Math 151A Homework #7 – practice homework.

1. Pivoting strategies (Section 6.2, problems 1c,2c,3c,4c)
Given the linear system

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 5 \\ -4x_1 + 2x_2 - 6x_3 &= 14 \\ 2x_1 + 2x_2 + 4x_3 &= 8 \end{aligned}$$

find the row interchanges required when using:

a. simple pivoting (algorithm 6.1)

Answer:
The coefficient matrix is

$$\begin{bmatrix} 2 & -3 & 2 \\ -4 & 2 & -6 \\ 2 & 2 & 4 \end{bmatrix}$$

So $a_{1,1} \neq 0$ and we do $(2E_1 + E_2) \to (E_2)$ and $(E_3 - E_1) \to (E_3)$ and get

$$\begin{bmatrix} 2 & -3 & 2 \\ 0 & -4 & -2 \\ 0 & 5 & 2 \end{bmatrix}$$

which has $a_{2,2} \neq 0$, so no row interchanges are required.

b. partial pivoting (algorithm 6.2)

Answer:
Since $|-4| > |2|$, we exchange rows 1 and 2. Then we do $(\frac{1}{2}E_1 + E_2) \to (E_2)$ and the same for E_3. We then get

$$\begin{bmatrix} -4 & 2 & -6 \\ 0 & -2 & -1 \\ 0 & 3 & 1 \end{bmatrix}$$

We see that $|3| > |-2|$ and so we exchange rows 3 and 2.
Thus, in total, we exchanged rows 1 and 2, and then rows 2 and 3.

c. scaled partial pivoting (algorithm 6.3)

Answer:
The scale factors are defined as $s_i = \max_{1 \leq j \leq n} |a_{i,j}|$. For this matrix they are $s_1 = 3$, $s_2 = 6$ and $s_3 = 4$. Then the desired pivot element is the first maximum of $|a_{1,1}|/s_1 = \frac{2}{3}$, $|a_{2,1}|/s_2 = \frac{2}{3}$, and $|a_{3,1}|/s_3 = \frac{1}{2}$. Since this is the first row, no interchange is need. One step of Gaussian elimination yields

$$\begin{bmatrix} 2 & -3 & 2 \\ 0 & -4 & -2 \\ 0 & 5 & 2 \end{bmatrix}$$

Now we see that $|-4|/6$ is less than $|5|/4$ so we exchange the last two rows.

d. complete pivoting

Answer:
The largest element in the coefficient matrix is $|a_{2,3}| = 6$, so we swap rows 1 and 2 and columns 1 and 3 to get

$$\begin{bmatrix} -6 & 2 & 4 \\ 2 & -3 & 2 \\ 4 & 2 & 2 \end{bmatrix}$$

Elimination yields

$$\begin{bmatrix} -6 & 2 & 4 \\ 0 & -\frac{7}{3} & \frac{10}{3} \\ 0 & \frac{10}{3} & \frac{14}{3} \end{bmatrix}$$

We see the largest element of the remaining submatrix is $a_{3,3}$ so we exchange rows 2 and 3 and columns 2 and 3.

2. LU factorization Consider the linear system

$$\begin{aligned} 2x_1 + x_2 + x_3 &= b_1 \\ -2x_1 - 1x_2 &= b_2 \\ 2x_1 + 3x_2 &= b_3 \end{aligned}$$

The coefficient matrix for the system does not have an LU factorization. Using an appropriate permutation matrix, factor the resulting matrix ($PA = LU$). Use this factorization to solve the system for $\mathbf{x}$ when $\mathbf{b} = (1, 2, 3)^T$.

Answer:
The coefficient matrix is

$$\begin{bmatrix} 2 & 1 & 1 \\ -2 & -1 & 0 \\ 2 & 3 & 0 \end{bmatrix}$$

After one elimination step, we get

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

which requires swapping rows 2 and 3. This means our permutation matrix should be

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

So now we factor the permuted matrix

$$PA = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 0 \\ -2 & -1 & 0 \end{bmatrix}$$

The first elimination step uses the multipliers $m_{2,1} = 1$ and $m_{3,1} = -1$ to get

$$\begin{bmatrix} 2 & 1 & 1 \\ -0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

No more elimination is needed, so $m_{3,2} = 0$ and thus

$$PA = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

To find the solution to the equation $Ax = b$ for $b = (1, 2, 3)^T$ we must solve $PAx = LUx = Pb = (1, 3, 2)^T$. First we let $Ux = y$ and solve $Ly = Pb$ for y:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \cdot y = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

Forward substitution gives $y = (1, 2, 3)^T$. Now we must solve $Ux = y$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Backward substitution gives the solution $x = (\frac{-9}{4}, \frac{5}{2}, 3)^T$. This is the solution to the permuted equation $PAx = Pb$, but is also the solution to the original set of equations $Ax = b$.

3. Special matrices

Which of the following matrices are

i. symmetric

Answer:

Only matrix a is symmetric.

ii. singular

Answer:

We can check singularity by checking the determinant. The determinant of matrix a is $2 \cdot 3 - 1 \cdot 1 = 5 \neq 0$ so it is non-singular. For matrix b we have $2(3 \cdot 4 - 0) - 1(0) = 24 \neq 0$, so it too is non-singular. For matrix c we have $4(0 - (-7)) - 2(-9 - (-14)) + 6(-3 - 0) = 28 - 10 - 18 = 0$, so matrix c is singular.

iii. strictly diagonally dominant

Answer:

Matrices a and b are s.d.d. since the largest (magnitude) element in each row occurs on the diagonal. Matrix c is not s.d.d., since the larges element in row 1 is off the diagonal. We didn't actually need to check matrix c since we already knew (from part ii) that it was singular. And we know that all s.d.d. matrices are non-singular.

a. $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 4 \end{bmatrix}$

c. $\begin{bmatrix} 4 & 2 & 6 \\ 3 & 0 & 7 \\ -2 & -1 & -3 \end{bmatrix}$