Math 151A Homework #5 – due Wednesday 11/29, in class

Show all your work!

1. Forward and backward differences

Suppose we use Newton's divided differences on the points $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$ to construct the following matrix

$$\begin{bmatrix} \boxed{3} & 0 & 0. \\ \boxed{4} & \boxed{1} & 0 \\ 6 & 2 & \frac{1}{2} \end{bmatrix}$$

a. Fill in the missing elements.

Answer:

$$\begin{aligned} f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \\ \frac{1}{2} &= \frac{2 - f[x_0, x_1]}{2 - 0} \end{aligned}$$

So then

$$f[x_0, x_1] = 2 - 1 = 1$$

Similarly, we get $f[x_0] = 3$ and $f[x_1] = 4$.

b. Use the Newton *foward* difference formula to estimate $f(0.1)$.

Answer:

The $n = 2$ forward difference formula (in expanded notation) is

$$P_2(x) = P_2(x_0 + sh) = f[x_0] + shf[x_0, x_1] + s(s-1)h^2 f[x_0, x_1, x_2]$$

In this case, $h = x_2 - x_1 = x_1 - x_0 = 1$ and $x = x_0 + sh = 0.1$, so $s = 0.1$. Thus

$$\begin{aligned} P_2(0.1) = P_2(0 + 0.1) &= 3 + (0.1)(1)(1) + (0.1)(-0.9)(1)^2(\frac{1}{2}) \\ &= 3.055 \end{aligned}$$

c. Use the Newton *backward* difference formula to estimate $f(1.9)$.

Answer:

The $n = 2$ backward difference formula (in expanded notation) is

$$P_2(x) = P_2(x_2 + sh) = f[x_2] + shf[x_1, x_2] + s(s-1)h^2 f[x_0, x_1, x_2]$$

In this case, $h = x_2 - x_1 = x_1 - x_0 = 1$ again and now $x = x_n + sh = 1.9$, so $s = -0.1$. Thus

$$\begin{aligned} P_2(1.9) = P_2(2 - 0.1) &= 6 + (-0.1)(1)(2) + (-0.1)(0.9)(1)^2(\frac{1}{2}) \\ &= 5.755 \end{aligned}$$

2. Numerical differentiation

Let $f(x) = e^x$. Approximate the value of $f'(1)$ using:

a. The 3-point forward difference formula with $h = 0.1$

Answer:

$$\begin{aligned}
f'(x) &= \frac{1}{2h}[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)] + \frac{h^2}{3}f^{(3)}(\xi) \\
f'(1) &\approx \frac{1}{0.2}[-3e^1 + 4e^{1.1} - e^{1.2}] \\
&\approx \frac{1}{0.2}[-8.15484548538 + 12.0166640958 - 3.32011692274] \\
&\approx 2.708508
\end{aligned}$$

b. The 3-point backward difference formula with $h = 0.1$

Answer:

$$\begin{aligned}
f'(x) &= \frac{1}{2h}[3f(x_0) - 4f(x_0-h) + f(x_0-2h)] + \frac{h^2}{3}f^{(3)}(\xi) \\
f'(1) &\approx \frac{1}{0.2}[3e^1 - 4e^{0.9} + e^{0.8}] \\
&\approx \frac{1}{0.2}[8.15484548538 - 9.83841244463 + 2.22554092849] \\
&\approx 2.70986984621
\end{aligned}$$

c. The 3-point centered difference formula with $h = 0.1$

Answer:

$$\begin{aligned}
f'(x) &= \frac{1}{2h}[-f(x_0-h) + f(x_0+h)] - \frac{h^2}{6}f^{(3)}(\xi) \\
f'(1) &\approx \frac{1}{0.2}[-e^{0.9} + e^{1.1}] \\
&\approx \frac{1}{0.2}[-2.45960311116 + 3.00416602395] \\
&\approx 2.72281456395
\end{aligned}$$

Which do you expect to be a better approximation, and why?

Answer:
Since the error term for centered difference is half of that for forward and backward differences, we expect the centered difference to produce a better approximation. This is realized in this case, since the actuall errors for forward, backward and centered are 0.00977, 0.00841 and 0.00453 respectively.

3. Richardson's extrapolation (problem 9 of section 4.2)
Suppose that $N(h)$ is an approximation to M at every $h > 0$ and that

$$M = N(h) + K_1 h + K_2 h^2 + K_3 h^3 + \ldots,$$

for some constants K_1, K_2, K_3, ... Use the values $N(h)$, $N(h/3)$, and $N(h/9)$ to produce an $\mathcal{O}(h^3)$ approximation to M.

