{v_1, v_2\}$ is a basis for $V_2$. And $Tv_2 \in V_2 = \text{span}\{v_1, v_2\}$ since $V_2$ is stable under $T$, hence we can write $Tv_2 = a_{12}v_1 + a_{22}v_2$ for some scalars $a_{12}, a_{22}$.

Finally, choose $v_3 \in V$ not in $V_2 = \text{span}\{v_1, v_2\}$. Then $\beta = \{v_1, v_2, v_3\}$ is a basis for $V$, and $Tv_3 = a_{13}v_1 + a_{23}v_2 + a_{33}v_3$ for scalars $a_{13}, a_{23}, a_{33}$. Thus
\[
[T]_\beta = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}
\]
is upper triangular, as desired.