

Math 3A Practice problems for midterm exam #1

1. Find an expression for a_n based on the values of a_0, a_1, a_2, \dots for

$$\{a_n\} = 0, -\frac{2}{2}, \frac{4}{3}, -\frac{6}{4}, \frac{8}{5}, -\frac{10}{6}, \dots$$

Answer: $a_n = (-1)^n \frac{2n}{n+1}, n = 0, 1, 2, \dots$

2. Consider the recursive sequence defined by $a_{n+1} = 2a_n - 1$ and $a_0 = 2$.

a. Write down the first five terms of this sequence.

Answer: $a_0 = 2, a_1 = 2(2) - 1 = 3, a_2 = 2(3) - 1 = 5, a_3 = 9, a_4 = 17$.

b. Write down an explicit (non-recursive) description of this sequence.

Answer: $a_n = 2^n + 1, n = 0, 1, 2, \dots$

c. Rewrite the explicit description so that the domain of the sequence is $n = 0, 1, 2, \dots$

Answer: $a_n = 2^{n-1} + 1, n = 1, 2, 3, \dots$

3. Compute

$$\lim_{n \rightarrow \infty} \frac{n}{3n+1}$$

Answer:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n}{3n+1} &= \lim_{n \rightarrow \infty} \frac{1}{3 + 1/n} \\ &= \frac{\lim_{n \rightarrow \infty} 1}{\lim_{n \rightarrow \infty} 3 + \lim_{n \rightarrow \infty} 1/n} \\ &= \frac{1}{3+0} = \frac{1}{3} \end{aligned}$$

4. Compute

$$\lim_{x \rightarrow 1} \frac{1-2x+x^2}{1-x}$$

Answer:

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{1-2x+x^2}{1-x} &= \lim_{x \rightarrow 1} \frac{(1-x)(1-x)}{1-x} \\ &= \lim_{x \rightarrow 1} (1-x) = 0 \end{aligned}$$

5. Use the limit laws to compute

$$\lim_{x \rightarrow 0} \frac{x + \sin(x)}{2x}$$

Answer:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{x + \sin(x)}{2x} &= \lim_{x \rightarrow 0} \frac{x}{2x} + \lim_{x \rightarrow 0} \frac{\sin(x)}{2x} \\ &= \lim_{x \rightarrow 0} \frac{1}{2} + \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin(x)}{x} \\ &= \frac{1}{2} + \frac{1}{2} \cdot 1 = 1\end{aligned}$$

6. Let $f(x)$ be defined by

$$f(x) = \begin{cases} \frac{x^2+2x-8}{x-2} & x \neq 2 \\ a & x = 2 \end{cases}$$

What value of a will make $f(x)$ continuous at $x = 2$?

Answer:

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^2 + 2x - 8}{x - 2} &= \lim_{x \rightarrow 2} \frac{(x - 2)(x + 4)}{x - 2} \\ &= \lim_{x \rightarrow 2} (x + 4) = 6\end{aligned}$$

So if we set $a = 6$ then $\lim_{x \rightarrow 2} f(x) = f(2)$, so the function is continuous.

7. For what x values is the function $f(x) = \sin^2(x)$ continuous?

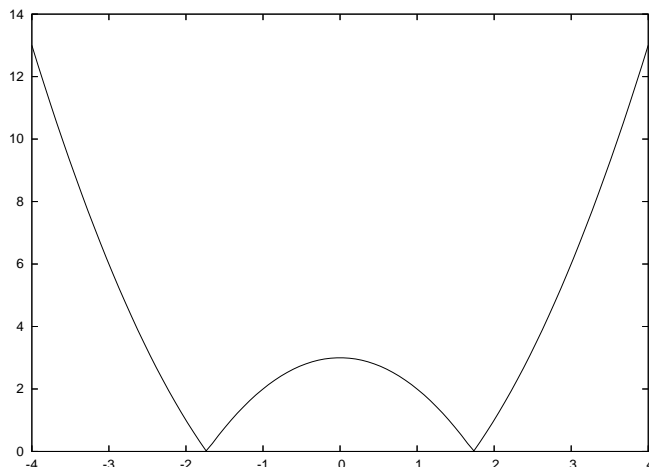
Answer: Trigonometric functions are continuous wherever they are defined. Since the sine function is defined everywhere, it is continuous everywhere. Positive powers of x are also continuous everywhere, so the composition of the two functions is also continuous everywhere.

8. Use the intermediate value theorem to show that $f(x) = x^5 + 3x^2 + 1 = 0$ has a root.

Answer: We see that $\lim_{x \rightarrow -\infty} f(x) = -\infty$ and $\lim_{x \rightarrow \infty} f(x) = \infty$. So we can always choose a small enough a and large enough b such that $f(a) < 0$ and $f(b) > 0$. Then the intermediate value theorem tells us that there exists at least one c in (a, b) with $f(c) = 0$. This is then a root of $f(x)$.

9. Graph the function $f(x) = |x^2 - 3|$, and use the graph to guess where the function is not differentiable.

Answer:



We can see there are corners at $\pm\sqrt{3}$, so we would guess the function is not differentiable there.

10. Let a be a constant, and let $\lim_{x \rightarrow c} f(x)$ and $\lim_{x \rightarrow c} g(x)$ exist. Use the limit laws to show that

$$\lim_{x \rightarrow c} (f(x) - a \cdot g(x)) = \lim_{x \rightarrow c} (f(x)) - a \cdot \lim_{x \rightarrow c} (g(x))$$

Answer:

$$\begin{aligned} \lim_{x \rightarrow c} (f(x) - a \cdot g(x)) &= \lim_{x \rightarrow c} (f(x)) + \lim_{x \rightarrow c} (-a \cdot g(x)) \\ &= \lim_{x \rightarrow c} (f(x)) - a \cdot \lim_{x \rightarrow c} (g(x)) \end{aligned}$$

11. Use the definition of the derivative to find the derivative of $f(x) = \frac{1}{x}$.

Answer:

$$\begin{aligned} \frac{d}{dx} \frac{1}{x} &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-h}{x(x+h)} \right) \\ &= \lim_{h \rightarrow 0} \frac{-1}{x(x+h)} \\ &= -\frac{1}{x} \lim_{h \rightarrow 0} \frac{1}{x+h} \\ &= -\frac{1}{x} \cdot \frac{1}{x} \\ &= -\frac{1}{x^2} \end{aligned}$$

12. Given that for $e \leq x \leq e^{\sqrt{x}}$ whenever $x > e$, use the sandwich theorem to show that

$$\lim_{x \rightarrow \infty} \frac{\ln(x)}{x} = 0$$

Answer: Taking the log of all sides of the inequality, we get

$$1 \leq \ln x \leq \sqrt{x}$$

Since we are only considering positive values of x , we can divide by x without affecting the inequality

$$\frac{1}{x} \leq \frac{\ln x}{x} \leq \frac{1}{\sqrt{x}}$$

We observe that $\lim_{x \rightarrow \infty} \frac{1}{x} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} = 0$. Since $\frac{\ln x}{x}$ is sandwiched between $\frac{1}{x}$ and $\frac{1}{\sqrt{x}}$, and they both go to zero, then by the sandwich theorem $\lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0$.

13. The limit

$$\lim_{h \rightarrow 0} \frac{\frac{1}{(2+h)^2+1} - \frac{1}{5}}{h}$$

represents the derivative of a function at the point $(a, f(a))$. Find the function $f(x)$ and the point a .

Answer: $f(x) = \frac{1}{x^2+1}$ and $a = 2$. So that

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{(2+h)^2+1} - \frac{1}{(2^2+1)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{(2+h)^2+1} - \frac{1}{5}}{h} \end{aligned}$$