Math 3A Homework #6 solutions

4.7:10 Let $f(x) = \sqrt{5+x^2}$, $x \geq 0$. Find $\frac{d}{dx} f^{-1}(3)$. [Note that $f(2) = 3$.]

Answer: We recall that

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Since $f(2) = 3$, then $f^{-1}(3) = 2$. Also $f'(x) = \frac{x}{\sqrt{5+x^2}}$. So

$$\begin{aligned} \frac{d}{dx} f^{-1}(x) &= \left.\frac{1}{\frac{x}{\sqrt{5+x^2}}}\right|_{x=2} \\ &= \frac{2}{\sqrt{5+2^2}} \\ &= \frac{2}{3} \end{aligned}$$

4.7:20 Let $f(x) = e^{-x^2/2} + x$. Find $\frac{d}{dx} f^{-1}(x)$.

Answer:

$$\frac{d}{dx} f^{-1}(x) = \left.\frac{1}{(-x)e^{-x^2/2} + 1}\right|_{x=f^{-1}(1)}$$

Notice that when $x = 0$, $f(x) = e^0 + 0 = 1$. So $f^{-1}(1) = 0$. Thus

$$\frac{d}{dx} f^{-1}(1) = \frac{1}{(-1)e^{-1^2/2} + 1} = \frac{-1}{e^{-1/2} + 1}$$

4.7:48 Differentiate $f(x) = x^2 \ln x^2$.

Answer:

$$f'(x) = 2x \cdot \ln x^2 + x^2 \cdot \frac{d}{dx} \ln x^2$$

We know that

$$\frac{d}{dx} \ln g(x) = \frac{g'(x)}{g(x)}$$

So then

$$\begin{aligned} f'(x) &= 2x \ln x^2 + x^2 \cdot \frac{2x}{x^2} \\ &= 2x(\ln(x^2) + 1) \end{aligned}$$

4.7:70 Differentiate $f(x) = x^{3/x}$.

Answer: Let $y = x^{3/x}$. Then

$$\begin{aligned}
\ln(y) &= \ln(x^{3/x}) \\
\ln(y) &= \frac{3}{x}\ln(x) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{-3}{x^2}\ln(x) + \frac{3}{x}\cdot\frac{1}{x} \\
\frac{dy}{dx} &= \left(\frac{-3\ln(x)}{x^2} + \frac{3}{x^2}\right)(x^{3/x}) \\
&= 3x^{3/x-2}(1-\ln(x))
\end{aligned}$$

4.7:76 Differentiate

$$y = \frac{e^{x-1}\sin^2 x}{(x^2+5)^{2x}}$$

Answer: While we could differentiate directly, it is easier to take the log of both sides and differentiate implicitly.

$$\begin{aligned}
\ln(y) &= \ln(e^{x-1}) + \ln(\sin^2(x)) - \ln((x^2+5)^{2x}) \\
&= (x-1) + 2\ln(\sin(x)) - 2x\ln(x^2+5) \\
\frac{1}{y}\frac{dy}{dx} &= 1 + 2\frac{\cos(x)}{\sin(x)} - 2\ln(x^2+5) - 2x\frac{2x}{x^2+5} \\
\frac{dy}{dx} &= \left(1 + 2\cot(x) - 2\ln(x^2+5) - \frac{4x^2}{x^2+5}\right)\left(\frac{e^{x-1}\sin^2 x}{(x^2+5)^{2x}}\right)
\end{aligned}$$

4.8:6 Use the formula $f(x) \approx f(a) + f'(a)(x-a)$ to approximate $\sin(0.01)$.

Answer: We use $a = 0$ and the fact that $\frac{d}{dx}\sin x = \cos x$ to get

$$\sin(x) \approx \sin(0) + \cos(0)(x-0) = x$$

So $\sin(0.01) \approx 0.01$. The actual value for $\sin(0.01)$ is $0.0099998\ldots$, so this is a very good approximation.

4.8:10 Use the formula $f(x) \approx f(a) + f'(a)(x-a)$ to approximate $1/0.99$.

Answer: We use $a = 1$ and the fact that $\frac{d}{dx}\frac{1}{x} = \frac{-1}{x^2}$ to get

$$\frac{1}{x} \approx \frac{1}{1} - \frac{1}{1^2}(x-1) = 2 - x$$

So $\frac{1}{0.99} \approx 2 - 0.99 = 1.01$. The actual value for $\frac{1}{0.99}$ is $1.010101\ldots$, so this too is a very good approximation.

4.8:12 Use the formula $f(x) \approx f(a) + f'(a)(x-a)$ to approximate $1/(1-x)$ at a=0.

Answer: With

$$\frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$$

we get

$$\begin{aligned}\frac{1}{1-x} &\approx \frac{1}{1-0}+\frac{1}{(1-0)^2}(x-0)\\ &= 1+x\end{aligned}$$

4.8:46 *See textbook*

Answer: Since v is proportional to R^2, we know that

$$\left|\frac{\Delta v}{v}\right| = 2\left|\frac{\Delta R}{R}\right|$$

So $\left|\frac{\Delta v}{v}\right| = 0.1$ or 10%.

4.8:48 *See textbook*

Answer: Given that $f(R) = a\frac{R}{k+R}$, we have

$$\begin{aligned}\Delta f &\approx f'(R)\Delta R\\ &= a\frac{k}{(k+R)^2}\Delta R\\ \left|\frac{\Delta f}{f}\right| &= \left|\frac{a\frac{k}{(k+R)^2}\Delta R}{a\frac{R}{k+R}}\right|\\ &= \frac{k}{(k+R)}\left|\frac{\Delta R}{R}\right|\end{aligned}$$