Math 151A Homework #3 – due Wednesday 10/25, in class

Show all your work!

1. Trinary computing.

Imagine we've invented a trinary (base 3) computer, which stores information as *trits* (**tri**nary b**its**) which can take on values $0, 1$ or 2. If (as with binary computers) the trinary computer uses 52 trits to hold the mantissa (a.k.a. the trinary fraction), how many decimal digits of precision can we expect?

Answer:

Each trit can take on three values, so 52 trits take on 3^{52} values. And $\log_{10}(3^{52}) = 52 \cdot \log_{10}(3) \approx 24.810$, so we can expect 24 or 25 digits of precision.

2. k-digit rounding approximation (Problem 24 of section 1.2)

Suppose that $fl(y)$ is a k-digit rounding approximation to y. Show that

$$\left|\frac{y - fl(y)}{y}\right| \leq 0.5 \times 10^{-k+1}.$$

[*Hint:* If $d_{k+1} < 5$, then $fl(y) = 0.d_1 d_2 \ldots d_k \times 10^n$. If $d_{k+1} \geq 5$, then $fl(y) = 0.d_1 d_2 \ldots d_k \times 10^n + 10^{n-k}$.]

Answer:

Let y be represented by $y = 0.d_1 d_2 d_3 \ldots d_k d_{k+1} \ldots \times 10^n$, where $1 \leq d_1 \leq 9$ and $0 \leq d_i \leq 9$. If $d_{k+1} < 5$, then $fl(y) = 0.d_1 d_2 \ldots d_k \times 10^n$ and so

$$\begin{aligned}
\left|\frac{y - fl(y)}{y}\right| &= \left|\frac{0.d_1 d_2 \ldots d_k d_{k+1} \ldots \times 10^n - 0.d_1 d_2 \ldots d_k \times 10^n}{0.d_1 d_2 d_3 \ldots d_k d_{k+1} \ldots \times 10^n}\right| \\
&= \left|\frac{0.d_{k+1} d_{k+2} \ldots \times 10^{n-k}}{0.d_1 d_2 \ldots d_{k+1} d_{k+2} \ldots \times 10^n}\right| \\
&= \left|\frac{0.d_{k+1} d_{k+2} \ldots}{0.d_1 d_2 \ldots}\right| \times 10^{-k}
\end{aligned}$$

The denominator of the above expression is bounded below by 0.1, since we know that $d_1 \geq 1$. The numerator is bounded above by 0.5, since we have assumed that $d_{k+1} < 5$. So we have

$$\begin{aligned}
\left|\frac{y - fl(y)}{y}\right| &\leq \frac{0.5}{0.1} \times 10^{-k} \\
&\leq 0.5 \times 10^{-k+1}.
\end{aligned}$$

So the error bound holds for $d_{k+1} < 5$.

Now if $d_{k+1} \geq 5$, then we round up by adding 1 to the digit d_k, which means that $fl(y) = 0.d_1 d_2 \ldots d_k \times 10^n + 10^{n-k}$ and thus

$$\begin{aligned}
\left|\frac{y - fl(y)}{y}\right| &= \left|\frac{0.d_1 d_2 \ldots d_k d_{k+1} \ldots \times 10^n - 0.d_1 d_2 \ldots d_k \times 10^n - 10^{n-k}}{0.d_1 d_2 d_3 \ldots d_k d_{k+1} \ldots \times 10^n}\right| \\
&= \frac{\left|0.d_{k+1} d_{k+2} \ldots \times 10^{n-k} - 10^{n-k}\right|}{\left|0.d_1 d_2 \ldots d_{k+1} d_{k+2} \ldots \times 10^n\right|} \\
&= \frac{\left|0.d_{k+1} d_{k+2} \cdots - 1\right|}{\left|0.d_1 d_2 \ldots d_{k+1} d_{k+2} \ldots\right|} \times 10^{-k}.
\end{aligned}$$

Again, the denominator is bounded below by 0.1. The first term of the numerator is bounded above by 1.0 and below by 0.5, which means the numerator is bounded above by 0.5. So we have

$$\begin{aligned} \left|\frac{y - fl(y)}{y}\right| &\leq \frac{0.5}{0.1}\times 10^{-k} \\ &\leq 0.5\times 10^{-k+1}. \end{aligned}$$

Therefor, the error bound also holds for $d_{k+1} \geq 5$.

3. Approximating π

Compute the absolute error and relative error for approximating π by 22/7. How many significant digits is the approximation good to?

Answer:

The absolute error is $|\pi - 22/7| \approx 1.26 \times 10^{-3}$ and the relative error is the absolute error divided by π, $(1.26 \times 10^{-3})/(\pi) = 4.02 \times 10^{-4}$. The number of significant digits is the largest integer t for which

$$\text{Relative Error} \leq 5 \times 10^{-t}.$$

Since $4.02 \times 10^{-4} \leq 5 \times 10^{-4}$ but $4.02 \times 10^{-4} > 5 \times 10^{-5}$, the approximation is good to 4 significant digits.

4. Approximating e

a. The number e can be approximated by the sequence

$$e \approx \sum_{k=0}^{n} \frac{1}{k!}.$$

If $n = 4$, what is the relative error and number of significant digits in this approximation of e? How about when $n = 8$?

Answer:

When $n = 4$, $e \approx \sum_{k=0}^{4} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 2.708\bar{3}$, which gives a relative error of

$$\left|\frac{e - 2.708\bar{3}}{e}\right| \approx 3.66 \times 10^{-3}$$

and 3 significant digits.

When $n = 8$, $e \approx \sum_{k=0}^{8} \frac{1}{k!} \approx 2.718278770$, which gives a relative error of

$$\left|\frac{e - 2.718278770}{e}\right| \approx 1.125 \times 10^{-6}$$

and 6 significant digits.

b. The number e can also be approximated by

$$e \approx n \cdot \left(\frac{\sqrt{2\pi n}}{n!}\right)^{1/n}$$

What is the relative error and number of significant digits when $n = 4$, $n = 8$, and $n = 20$?

Answer:
When $n = 4$,

$$e \approx 4 \cdot \left(\frac{\sqrt{2\pi 4}}{4!} \right)^{1/4} \approx 2.704190$$

which gives a relative error of 5.18×10^{-3} and 2 (but *nearly* 3) significant digits.
For $n = 8$, we get $e \approx 2.7147465$, with relative error 1.30×10^{-3} and 3 significant digits.
For $n = 20$, we get $e \approx 2.7177156$, with relative error 2.08×10^{-4} and 4 significant digits.