Answer:

$$
\begin{aligned}
M &= N(h) + K_1 h + K_2 h^2 + K_3 h^3 + \dots \\
M &= N\big(\frac{h}{3}\big) + K_1\frac{h^3}{3} + K_2\frac{h^2}{9} + K_3\frac{h^3}{27} + \dots
\end{aligned}
$$

Subtracting the first M from thrice the second M gives

$$
\begin{aligned}
3M - M = 2M &= 3N\big(\frac{h}{3}\big) - N(h) - \frac{2}{3}K_2 h^2 - \frac{8}{9}K_3 h^3 \\
M &= \frac{3N\big(\frac{h}{3}\big) - N(h)}{2} - \frac{1}{3}K_2 h^2 - \frac{4}{9}K_3 h^3
\end{aligned}
$$

Repeat the process once more by evaluating this M at $h/3$:

$$
M = \frac{3N\big(\frac{h}{9}\big) - N\big(\frac{h}{3}\big)}{2} - \frac{1}{3}K_2\frac{h^2}{9} - \frac{4}{9}K_3\frac{h^3}{27}
$$

And cancel the $\mathcal{O}(h^2)$ error terms:

$$
\begin{aligned}
9M - M &= \frac{1}{2}\big[27N\big(\frac{h}{9}\big) - 9N\big(\frac{h}{3}\big) - 3N\big(\frac{h}{3}\big) + N(h)\big] \\
M &= \frac{N(h) - 12N\big(\frac{h}{3}\big) + 27N\big(\frac{h}{9}\big)}{16} - \frac{1}{9}K_3 h^3
\end{aligned}
$$

4. Numerical integration (problems 1a, 3a, 5a, 7a, 9a, and 11a of section 4.3)
Approximate the integral

$$
\int_{0.5}^{1} x^4 dx
$$

using

a. The trapezoidal rule

Answer:
The trapezoidal rule states that

$$
\int_a^b f(x)dx = \frac{h}{2}[f(x_0) + f(x_1)] - \frac{h^3}{12}f''(\xi) \qquad a < \xi < b
$$

So

$$
\begin{aligned}
\int_{0.5}^{1.0} x^4 dx &= \frac{0.5}{2}[0.5^4 + 1^4] - \frac{0.5^3}{12}\cdot 12\xi^2 \\
&= \frac{1}{4}[\frac{1}{16} + 1] - \frac{1}{8}\xi^2 \\
&= \frac{17}{64} - \frac{1}{8}\xi^2
\end{aligned}
$$

So the integral is approximated by $17/64 = 0.265625$, which gives an actual error of 0.071875. The error bound is $\max_{x\in[0.5,1]} \xi^2/8 = 1/8 = 0.125$.

b. Simpson's rule

Answer:
Simpson's rule states that

$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(\xi) \qquad x_0 < \xi < x_2$$

So

$$\begin{aligned}\int_{0.5}^{1.0} x^4 dx &= \frac{1}{12}[0.5^4 + 4(0.75)^4 + 1^4] - \frac{0.25^5}{90}\cdot 24 \\ &= \frac{1}{12}\left[\frac{1}{16} + 4\cdot\frac{81}{256} + 1\right] - \frac{1}{30\cdot 128} \\ &= \frac{1}{12}\left[\frac{4}{64} + \frac{81}{64} + \frac{64}{64}\right] - \frac{1}{30\cdot 128} \\ &= \frac{149}{768} - \frac{1}{30\cdot 128}\end{aligned}$$

So the integral is approximated by $149/768 = 0.194014167$, which gives an actual error of 0.0002604. The error bound is $\frac{1}{30\cdot 128} = 0.0002604$. Notice that the error term is exact, since the fourth derivative of $f(x)$ is a constant.

c. The midpoint rule

Answer:
The midpoint rule states that

$$\int_{x_{-1}}^{x_1} f(x)dx = 2hf(x_0) - \frac{h^3}{3} f''(\xi) \qquad x_{-1} < \xi < x_1$$

So

$$\begin{aligned}\int_{0.5}^{1.0} x^4 dx &= \frac{1}{2}(0.75)^4 - \frac{1}{3\cdot 64}\cdot 12\xi^2 \\ &= \frac{1}{2}\left(\frac{81}{256}\right) - \frac{1}{16}\xi^2 \\ &= \frac{81}{512} - \frac{1}{16}\xi^2\end{aligned}$$

So the integral is approximated by $81/512 = 0.158203$, which gives an actual error of 0.0355469. The error bound is $\max_{x\in[0.5,1]} \xi^2/16 = 1/16 = 0.0625$.

For each of the above, find a bound for the error using the error formula, and compare this to the actual error.

Answer:
See above.