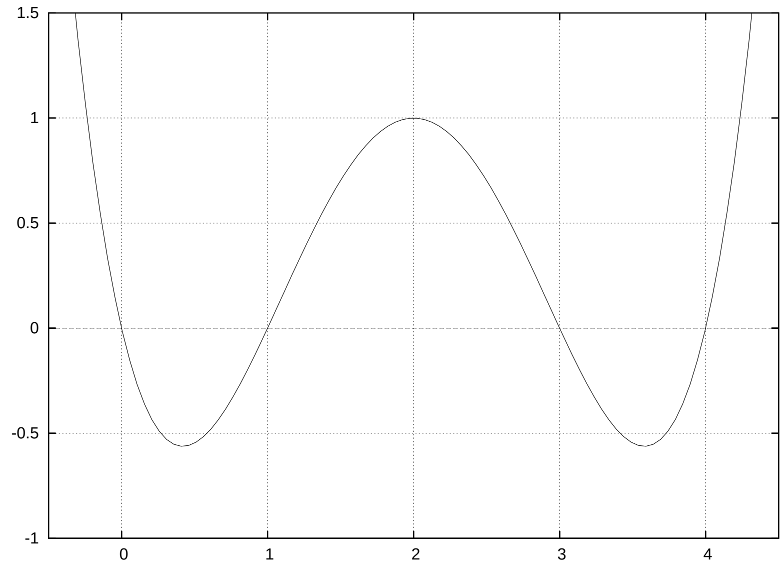
Math 151A Midterm Exam Name: ____________________________

Friday November 3, 2006

Show all your work!

1. Lagrange polynomial interpolation

Below is a plot of a Lagrange basis polynomial $L_{n,k}(x)$ based on the points $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ and $x_4 = 4$.



a. What is the order n of this Lagrange polynomial?

Answer:

The polynomial goes to zero at four (of five) points, so $n = 4$.

b. What is k?

Answer:

The one point which doesn't go to zero (and instead, goes to one) is $x2$, so $k = 2$.

c. Write down this Lagrange basis polynomial $L_{n,k}(x)$ (you don't need to simplify).

Answer:

$$\begin{aligned} L_{4,2} &= \prod_{\substack{i=0 \\ i\neq 2}}^{4} \frac{(x - x_i)}{(x_2 - x_i)} \\ &= \frac{(x-0)(x-1)(x-3)(x-4)}{(2-0)(2-1)(2-3)(2-4)} \end{aligned}$$

2. Newton divided differences Suppose we run the Newton's divided difference algorithm on the points $x_0 = 1$, $x_1 = 2$, $x_2 = 3$ and $x_3 = 4$. The program outputs the following matrix

$$\begin{bmatrix} 3.000 & 0.0000 & 0.0000 & 0.0000 \\ 1.000 & -2.0000 & 0.0000 & 0.0000 \\ 0.000 & \boxed{\text{-1.0}} & \boxed{0.5} & 0.0000 \\ 2.000 & \boxed{2.0} & 1.5000 & 0.3333 \end{bmatrix}$$

Fill in the missing elements.

Answer:
Each coefficient is given by

$$\begin{aligned} F_{i,j} &= f[x_{i-j} \dots x_i] \\ &= \frac{f[x_{i-j+1}, \dots, x_i] - f[x_{i-j}, \dots, x_{i-1}]}{x_i - xi - j} \\ &= = \frac{F_{i,j-1} - F_{i-1,j-1}}{x_i - x_{i-j}} \end{aligned}$$

So $F_{2,1} = (0.0 - 1.0)/(3-2) = -1.0$, etc.

Write down the interpolating polynomial, given these values. (you don't need to simplify)

Answer:

$$\begin{aligned} P(x) &= F_{0,0} + F_{1,1}(x - x_1) + F_{2,2}(x - x_1)(x - x_2) + F_{3,3}(x - x_1)(x - x_2)(x - x_3) \\ &= 3.0 - 2.0(x-1) + 0.5(x-1)(x-2) + 0.3333(x-1)(x-2)(x-3) \end{aligned}$$

3. Iterative methods, pros and cons

a. One of the advantages of Newton's method over the Bisection method is faster convergence. However, there are some advantages of the bisection method over Newton's method. List two advantages of the bisection method.

Answer:
Some acceptable answers:
- Bisection method always brackets the root.
- Bisection method does not require evaluation of the derivative.
- Bisection method works even when the derivative goes to zero.

b. The secant method is very similar to Newton's method. In fact, we derived the secant method to circumvent a weakness in Newton's method. What weakness was this?

Answer:
Newton's method requires evaluating the derivative at each point. The secant method was derived as an approximation to Newton's method that does not require evaluating the derivative.

4. Multiple zeros

a. Show that the function $g(x) = e^x - x - 1$ has a zero of multiplicity two at $p = 0$.

Answer:
The function g has a zero at $p = 0$, since $g(0) = e^0 - 0 - 1 = 0$. It is not a simple zero, since $g'(0) = e^0 - 1 = 0$. But $g''(0) = e^0 \neq 0$, so it is a zero of multiplicity 2.
Alternately, we can rewrite $g(x)$ as $g(x) = (x-0)^2(e^x - x - 1)/(x^2)$, and we see by L'hopital's rule that

$$\lim_{x\to 0} \frac{e^x - x - 1}{x^2} \quad = \quad \lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2} \neq = 0$$

b. We showed in class that, given a function $f(x)$ with a zero p of multiplicity m, we could construct a function with a simple zero at p as follows:

$$\mu(x) = \frac{f(x)}{f'(x)}$$

Use this to find $\mu(x)$ for the function $g(x)$ from part (a), and show that this $\mu(x)$ has a simple zero at $p = 0$.

Answer:

$$\mu(x) = \frac{f(x)}{f'(x)} = \frac{e^x - x - 1}{e^x - 1}$$

$$\lim_{x\to 0} \mu(x) = \lim_{x\to 0} \frac{e^x - x - 1}{e^x - 1} = \lim_{x\to 0} \frac{e^x - 1}{e^x} = \frac{0}{1} = 0$$

So $p = 0$ is indeed a zero of μ. The easiest way to show that it is a simple zero is to see that we can write

$$\mu(x) = (x-0)\frac{e^x - x - 1}{xe^x - x}$$

where

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - x - 1}{xe^x - x} &= \lim_{x\to 0} \frac{e^x - 1}{xe^x + e^x - 1} \\
&= \lim_{x\to 0} \frac{e^x}{xe^x + e^x + e^x} \\
&= \frac{1}{1+1} \\
&\neq 0
\end{aligned}$$

5. Truncation error

We showed in class that given a number y and its k-digit floating point representation $fl(y)$, the truncation error for chopping is given by

$$\frac{|y - fl(y)|}{|y|} \leq 10^{-k+1}.$$

Suppose that, instead of chopping (i.e. rounding down), we round *up*. Show that we get the same truncation error.

Answer:

Let $y = 0.d_1d_2\ldots\times 10^n$. Then rounding up will give us $fl(y) = 0.d_1d_2\ldots d_k\times 10^n + 10^{n-k}$. So the relative error is

$$\begin{aligned}
\frac{|y - fl(y)|}{|y|} &= \frac{|0.d_1d_2\ldots\times 10^n - 0.d_1d_2\ldots d_k\times 10^n - 10^{n-k}|}{|0.d_1d_2\ldots\times 10^n|} \\
&= \frac{|0.d_{k+1}d_{k+2}\ldots\times 10^{n-k} - 10^{n-k}|}{|0.d_1d_2\ldots\times 10^n|} \\
&= \frac{|0.d_{k+1}d_{k+2}\ldots - 1|}{|0.d_1d_2\ldots|}\times 10^{-k}
\end{aligned}$$

Since $d_1 \geq 1$, the denominator is no smaller than 0.1. Since $0 \leq d_j \leq 9$ for $j = k+1, k+2, \ldots$, the decimal in the numerator is between 0 and 1, which means the numerator is no larger than 1. So then

$$\frac{|y - fl(y)|}{|y|} \leq \frac{1}{0.1}\times 10^{-k} = 10^{-k+1}$$

6. Quadratic convergence of Newton's method
Newton's method generates a sequence of points $p_n = g(p_{n-1})$ where

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Show that Newton's method converges at least quadratically (assuming of course that $f'(p) \neq 0$).

[HINT: Don't *directly* use the definition of order of convergence – there is an easier way.]

Answer:
In class we discussed (and proved) a theorem that says that if p is a solution to the fixed point problem $g(x) = x$, and $g'(p) = 0$, then the sequence generated by iterating $g(x)$ converges at least quadratically (on some interval around p). So we just need to check if $g'(p)$ is zero.

$$\begin{aligned} g'(x) &= 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} \\ &= 1 - 1 + \frac{f(x)f''(x)}{[f'(x)]^2} \\ &= \frac{f(x)f''(x)}{[f'(x)]^2} \end{aligned}$$

But if p is a solution to $g(x) = x$, then

$$g(p) = p = p - \frac{f(p)}{f'(p)}$$

which means $f(p) = 0$. So then

$$g'(p) = \frac{f(p)f''(p)}{[f'(p)]^2} = 0$$