Math 3A Midterm #2 practice problems

1. The volume of sphere is given by

$$V(r) = \frac{4}{3}\pi r^3$$

Express the rate of change of the volume in terms of the rate of change of the radius.

Answer:

$$\frac{d}{dt}V(r) = 4\pi r^2 \frac{dr}{dt}$$

2. Suppose we measure x to be 20, with an accuracy of 2%. What is the accuracy of the calculation of f, if f is given by $f(x) = \ln x$.

Answer: We can make the linear approximation for $f(x)$ as

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

Letting x_0 be the actual value of the measured quantity, we have that $\Delta x = 20 - x_0$ and

$$\Delta f = f(x) - f(x_0) \approx f'(x_0)\Delta x$$

With $f(x) = \ln x$, we have $f'(x) = 1/x$ and thus

$$\begin{aligned}
\Delta f &= \frac{1}{x_0}\Delta x \\
\left|\frac{\Delta f}{f}\right| &= \frac{1}{\ln(x)}\left|\frac{\Delta x}{x_0}\right| \\
&= \frac{0.02}{\ln(20)} = 0.00668 = 0.668\%
\end{aligned}$$

3. Compute the derivative of each of the following functions:

a.

$$f(x) = 2^{x^3}$$

Answer:

$$\begin{aligned}
f(x) &= \exp[\ln(2^{x^3})] = e^{x^3 \ln(2)} \\
f'(x) &= 3\ln(2)x^2 e^{x^3 \ln(2)} \\
&= 3\ln(2)x^2 2^{x^3}
\end{aligned}$$

b.

$$f(x) = \log_{10}(e^x + 1)$$

Answer:

$$\begin{aligned} f(x) &= \frac{1}{\ln(10)}\ln(e^x+1) \\ f'(x) &= \frac{1}{\ln(10)}\frac{e^x}{e^x+1} \end{aligned}$$

c.

$$f(x) = x^{3x+1}$$

Answer: Let $y = x^{3x+1}$. Then

$$\begin{aligned} \ln(y) &= \ln(x^{3x+1}) \\ \frac{1}{y}\frac{dy}{dx} &= \frac{d}{dx}[(3x+1)\ln(x)] \\ &= 3\ln(x) + (3x+1)\frac{1}{x} \\ \frac{dy}{dx} &= \left[3(\ln(x)+1) + \frac{1}{x}\right]x^{3x+1} \end{aligned}$$

d.

$$f(x) = \frac{3x^2+2}{x-3}$$

Answer:

$$\begin{aligned} f'(x) &= \frac{(x-3)(6x)-(3x^2+2)(1)}{(x-3)^2} \\ &= \frac{3x^2-18x+2}{(x-3)^2} \end{aligned}$$

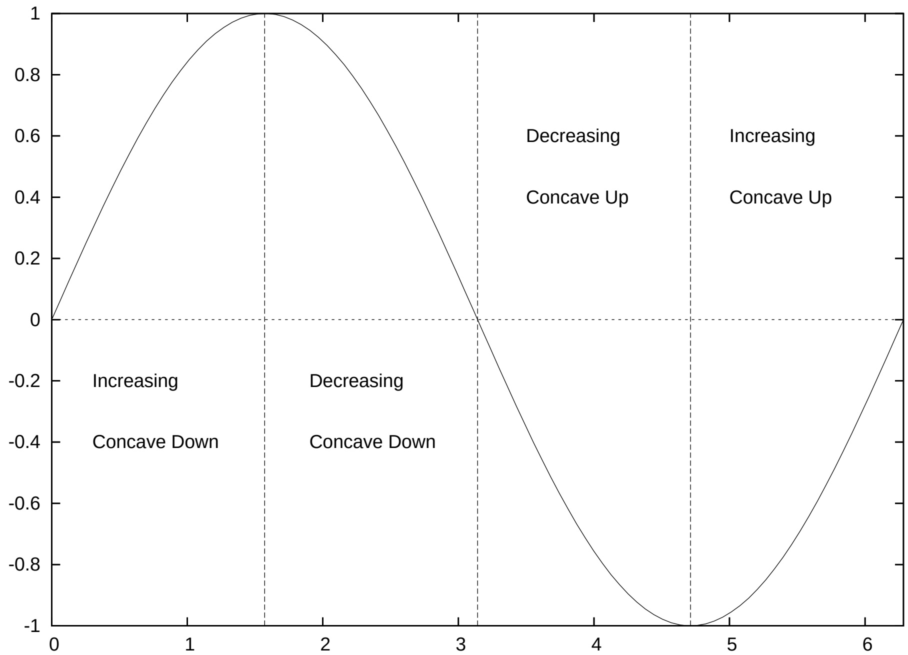
4. Find the linear approximation of $f(x) = \sqrt{x}$ at $x = 1$.

Answer:

$$\begin{aligned} f(x) &\approx f(1) + f'(1)(x-1) \\ &= \sqrt{1} + \frac{1}{2\sqrt{1}}(x-1) \\ &= 1 + \frac{1}{2}(x-1) \\ &= \frac{x+1}{2} \end{aligned}$$

5. Sketch the graph of $y = \sin(x)$ on the domain $[0, 2\pi]$ and determine where the function is increasing, decreasing, concave up, and concave down.

Answer:



6. Let $f(x) = \frac{4}{3x^2}$.

a. Differentiate $f(x)$ using the quotient rule.

Answer:

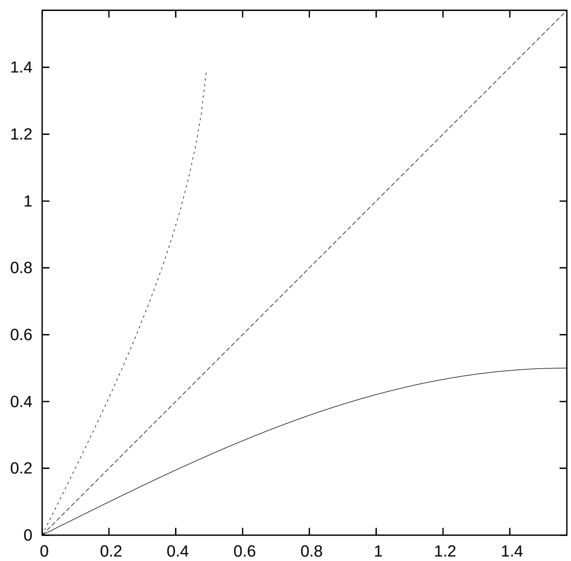
$$\begin{aligned} f'(x) &= \frac{(3x^2)(0) - (4)(6x)}{9x^4} \\ &= -\frac{24x}{9x^4} \\ &= -\frac{8x}{3x^3} \end{aligned}$$

b. Differentiate $f(x)$ using the power rule.

Answer:

$$\begin{aligned} f'(x) &= \frac{4}{3}(-2)x^{-3} \\ &= -\frac{8}{3x^3} \end{aligned}$$

7. Below is a plot of $f(x) = \frac{1}{2}\sin(x)$.



a. On the above plot, draw the graph of $f^{-1}(x)$.

b. Compute $f'(\frac{\pi}{4})$.

Answer:

$$\begin{aligned} f'(x) &= \frac{1}{2}\cos(x) \\ f'(\frac{\pi}{4}) &= \frac{1}{2}\cos(\frac{\pi}{4}) = \frac{1}{2\sqrt{2}} \end{aligned}$$

c. Compute the derivative of $f^{-1}(x)$ at $x = \frac{1}{2\sqrt{2}}$.

Answer:

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Since $\frac{1}{2}\sin(\frac{\pi}{4}) = \frac{1}{2\sqrt{2}}$, then $f^{-1}(\frac{1}{2\sqrt{2}}) = \frac{\pi}{4}$. So then

$$\begin{aligned} \frac{d}{dx}f^{-1}(\frac{1}{2\sqrt{2}}) &= \frac{1}{f'(f^{-1}(\frac{1}{2\sqrt{2}}))} \\ &= \frac{1}{f'(\frac{\pi}{4})} \\ &= \frac{1}{\frac{1}{2}\cos(\frac{\pi}{4})} \\ &= 2\sqrt{2} \end{aligned}$$

Alternately, we could recognize that since $f^{-1}(\frac{1}{2\sqrt{2}}) = \frac{\pi}{4}$, the slope of the line tangent to $f(x)$ at $\frac{\pi}{4}$ must be the reciprocal of the slope tangent to $f^{-1}(x)$ at $\frac{1}{2\sqrt{2}}$, which gives us the above answer.