A Formalism of Mixed Sheaves in Positive Characteristic

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Joint work with Shane Kelly.
Motivation

To a complex variety $X/\mathbb{C}$ one associates categories of:

- Mixed sheaves on $X$ (mixed $\ell$-adic sheaves, mixed Hodge modules, ...)
- Constructible sheaves on $X$ ($\mathbb{C}(X)$ equipped with metric topology)
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To a complex variety $X/\mathbb{C}$ one associates categories of:

- **Constructible sheaves** on $X(\mathbb{C})$
- $(X(\mathbb{C})$ equipped with metric topology)
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  (mixed $\ell$-adic sheaves, mixed Hodge modules, ...)

- **Constructible sheaves** on $X(\mathbb{C})$
  ($X(\mathbb{C})$ equipped with metric topology)
Motivation

Mixed $\ell$-adic sheaves, mixed Hodge modules come with:

- Grothendieck's six functor formalism ($f_\ast$, $f^\ast$, $f_!$, $f^!$, $\otimes$, $\text{Hom}$)
- Deligne's Yoga of weights
- BBDG/Saito: decomposition theorem for perverse sheaves

But only work with characteristic zero coefficients.
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Mixed \( \ell \)-adic sheaves, mixed Hodge modules come with:

- Grothendieck *six functor formalism* \( (f^*, f_*, f!, f^!, \otimes, \mathcal{H}om) \)
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Frobenius acting on $\mathcal{H}^i_{\text{ét}}(X/\overline{\mathbb{F}}_p, \mathbb{Z}/\ell)$
Our proposal for *mixed sheaves* with coefficients in a field \( \mathbb{k} \) (\( \text{char}\mathbb{k} = p \)): 

Theorem (E.-K. 2016)

There is a system of monoidal, \( \mathbb{k} \)-linear, triangulated categories of motives \( H(X, \mathbb{k}) \) for quasi-projective varieties \( X/\mathbb{F}_p \).

Which has a full six functor formalism (using Ayoub, Cisinski–Déglise), a formalism of weights (after Bondarko), and computes higher Chow groups \( \text{CH}_n(X, 2n-i; \mathbb{k}) \approx \text{Hom}_{H(X, \mathbb{k})}(1_{\mathbb{X}}, 1_{\mathbb{X}}(n)[i]) \) for \( X/\mathbb{F}_p \) smooth (using Geisser-Levine).
Proposal

Our proposal for *mixed sheaves* with coefficients in a field $\mathbb{k}$ ($\text{char } \mathbb{k} = p$):

**Theorem (E.-K. 2016)**

*There is a system of monoidal, $\mathbb{k}$-linear, triangulated categories of motives* 

$$H(X, \mathbb{k})$$

*for quasi-projective varieties $X/\overline{\mathbb{F}}_p$.***
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Our proposal for *mixed sheaves* with coefficients in a field $k$ (char $k = p$):

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There is a system of monoidal, $k$-linear, triangulated categories of motives

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for quasi-projective varieties $X / \overline{F}_p$. Which has

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There is a system of monoidal, \( k \)-linear, triangulated categories of motives

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- a full six functor formalism (using Ayoub, Cisinski–Déglise),
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- and computes higher Chow groups

\[ \text{CH}^n(X, 2n-i; k) \cong \text{Hom}_{H(X, k)}(\mathbb{1}_X, \mathbb{1}_X(n)[i]) \]

for \( X/\overline{F}_p \) smooth (using Geisser-Levine).
$G/\mathbb{k}$ split reductive group, $X^\vee/\overline{\mathbb{F}}_p$ Langlands dual flag variety.
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\[ \text{Derived graded modular category } \mathcal{O} \text{ (subquotient of } \text{Rep}_0^Z(G), \text{ defined by Soergel)} \]
Applications in Representation Theory

$G/\mathbb{k}$ split reductive group, $X^\vee/\overline{\mathbb{F}}_p$ Langlands dual flag variety. Using results of Soergel (2001) and ideas of Soergel–Wendt (2015) we prove:

$$\text{MTDer}_{(B)}(X^\vee/\overline{\mathbb{F}}_p, \mathbb{k}) \sim \text{Der}^b(\mathcal{O}^\mathbb{Z}(G))$$

Stratified mixed Tate motives (full subcategory of $\mathcal{H}(X^\vee, \mathbb{k})$, defined as in Soergel’s talk.)

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\textit{Shadow of graded Finkelberg-Mirkovic conjecture.}