Math 3A Final Practice Problems

1. Sequences
Let the sequence $\{a_n\}$ be defined explicity by $a_n = 2^n + 1$ on the domain $n = 0, 1, 2, \ldots$.

a. Rewrite $\{a_n\}$ as a recursively defined sequence.

Answer:

$$\begin{aligned} a_n &= 2^n + 1 \\ &= 2 \cdot 2^{n-1} + 1 \\ &= 2 \cdot 2^{n-1} + 2 - 1 \\ &= 2(2^{n-1} + 1) - 1 \\ &= 2 \cdot a_{n-1} - 1 \end{aligned}$$

with $a_0 = 2^0 + 1 = 2$.

b. Rewrite $\{a_n\}$ as an explicitly defined sequence on the the domain $n = 1, 2, 3, \ldots$.

Answer: $a_n = 2^{n-1} + 1$, $n = 1, 2, 3, \ldots$.

2. Continuity
What does it mean for a function $f(x)$ to be continuous at $x = c$? What does it mean for a function to be continuous everywhere?

Answer: A function $f(x)$ is continuous at $x = c$ if $\lim_{x \to c} f(x) = f(c)$. A function is continous everywhere, if the function is continuous at every $x \in \mathbf{R}$.

3. Sandwich Theorem
When we used the sandwich theorem to compute

$$\lim_{x \to 0} \frac{\sin x}{x}$$

we used some tricky geometry to get the correct inequality. Why didn't we use the fact that $-1 \leq \sin(x) \leq 1$?

Answer: Because then we'd end up with $-\frac{1}{x} \leq \sin(x) \leq \frac{1}{x}$, but $\lim_{x \to 0} \pm\frac{1}{x}$ doesn't exist (left- and right-handed limits differ), so the Sandwich theorem tells us nothing.

4. Mean Value Theorem
Use the Mean Value Theorem to show that $f(x) = x^2 - x + 2$ has a critical point somewhere in the interval $(0, 1)$.

Answer: $f(0) = 2 = f(1)$, so

$$\frac{f(1) - f(0)}{1 - 0} = 0$$

Since $f(x)$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$, the MVT tells us that there exists a point $c \in (0, 1)$ such that $f'(c) = \frac{f(1) - f(0)}{1 - 0} = 0$. Thus c is a critical point.

5. Some limits
Compute the following limits

a.

$$\lim_{h\to 0} \frac{2^{5h}-1}{h}$$

Answer: Let $u = 5h$. Then $u \to 0$ as $h \to 0$, so

$$\begin{aligned}\lim_{h\to 0} \frac{2^{5h}-1}{h} &= \lim_{u\to 0} \frac{2^{u}-1}{u/5} \\ &= 5 \cdot \lim_{u\to 0} \frac{2^{u}-1}{u} \\ &= 5 \ln 2\end{aligned}$$

b.

$$\lim_{x\to \infty} \frac{3x-2x^3}{2x^2+4}$$

Answer:

$$\begin{aligned}\lim_{x\to \infty} \frac{3x-2x^3}{2x^2+4} &= \lim_{x\to \infty} \frac{3/x-2x}{2+4/x^2} \\ &= \lim_{x\to \infty} \frac{2x}{2} = \infty\end{aligned}$$

c.

$$\lim_{x\to \infty} e^{-x}$$

Answer: 0

6. Some derivatives
Compute the derivative of each of the following functions

a.

$$3^{\sin(x)}$$

Answer:

$$\begin{aligned}\frac{d}{dx} 3^{\sin(x)} &= \frac{d}{dx} e^{\sin(x)\ln 3} \\ &= \ln(3)\cos(x) e^{\sin(x)\ln 3} \\ &= \ln(3)\cos(x) 3^{\sin(x)}\end{aligned}$$

b.

$$\tan(\cos(x))$$

Answer:

$$\frac{d}{dx}\tan(\cos(x)) \quad = \quad \sec^2(\cos(x)) \cdot (-\sin(x))$$

c.

$$\log_2(x)$$

Answer:

$$\begin{aligned} \frac{d}{dx}\log_2(x) &= \frac{d}{dx}\frac{\ln(x)}{\ln(2)} \\ &= \frac{1}{\ln(2)\cdot x} \end{aligned}$$

7. L'Hospital's rule Compute the following limits.

a.

$$\lim_{x\to\infty}\frac{x^2+2}{3x-x^3}$$

Answer:

$$\begin{aligned} \lim_{x\to\infty}\frac{x^2+2}{3x-x^3} &= \lim_{x\to\infty}\frac{2x}{3-3x^2} \\ &= \lim_{x\to\infty}\frac{2}{6x} = 0 \end{aligned}$$

b.

$$\lim_{x\to\infty}[\ln x]^{1/x}$$

Answer:

$$\begin{aligned} \lim_{x\to\infty}[\ln x]^{1/x} &= \lim_{x\to\infty}\exp[(1/x)\cdot\ln(\ln x)] \\ &= \exp\Big[\lim_{x\to\infty}\frac{\ln(\ln(x))}{x}\Big] \\ &= \exp\Big[\lim_{x\to\infty}\frac{1/(x\ln x)}{1}\Big] \\ &= \exp\Big[\lim_{x\to\infty}\frac{1}{x\ln x}\Big] \\ &= \exp(0) \\ &= 1 \end{aligned}$$

8. Antidifferentiation
Consider the differential equation $\frac{dy}{dx} = \frac{1}{x}$.

a. Find the general solution to this differential equation.

Answer: $y = \ln x + C$

b. Find the solution that fits the initial condition $y(1) = 1$.

Answer: $y(1) = 0 + C = 1$, so $C = 1$ and $y = \ln x + 1$.

9. More Antiderivatives Antidifferentiate the following functions:

a.

$$\frac{1}{1+x}$$

Answer: $\ln(1+x)$

b.

$$2e^{2x}$$

Answer: e^{2x}

c.

$$\cos(\frac{x}{5})$$

Answer: $5\sin(\frac{x}{5})$