Math 3A Homework #4 solutions

4.2:4 Differentiate $f(x) = -3x^4 + 6x^2 - 2$.

Answer:

$$\begin{aligned} \frac{d}{dx}[-3x^4 + 6x^2 - 2] &= (-3)(4)x^3 + (6)(2)x - 0 \\ &= -12x^3 + 12x \end{aligned}$$

4.2:26 Differentiate $f(x) = a^2x^4 - 2ax^2$.

Answer:

$$\begin{aligned} \frac{d}{dx}[a^2x^4 - 2ax^2] &= (a^2)(4)x^3 - (2a)(2)x \\ &= 4a^2x^3 - 4ax \end{aligned}$$

4.2:44 Find the line tangent to $y = -2x^3 - 3x + 1$ at $x = 1$.

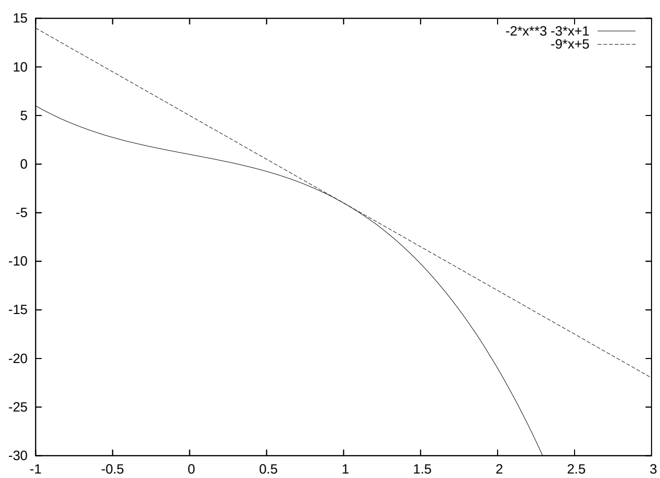
Answer: The slope of the tangent line is

$$m = \frac{dy}{dx} = -6x^2 - 3$$

When $x = 1$, $m = -6(1)^2 - 3 = -9$ and $f(1) = -2(1)^3 - 3(1) + 1 = -4$. Using the point-slope formula, we see

$$(y - (-4)) = -9 \cdot (x - 1)$$

which yields $y = -9x + 5$ for the tangent line. We can verify this graphically:



4.2:50 Find the line normal to $y = 1 - 3x^2$ at $x = -2$.

Answer: The slope of the tangent line is

$$m = \frac{dy}{dx} = -6x$$

When $x = -2$, $m = -6(-2) = 12$ and $f(-2) = 1 - 3(-2)^2 = -11$. The normal line has a slope that is the negative reciprocal of the tangent line's slope, or $-1/12$. Using the point-slope formula, we see

$$(y - (-11)) = -\frac{1}{12} \cdot (x - (-2))$$

which yields $y = -\frac{1}{12}x - \frac{67}{6}$ for the normal line.

4.3:2 Differentiate $f(x) = (2x-1)(2-x^2)$.

Answer: Using the product rule

$$\begin{aligned} \frac{d}{dx}f(x) &= (2x-1)\frac{d}{dx}(2-x^2) + (2-x^2)\frac{d}{dx}(2x-1) \\ &= (2x-1)(-2x) + (2-x^2)(2) \\ &= -6x^2 + 2x + 4 \end{aligned}$$

4.3:48 Differentiate $R(x) = k(a-x)(b-x)$.

Answer: Using the product rule

$$\begin{aligned} \frac{d}{dx}f(x) &= k(a-x)\frac{d}{dx}(b-x) + k(b-x)\frac{d}{dx}(a-x) \\ &= k(a-x)(-1) + k(b-x)(-1) \\ &= k(2x-a-b) \end{aligned}$$

4.3:50 Differentiate $f(x) = \frac{3x^2-2x+1}{2x+1}$.

Answer: By the quotient rule

$$f'(x) = \frac{(2x+1)(6x-2) - (3x^2-2x+1)(2)}{(2x+1)^2} = \frac{6x^2-2x}{(2x+1)^2}$$

4.3:78 Differentiate $f(x) = \frac{ax^2}{k^2+x^2}$.

