Math 3A Homework #8 solutions

5.3:12 Find the local and global extrema, and where the function is increasing and decreasing for $y = e^{-x^2/4}$, $x \in \mathbf{R}$.

Answer: With $y' = -\frac{1}{2}xe^{-x^2/4}$, we find one critical point at $x = 0$. When $x < 0$, $y' > 0$, and when $x > 0$, $y' < 0$. So the function is increasing left of zero, and deacreasing right of zero. Thus $x = 0$ is a local maximum. $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$, so $x = 0$ is also a global maximum. Since the minima are attained only at $\pm\infty$, there is no global minimum.

5.3:16 Find the local and global extrema, and where the function is increasing and decreasing for $y = \sqrt{1+x^2}$.

Answer: With $y' = \frac{x}{\sqrt{1+x^2}}$, we find one critical point when $x = 0$. Since the denominator is everywhere positive, we find that the function is decreasing left of zero and increasing right of zero. Thus $x = 0$ is a local minimum. With $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty$, we see that the function has no global maxima, but does have a global minimum at $x = 0$.

5.3:20 Find the inflection points of $f(x) = \cos x$, $0 \leq x \leq \pi$.

Answer: $f'(x) = -\sin x$ and $f''(x) = -\cos x$, so the concavity goes to zero when $-\cos x = 0$, which means $x = \frac{\pi}{2}$. Since $-\cos(x) < 0$ when x is slightly less than $\frac{\pi}{2}$ and $-\cos(x) > 0$ when x is slightly greater than $\frac{\pi}{2}$. This means the concavity switches signs, and thus $\frac{\pi}{2}$ is an inflection point.

5.3:24 Find the inflection points of $f(x) = \ln x + \frac{1}{x}$, $x > 0$.

Answer: $f'(x) = \frac{1}{x} - \frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3} - \frac{1}{x^2}$, so the concavity goes to zero when $\frac{2}{x^3} = \frac{1}{x^2}$, which means $x = 2$. We see that when x is slightly less than 2, the function is concave up. And when $x > 2$, the second derivative is negative. So the concavity changes at $x = 2$ and so $x = 2$ is an inflection point.

5.3:36 Let

$$f(x) = -\frac{2}{x^2-1}, \quad x \neq -1, 1$$

Answer: Since x only appears as an even power, the limits to $\pm\infty$ are the same.

$$\lim_{x\to\pm\infty} -\frac{2}{x^2-1} = -\lim_{x\to\pm\infty} \frac{2/x^2}{1-1/x^2} = 0$$

And now:

$$\begin{aligned}
\lim_{x\to-1^-} -\frac{2}{x^2-1} &= \lim_{x\to-1^-} -\frac{2}{(x-1)(x+1)} \\
&= -\frac{2}{-1-1} \lim_{x\to-1^-} \frac{1}{1+x} \\
&= -\infty
\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to -1^+} -\frac{2}{x^2-1} &= \lim_{x\to -1^+} -\frac{2}{(x-1)(x+1)}\\ &= -\frac{2}{-1-1}\lim_{x\to -1^+}\frac{1}{1+x}\\ &= \infty\end{aligned}$$

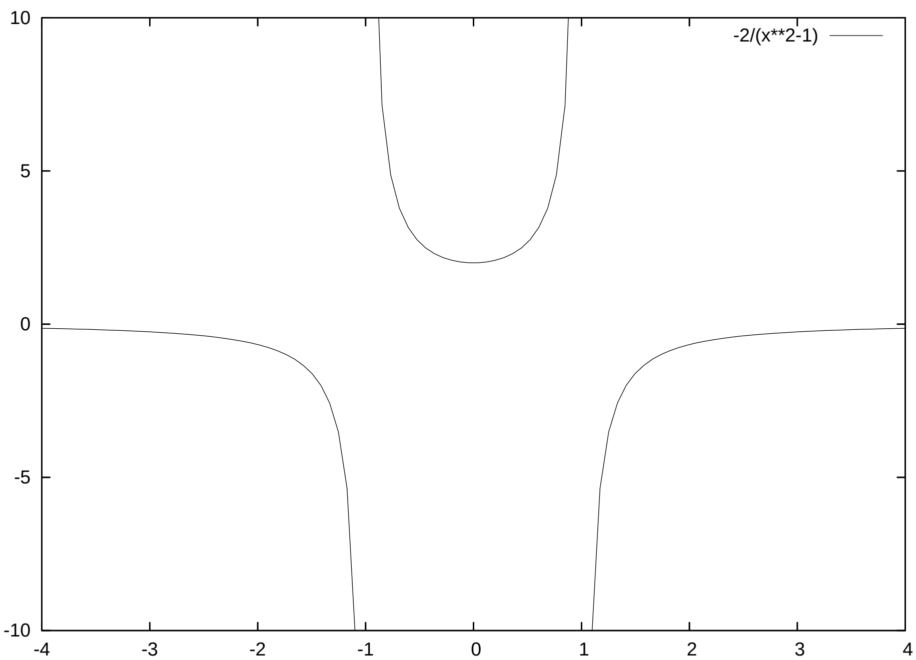
and

$$\begin{aligned}\lim_{x\to 1^-} -\frac{2}{x^2-1} &= \lim_{x\to 1^-} -\frac{2}{(x-1)(x+1)}\\ &= -\frac{2}{1+1}\lim_{x\to 1^-}\frac{1}{1-x}\\ &= \infty\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to 1^+} -\frac{2}{x^2-1} &= \lim_{x\to 1^+} -\frac{2}{(x-1)(x+1)}\\ &= -\frac{2}{1+1}\lim_{x\to 1^+}\frac{1}{1-x}\\ &= -\infty\end{aligned}$$

$f'(x) = \frac{4x}{(x^2-1)^2}$ and $f'\prime = \frac{4(x^2-1)^2-(8x)(x^2-1)(2x)}{(x^2-1)^4} = -\frac{12x^2+4}{(x^2-1)^3}$. Critical points at $x = \pm 1$ (undefined) and $x = 0$ (zero). Increasing when $x > 0$ and decreasing when $x < 0$. Concave down when $x^2 - 1 < 0$ or $|x| > 1$. And concave up when $|x| < 1$. No inflection points, since concavity changes where $f(x)$ is undefined.



5.3:42 Let

$$f(x) = \frac{x^2}{a^2+x^2}, \quad x \geq 0$$

where a is a positive constant.

Answer:

a. $f'(x) = \frac{(2x)(a^2+x^2)-(x^2)(2x)}{(a^2+x^2)^2} = \frac{2ax}{(a^2+x^2)^2}$ which is negative for $x < 0$ and positive for $x > 0$. So it is increasing to the right of zero and decreasing to the left. Since the domain of the function is $x \geq 0$, the function is increasing on the whole domain.

b. $f''(x) = \frac{2a^2-6a^2x^2}{(a^2+x^2)^3}$. So the concavity goes to zero when $2a^4 = 6a^2x^2$ or $x = a\sqrt{\frac{1}{3}}$. When $x > a\sqrt{\frac{1}{3}}$, the function is concave down, and when $x < a\sqrt{\frac{1}{3}}$, the function is concave up. Thus $x = a\sqrt{\frac{1}{3}}$ is an inflection point.

c. $\lim_{x\to\infty} \frac{x^2}{a^2+x^2} = \lim_{x\to\infty} \frac{1}{a^2/x^2+1} = 1$. So there is a horizontal asymptote at $y = 1$.

d.