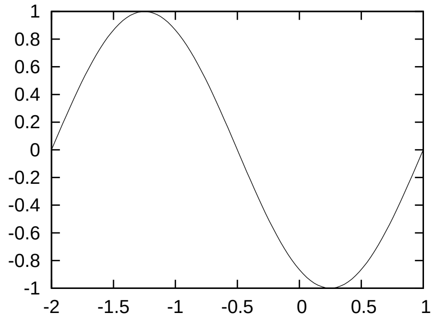
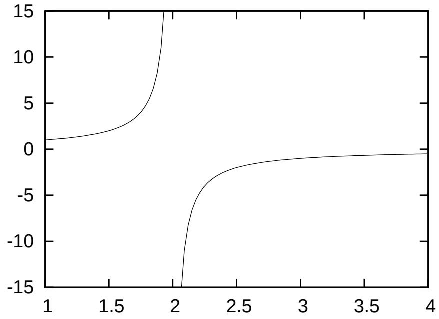
Solutions to Homework problems 7

5.1.10

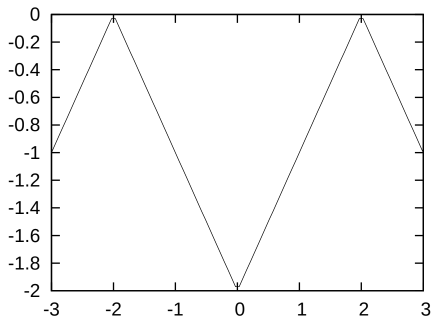


5.1.12



5.1.20 We note that $f(x) = (x-1)^2 \geq 0$ for all $x \in \mathbf{R}$ but $f(1) = 0$ and if $x \neq 1$ then $f(x) > 0$. Therefore $f(x)$ has a global (and thus a local) minimum at $x = 1$. Then we calculate $f'(x) = 2(x-1)$ so $f'(1) = 0$. Since the derivative was 0 at this point we see that 1 was a candidate point for an extremum.

5.1.32



From the graph we see that $x = \pm 2$ are local and global maxima, $x = \pm 3$ are local minima and $x = 0$ is a global minimum.

5.1.36 Let $f(x) = 1/x$. Then the secant line through $(1, 1)$ and $(2, 1/2)$ has slope $\frac{(1/2)-1}{2-1} = \frac{-1/2}{1} = -\frac{1}{2}$. The derivative of f is $f'(x) = -1/x^2$ and we note that if $c = \sqrt{2}$ then $f'(c) = -1/(\sqrt{2})^2 = -1/2$. Therefore the slope of the tangent line at $c = \sqrt{2}$ is equal to the slope of the secant line through $(1, 1)$ and $(2, 1/2)$. Since the function is continuous in $[1, 2]$ and differentiable in $(1, 2)$ the mean value theorem told us that such a c had to exist.

5.1.42 Let $f(x) = x^3$. Then $f(-1) = -1$ and $f(1) = 1$ so the secant line through $(-1, f(-1))$ and $(1, f(1))$ has slope $\frac{f(1)-f(-1)}{1-(-1)} = \frac{1-(-1)}{1-(-1)} = \frac{2}{2} = 1$. Since f is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$ the mean value theorem tells us that there must exist a point $c \in (-1, 1)$ such that $f'(c) = 1$ (the slope of the tangent line at c equals the slope of the secant line through $(-1, f(-1))$ and $(1, f(1))$).

5.1.46 Let f be a function which is continuous on $[a, b]$ and differentiable on (a, b). Also assume that $f(a) < f(b)$, so $f(b) - f(a) > 0$. We have that $a < b$ (since a is the left endpoint and b the right one) and so $b - a > 0$. Thus the slope of the secant line through $(a, f(a))$ and $(b, f(b))$ which is given by $\frac{f(b)-f(a)}{b-a}$ is positive since both the numerator and denominator are positive.

Now, by the mean value theorem (note that the conditions for the mean value theorem to hold are satisfied), we have that there exists a $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$ and this is a positive number, as we have justified, so for this c we have $f'(c) > 0$.

5.1.52 Let f be differentiable for all $x \in \mathbf{R}$ and assume that $f(0) = 0$ and $1 \leq f'(x) \leq 2$ for all $x > 0$.