Answer: By the quotient rule

$$f'(x) = \frac{(k^2+x^2)(2ax) - (ax^2)(2x)}{(k^2+x^2)^2} = \frac{2k^2ax}{(k^2+x^2)^2}$$

4.4:8 Differentiate $h(x) = \sqrt{5x+3x^2}$.

Answer: Let $g(x) = 5x + 3x^2$ and $f(u) = \sqrt{u}$. Then $g'(x) = 5 + 6x$ and $f'(u) = \frac{1}{2\sqrt{u}}$. So then

$$\begin{aligned} h'(x) &= f'(g(x)) \cdot g'(x) \\ &= \frac{5+6x}{2\sqrt{5x+3x^2}} \end{aligned}$$

4.4:18 Differentiate $h(s) = \left(\frac{2s}{s+1}\right)^4$.

Answer: Let $g(s) = \frac{2s}{s+1}$ and $f(u) = u^4$. Then $g'(s) = \frac{2(s+1)-2s}{(s+1)^2} = \frac{2}{(s+1)^2}$ and $f'(u) = 4u^3$. So then

$$\begin{aligned} h'(x) &= f'(g(x)) \cdot g'(x) \\ &= 4\Big(\frac{2s}{(s+1)^2}\Big)^3 \cdot \frac{2}{(s+1)^2} \end{aligned}$$

4.4:34 Let $f'(x) = 2x+1$. Find $\frac{d}{dx}f(x^2)$ at $x = -1$, and $\frac{d}{dx}f(\sqrt{x})$ at $x = 4$.

Answer:

$$\begin{aligned} \frac{d}{dx}f(x^2)\Big|_{x=-1} &= f'(x^2)\cdot(2x)\Big|_{x=-1} \\ &= (2(x^2)+1)\cdot(2x)\Big|_{x=-1} \\ &= (4x^3+2x)\Big|_{x=-1} \\ &= -6 \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dx}f(\sqrt{x})\Big|_{x=4} &= f'(\sqrt{x})\cdot\big(\frac{1}{2\sqrt{x}}\big)\Big|_{x=4} \\ &= (2(\sqrt{x})+1)\cdot\big(\frac{1}{2\sqrt{x}}\big)\Big|_{x=4} \\ &= \big(1+\frac{1}{2\sqrt{x}}\big)\Big|_{x=4} \\ &= \frac{5}{4} \end{aligned}$$

4.4:50 Find $\frac{dy}{dx}$ given $xy - y^3 = 1$.

Answer:

$$\begin{aligned} \frac{d}{dx}xy - \frac{d}{dx}y^3 &= \frac{d}{dx}1 \\ y + x\frac{dy}{dx} - 3y^2\frac{dy}{dx} &= 0 \\ \frac{dy}{dx}(x-3y^2) &= -y \\ \frac{dy}{dx} &= \frac{-y}{x-3y^2} \end{aligned}$$

4.4:56 Find the normal and tangent lines to $\frac{x^2}{4} + \frac{y^2}{9} = 1$ at the point $(1, \frac{3}{2}\sqrt{3})$.

Answer:

$$\begin{aligned} \frac{d}{dx}\frac{x^2}{4} + \frac{d}{dx}\frac{y^2}{9} &= \frac{d}{dx}1 \\ \frac{1}{2}x + \frac{2}{9}y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} = -\frac{9}{4}\frac{y}{x} & \end{aligned}$$

At $(1, \frac{3}{2}\sqrt{3})$, we get the slope of the tangent line to be $m = -\frac{9}{4}\frac{1}{\frac{3}{2}\sqrt{3}} = -\frac{\sqrt{3}}{2}$. Then the slope of the normal line is $\frac{2}{\sqrt{3}}$.
Use the point slope formula to get the equations for the tangent

$$\begin{aligned}(y - \frac{3}{2}\sqrt{3}) &= -\frac{\sqrt{3}}{2}(x-1) \\ y &= \frac{\sqrt{3}}{2}x + 2\sqrt{3}\end{aligned}$$

and the normal

$$\begin{aligned}(y - \frac{3}{2}\sqrt{3}) &= \frac{2}{\sqrt{3}}(x-1) \\ y &= \frac{2}{\sqrt{3}}x + \frac{5}{6}\sqrt{3}\end{aligned}$$

We can confirm these by plotting them: