Solutions to Homework problems 9

5.4.6 Let x and y denote the lengths of the short sides of the triangle. Then $A = \frac{1}{2}xy$ is the area of the triangle and $\sqrt{x^2+y^2}$ is the length of the hypotenuse which is given as 4. We use this to eliminate y, so $y = \sqrt{16-x^2}$. We thus want to maximize $A = \frac{1}{2}x\sqrt{16-x^2}$ for $0 \le x \le 4$. We notice that A is 0 for $x = 0$ and $x = 4$ and A is differentiable for all $x \in (0,4)$ so the global maximum of A must occur at a point where $\frac{dA}{dx} = 0$. Now

$$\frac{dA}{dx} = \frac{1}{2}\left(\sqrt{16-x^2} + \frac{x\cdot(-2x)}{2\sqrt{16-x^2}}\right) = \frac{1}{2\sqrt{16-x^2}}(16-x^2-x^2)$$
$$= \frac{1}{\sqrt{16-x^2}}(8-x^2).$$

From this we see that $\frac{dA}{dx} = 0$ when $x^2 = 8$, that is when $x = 2\sqrt{2}$. Then $y = 2\sqrt{2}$ so the short sides have equal length and $A = 4$.

5.4.10 Consider the top left corner of the rectangle. If that point has x-coordinate x then its y-coordinate is given by $y = \sqrt{4-x^2}$ since it lies on the graph of that function. Thus the area of the rectangle is given by $A = 2x\sqrt{4-x^2}$ and we are looking for the maximum area A. From this point on the question is very similar to the last question.

5.4.12 We are looking for the point where the curve $y = \frac{1}{x}$ comes closest to the origin. A point on this curve has coordinates $(x, \frac{1}{x})$ and the square of its distance from the origin is given by $D = x^2 + \frac{1}{x^2}$. Now if you draw a sketch of $y = \frac{1}{x}$ then it is clear that $\lim_{x\to 0^+} D = \infty$ and $\lim_{x\to\infty} D = \infty$ so since D is a differentiable function on $(0,\infty)$ we see that the minimum distance must occur at a point where $\frac{dD}{dx} = 0$. Now

$$\frac{dD}{dx} = 2x - 2\frac{1}{x^3} = \frac{2}{x^3}(x^2-1)$$

so $\frac{dD}{dx} = 0$ only when $x = 1$ (since $x > 0$). Therefore the point on $y = \frac{1}{x}$ which comes closest to the origin is $(1,1)$ and its distance from the origin is $\sqrt{2}$.

5.4.18 We are studying a circular sector with area A, radius r and angle θ. We know that $A = \frac{1}{2}r^2\theta$ and the perimeter of the sector is $P = 2r + r\theta$. We want to minimize P. We note that from the formula for the area we can see that $\theta = \frac{2A}{r^2}$ so $P = 2r + \frac{2Ar}{r^2} = 2r + \frac{2A}{r}$.

Note we must have $\theta \le 2\pi$ which implies $\frac{2A}{r^2} \le 2\pi$ or $\sqrt{\frac{A}{\pi}} \le r$.

Now, $\frac{dP}{dr} = 2 - \frac{2A}{r^2}$ so $\frac{dP}{dr} = 0$ when $r = \sqrt{A}$. We therefore have two candidates for the minimum of P. They are firstly $r = \sqrt{\frac{A}{\pi}}$ which gives

$$P = 2\sqrt{\frac{A}{\pi}} + \frac{2A}{\sqrt{A/\pi}} = 2\sqrt{\frac{A}{\pi}} + 2\sqrt{A\pi} = 2\left(\sqrt{\frac{1}{\pi}} + \sqrt{\pi}\right)\sqrt{A}$$

and secondly $r = \sqrt{A}$ which gives

$$P = 2\sqrt{A} + 2\frac{A}{\sqrt{A}} = 4\sqrt{A}.$$

It is easy to check (using a calculator) that $2\left(\sqrt{\frac{1}{\pi}} + \sqrt{\pi}\right) > 4$ so the second choice gives the lower value of P. From this we deduce that $r = \sqrt{A}$ and $\theta = \frac{2A}{A} = 2$ gives the smallest value of the perimeter.

5.5.2 $\lim_{x\to 1} \frac{x-1}{x^2-1}$ is of the form $\frac{0}{0}$ so we can use L'Hospital's rule to calculate $\lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{1}{2x} = \frac{1}{2}$

5.5.6 $\lim_{x\to 0} \frac{3-\sqrt{2x+9}}{2x}$ is of the form $\frac{0}{0}$ so we can use L'Hospital's rule to calculate

$$\begin{aligned}\lim_{x\to 0} \frac{3-\sqrt{2x+9}}{2x} &= \lim_{x\to 0} \frac{-\frac{2}{2\sqrt{2x+9}}}{2} \\ &= \frac{-\frac{1}{\sqrt{2\cdot 0+9}}}{2} = -\frac{1}{6}\end{aligned}$$

5.5.12 $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$ is of the form $\frac{\infty}{\infty}$ so we can use L'Hospital's rule to calculate

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$

5.5.18 $\lim_{x\to\infty} \frac{x^4}{e^x}$ is of the form $\frac{\infty}{\infty}$ so we can use L'Hospital's rule. In fact we must use it several times to get to a manageable expression

$$\lim_{x\to\infty} \frac{x^4}{e^x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{4x^3}{e^x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{4\cdot 3x^2}{e^x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{4\cdot 3\cdot 2x}{e^x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{4\cdot 3\cdot 2}{e^x} = 0$$

5.5.32 $\lim_{x\to 0^+} \left(\frac{1}{\sin^2 x} - \frac{1}{x}\right)$ is of the form $\infty - \infty$. We try and convert it into a fraction.

$$\lim_{x\to 0^+} \left(\frac{1}{\sin^2 x} - \frac{1}{x}\right) = \lim_{x\to 0^+} \frac{x-\sin^2 x}{x\sin^2 x} \quad = \lim_{x\to 0^+} \frac{1-2\sin x\cos x}{\sin^2 x + 2x\sin x\cos x} = +\infty.$$

We arrived at the final equality since for x positive but close to 0, both the numerator and denominator are positive numbers but as x approaches 0 we see that the numerator approaches 1 whereas the denominator approaches 0 so the limit is $+\infty$.

5.5.34 $\lim_{x\to 0^+} x^{\sin x}$ is of the form 0^0. We rewrite $x^{\sin x} = \exp(\sin x \ln x)$. Since the exponential function is a continuous function we concentrate on the limit of the function inside exponential function which is $\lim_{x\to 0^+}(\sin x \ln x)$ and is of the form $0 \cdot \infty$. We proceed as follows:

$$\begin{aligned}\lim_{x\to 0^+}(\sin x \ln x) &= \lim_{x\to 0^+} \frac{\ln x}{\csc x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to 0^+} \frac{1/x}{-\csc x \cot x} = \lim_{x\to 0^+} \frac{1}{-x \csc x \cot x} \\ &= \lim_{x\to 0^+} -\frac{\sin^2 x}{x \cos x} = \lim_{x\to 0^+} -\frac{\sin x}{x} \sin x \frac{1}{\cos x} = -1 \cdot 0 \cdot 1 = 0.\end{aligned}$$

Thus $\lim_{x\to 0^+} x^{\sin x} = \exp(0) = 1$.