

## MATH 31A DISCUSSION

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### 1. MORE APPLICATIONS OF THE DERIVATIVE

#### 1.1. Recall.

1.1.1. *Concavity.* If  $f'(x)$  is increasing (or  $f''(x) > 0$ ), then  $f$  is *concave up* at  $x$ . If  $f'(x)$  is decreasing (or  $f''(x) < 0$ ), then  $f$  is *concave down* at  $x$ .

1.1.2. *Inflection.* If  $f''(c) = 0$  and  $f''(x)$  changes sign at  $x = c$ , then  $f(x)$  has a point of inflection at  $x = c$ .

1.1.3. *Second Derivative Test.* Let  $f$  be differentiable and  $c$  a critical point ( $f'(c) = 0$ ).

- (a) If  $f''(c) > 0$  then  $f(c)$  is a local minimum.
- (b) If  $f''(c) < 0$  then  $f(c)$  is a local maximum.
- (c) If  $f''(c) = 0$  then it is inconclusive,  $f(c)$  may be a local min, max, or neither.

1.2. **Exercise 4.4.24.** Find the critical points of  $f(x) = \sin^2 x + \cos x$ ,  $x$  in  $[0, \pi]$ , and use the Second Derivative Test to determine whether each corresponds to a local minimum or maximum.

*Solution.* Notice  $f'(x) = 2 \sin x \cos x - \sin x$ . Setting  $f'(x) = 0$  and solving, we get  $\sin x(2 \cos x - 1) = 0$ , so  $x = 0, \pi/3, \pi$ . Now  $f''(x) = 2 \cos^2 x - 2 \sin^2 x - \cos x$ . So  $f''(0) = 1$ ,  $f''(\pi/3) = -\frac{3}{2}$ ,  $f''(\pi) = 3$ , yielding local minima at  $x = 0, \pi$  and maximum at  $x = \pi/3$ .  $\square$

1.3. **Exercise 4.4.53.** If  $f'(c) = 0$  and  $f(c)$  is neither a local min or max, must  $x = c$  be a point of inflection? This is true of most “reasonable” examples, but it is not true in general. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

- Use the limit definition of the derivative to show that  $f'(0)$  exists and  $f'(0) = 0$ .
- Show that  $f(0)$  is neither a local min nor max.
- Show that  $f'(x)$  changes sign infinitely often near  $x = 0$  and conclude that  $f(x)$  does not have a point of inflection at  $x = 0$ .

*Solution.* Recall Exercise 3.7.92 from 10/20.

Recall  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ . So by definition,  $f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} h \sin \frac{1}{h}$ . Using Squeeze Theorem and  $-|h| \leq h \sin \frac{1}{h} \leq |h|$ , we get that  $f'(0) = 0$ . Away from  $x = 0$ , we can use the formula and get  $f'(x) = 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \cdot (-1) \frac{1}{x^2} = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$ . Now  $\lim_{x \rightarrow 0} f'(x)$  does not exist since  $\lim_{x \rightarrow 0} 2x \sin \frac{1}{x} = 0$  by Squeeze Theorem but  $\lim_{x \rightarrow 0} \cos \frac{1}{x}$  does not exist.  $\square$

1.4. **Exercise 4.5.59.**

1.5. **Exercise 4.5.68.**

1.6. **Bonus Question.** Assume  $f''(x)$  exists and  $f''(x) > 0$  for all  $x$ . Show that  $f(x)$  cannot be always negative.

*Solution.* If  $f'(x) \equiv 0$ , then  $f''(x) \equiv 0$ , a contradiction. So there exists  $b$  such that  $f'(b) \neq 0$ . Consider the tangent line at  $x = b$  to  $f(x)$ . It is given by the equation  $y = f'(b)(x - b) + f(b)$ . Consider  $g(x) = f(x) - f'(b)(x - b) - f(b)$ . Notice that  $g'(x) = f'(x) - f'(b)$  and  $g''(x) = f''(x)$ . So  $g(b) = g'(b) = 0$ , and  $g''(x) > 0$  for all  $x$ . Hence  $g'(x)$  is increasing. In particular,  $g'(x) < 0$  for  $x < b$  and  $g'(x) > 0$  for  $x > b$ . If  $x > b$ , then by MVT, we get

$$\frac{g(x) - g(b)}{x - b} = g'(c)$$

for some  $c$  in the interval  $(b, x)$ . In otherwords, since  $c > b$ , we have that  $g'(c) > 0$  and  $x - b > 0$ , hence  $g(x) - g(b) > 0$ . Similarly, if  $x < b$ , we get  $g'(c) < 0$ ,  $x - b < 0$ , so  $g(x) - g(b) > 0$  as well. We thus conclude  $g(x) \geq g(b)$  for all  $x$ . So  $f(x) \geq f'(b)(x - b) + f(b)$  for all  $x$ . Since  $f'(b) \neq 0$ , there exists  $x$  far enough from the origin such that  $f'(b)(x - b) + f(b) > 0$ , as desired.  $\square$