Math 3A Homework #5 solutions

4.4:66 Assume that the radius r and the area $A = \pi r^2$ of a circle are differentiable functions of t. Express dA/dt in terms of dr/dt.

Answer:

$$\begin{aligned} \frac{dA}{dt} &= \frac{d}{dt}\pi r^2 \\ &= 2\pi r \frac{dr}{dt} \end{aligned}$$

4.4:70 Suppose that we pump water into an inverted right-circular conical tank at the rate of 5 cubic feet per minute (i.e., the tank stands point down). The tank has a height of 6 ft and the radius on top is 3 ft. What is the rate at which the water level is rising when the water is 2 ft deep. (Note that the volume of a right-circular cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$).

Answer: We know dV/dt ($5\ ft^3/min$) and we want to find dh/dt. So we need to express V as a function of h. We notice that the radius of the tank at a given height h is $r = \frac{1}{2}h$. So we can right $V = \frac{1}{3}\pi(h/2)^2 h = \frac{1}{12}\pi h^3$. Then

$$\begin{aligned} \frac{dV}{dt} &= \frac{1}{12}\pi(3h^2)\frac{dh}{dt} \\ &= \frac{1}{4}\pi h^2 \frac{dh}{dt} \end{aligned}$$

Solving for dh/dt we get

$$\begin{aligned} \frac{dh}{dt} &= \frac{4}{\pi h^2}\frac{dV}{dt} \\ &= \frac{4}{\pi(2ft)^2} 5ft^3/min \\ &= 3.18 ft/min \end{aligned}$$

4.4:76 Find the first and second derivative of $h(s) = 1/(s^2+2)$.

Answer:

$$\begin{aligned} h'(s) &= \frac{d}{ds}(s^2+2)^{-1} = (-1)(s^2+2)^{-2}(2s) = \frac{-2s}{(s^2+2)^2} \\ h''(s) &= \frac{d}{ds}(-2s)(s^2+2)^{-2} = (-2)(s^2+2)^{-2} - 2(-2)(s^2+2)^{-3}(2s) = -\frac{2}{(s^2+2)^{-2}} + \frac{8s}{(s^2+2)^{-3}} \end{aligned}$$

4.4:80 Find the first and second derivative of $f(x) = x/(x+1)$.

Answer:

$$\begin{aligned} f'(x) &= \frac{(x+1)-x}{(x+1)^2} = \frac{1}{(x+1)^2} \\ f''(x) &= (-2)(x+1)^{-3} = -\frac{2}{(x+1)^3} \end{aligned}$$

4.5:28 Differentiate $f(x) = \sqrt{\sin x}$.

Answer: Let $u = g(x) = \sin x$ and $h(u) = \sqrt{u}$. Then $g'(x) = \cos x$ and $h'(u) = \frac{1}{2\sqrt{u}}$. So we get

$$\begin{aligned} f'(x) &= h'(u) \cdot g'(x) \\ &= \frac{1}{2\sqrt{u}} \cdot \cos x \\ &= \frac{1}{2\sqrt{\cos x}} \cdot \cos x \\ &= \frac{1}{2}\sqrt{\cos x} \end{aligned}$$

4.5:38 Differentiate $f(x) = \tan x \cot x$.

Answer: Since $\tan x = \frac{1}{\cot x}$, $f(x) = \frac{\tan x}{\tan x} = 1$. So $f'(x) = 0$.

4.5:66 Differentiate $f(x) = \cos\sqrt{x^2+1}$.

Answer: Let $u = g(x) = \sqrt{x^2+1}$ and $h(u) = \cos u$. Then $g'(x) = \frac{2x}{2\sqrt{x^2+1}}$ and $h'(u) = -\sin u$. So we get

$$\begin{aligned} f'(x) &= -\sin u \cdot \frac{2x}{2\sqrt{x^2+1}} \\ &= -\sin\sqrt{x^2+1} \cdot \frac{2x}{2\sqrt{x^2+1}} \\ &= -\frac{x\sin\sqrt{x^2+1}}{\sqrt{x^2+1}} \end{aligned}$$

4.5:72 Differentiate $f(x) = \sec^2(2x^2-2)$.

Answer: Let $u = g(x) = 2x^2 - 2$ and $h(u) = \sec^2(u)$. Then $g'(x) = 4x$ and $h'(u) = 2\sec(u)\frac{d}{dx}\sec(u) = 2\sec^2(u)\tan(u)$. So then

$$\begin{aligned} f'(x) &= 2\sec^2(2x^2-2)\tan(2x^2-2) \cdot 4x \\ &= 8x\sec^2(2x^2-2)\tan(2x^2-2) \end{aligned}$$

4.6:10 Differentiate $f(x) = 2xe^{3x}$.

Answer:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(2x) \cdot e^{3x} + 2x \cdot \frac{d}{dx}(e^{3x}) \\ &= 2e^{3x} + 6xe^{3x} \end{aligned}$$

4.6:34 Differentiate $f(x) = 3^x$.

Answer: By rule 4.9, we get $f'(x) = (\ln 3)3^x$.

4.6:44 Differentiate $f(x) = 3^{\sqrt{x+1}}$.

Answer:

$$f'(x) = \frac{(\ln 3)}{2\sqrt{x+1}} 3^{\sqrt{x+1}}$$

4.6:56 Compute $\lim_{h \to 0} \frac{2^h - 1}{h}$.

Answer:

$$\begin{aligned} \frac{2^h - 1}{h} &= \frac{e^{\ln(2^h)} - 1}{h} \\ &= \frac{e^{h \ln(2)} - 1}{h} \\ &= \ln(2) \frac{e^{h \ln(2)} - 1}{h \ln(2)} \end{aligned}$$

Now let $u = h \ln(2)$ and notice that as $h \to 0$ $u \to 0$. Then

$$\begin{aligned} \lim_{h \to 0} \frac{2^h - 1}{h} &= \ln(2) \lim_{h \to 0} \frac{e^{h \ln(2)} - 1}{h \ln(2)} \\ &= \ln(2) \lim_{u \to 0} \frac{e^u - 1}{u} \\ &= \ln(2) \end{aligned}$$