We apply Corollary 1 to the mean value theorem to f and the interval $[0, x]$ where $x > 0$. The corollary applies since f is continuous on $[0, x]$

and differentiable for all $y \in (0, x)$ and furthermore $1 \leq f'(y) \leq 2$ for all $y \in (0, x)$ so we can take m and M in the corollary as $m = 1$ and $M = 2$. Then the conclusion of the corollary tells us that

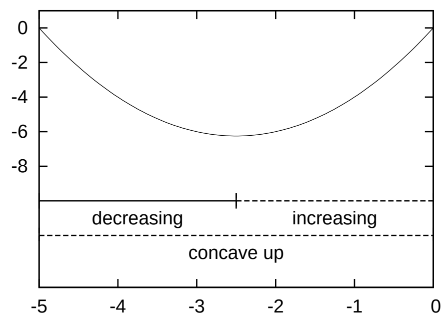
$$1 \cdot (x - 0) \leq f(x) - f(0) \leq 2 \cdot (x - 0)$$

which by simplifying and recalling that $f(0) = 0$ gives

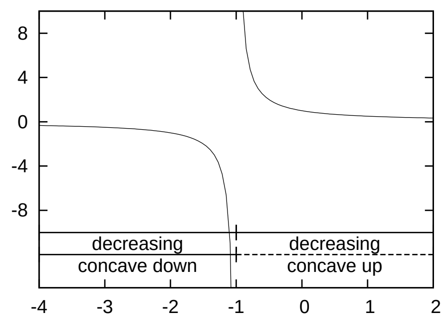
$$x \leq f(x) \leq 2x.$$

In particular we note that if $x = 1$ we get $1 \leq f(1) \leq 2$ so it is not possible that $f(1) = 3$, in fact the upper and lower bound for $f(1)$ is 1 and 2 respectively.

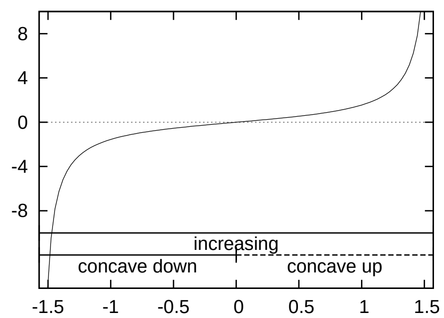
5.2.2 Let $y = x^2 + 5x$ for $x \in \mathbf{R}$. Then $y' = 2x + 5$ and $y'' = 2$. Since $y'' > 0$ for all x we have that the graph is concave up for all x. Since $y' > 0$ if $x > -5/2$ and $y' < 0$ if $x < -5/2$ we have that the function is increasing if $x > -5/2$ and decreasing if $x < -5/2$. We then sketch the graph.



5.2.12 Let $y = \frac{1}{1+x}$ for $x \neq -1$. Then $y' = -\frac{1}{(1+x)^2}$ and $y'' = 2\frac{1}{(1+x)^3}$. Since $y' < 0$ for all $x \neq -1$ we see that the function is decreasing for $x < -1$ and also decreasing for $x > -1$. Note that because of the discontinuity at $x = -1$, the function is not increasing on the whole of $\mathbf{R}$. Since $y'' < 0$ for $x < -1$ we have that the function is concave down for $x < -1$ and since $y'' > 0$ for $x > -1$ we have that the function is concave up for $x > -1$. We then sketch the graph.



5.2.16 Let $y = \tan x$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $y' = \sec^2 x$ and $y'' = 2\sec x \sec x \tan x = 2\sec^2 x \tan x$. Thus we see that $y' > 0$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ whereas $y'' < 0$ for $x \in (-\frac{\pi}{2}, 0)$ and $y'' > 0$ for $x \in (0, \frac{\pi}{2})$. From this we deduce that y is increasing for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, concave down for $x \in (-\frac{\pi}{2}, 0)$ but concave up for $x \in (0, \frac{\pi}{2})$. We then sketch the graph.



5.2.30 Let $f(N) = \frac{aN}{k^2+N^2}$. Then $f'(N) = \frac{a\cdot(k^2+N^2)-aN\cdot 2N}{(k^2+N^2)^2} = a\frac{k^2-N^2}{k^2+N^2}$. We should only consider this function for $N > 0$ (negative density does not make sense) and then $f'(N) > 0$ if $0 < N < k$ and $f'(N) < 0$ if $k < N$. Therefore the predation rate is increasing for $0 < N < k$ and decreasing for $k < N$.