Math 151A Homework #1 – due Wednesday 10/11, in class

Show all your work!

1. On the importance of stopping criteria. (section 2.1, problem 16)
Let $f(x) = (x-1)^{10}$ (with a root at $p = 1$), and consider the sequence of approximations to the root p given by $p_n = 1 + (1/n)$. Show the the stopping criterion $|f(p_n)| < 10^{-3}$ is met for $n \geq 2$, but $|p - p_n| < 10^{-3}$ requires that $n > 1000$.

Answer:
The first stopping criterion requires that

$$\begin{aligned} |f(p_n)| = |(p_n - 1)^{10}| = \left|\left(\frac{1}{n}\right)^{10}\right| = \left(\frac{1}{n}\right)^{10} &< 10^{-3} \\ -10\log(n) &< -3 \\ n &> 10^{0.3} \approx 1.995 \end{aligned}$$

which is satisfied when $n \geq 2$.
The second stopping criterion requires that

$$|p - p_n| = |1/n| = 1/n < 10^{-3}$$

which is satisfied when $n > 10^3$.

2. On the importance of stopping criteria, II. (section2.1, problem 17)
Let p_n be the sequence defined by $p_n = \sum_{k=1}^{n}(1/k)$. Show that p_n diverges even though $\lim_{n\to\infty}(p_n - p_{n-1}) = 0$.

Answer:

$$\begin{aligned} p_n - p_{n-1} &= \sum_{k=1}^{n}\left(\frac{1}{k}\right) - \sum_{k=1}^{n-1}\left(\frac{1}{k}\right) \\ &= \frac{1}{n} \end{aligned}$$

So

$$\lim_{n\to\infty}(p_n - p_{n-1}) = \lim_{n\to\infty}\frac{1}{n} = 0$$

so the difference between consectutive terms vanishes. But $\{p_n\}$ is the well-known harmonic series, which diverges.

3. The sqare root of two.
Use the bisection method to approximate the value of $\sqrt{2}$, given that it lies in the interval $[1, 2]$. Do this in two parts:

a. Calculate the maximum number of iterations required to estimate $\sqrt{2}$ to within 10^{-4}, starting with the interval $[1, 2]$.

b. Use the bisection method to compute the value of $\sqrt{2}$ to within 10^{-4}. Print out and submit (along with your code) the intermediate approximations at each iteration.

[Hint: Consider $f(x) = x^2 - 2$]

Answer:

Part a.

From theorem 2.1, we know that

$$|p_n - p| \leq \frac{b-a}{2^n},$$

when $n \geq 1$. In this case, $a = 1$, $b = 2$, and we need an answer to within 10^{-4}. This means that

$$\begin{aligned} 10^{-4} &\leq \frac{2-1}{2^n} = \frac{1}{2^n} \\ -4 &\leq -n\log(2) \\ n &\geq \frac{4}{\log(2)} \approx 13.29 \end{aligned}$$

To meet this, we require 14 iterations.

Part b.

Create a function file called *f.m* containing:

```
function f = f(x)
    f = x.^2 - 2;
end
```

and script file called *bisection.m* containing, for example:

```
%-----------------------------------------------------------
% Matlab/Octave program to find the root of a function
%   - the function must be defined in f.m
%-----------------------------------------------------------

% set the endpoints a and b, and the desired precision
a=1;
b=2;
tol = 1.0e-4;

% prevent infinite loops, by setting a maximum
% number of iterations
maxits = 1000;

for i = 1:maxits

    % compute the midpoint
    p = (a+b)/2.0;
```

```
        % nicely formatted printing of each approximation
        printf("%.2d:  p = %.8f\n",i,p);

        % check to see if we are at the root
        if f(p)==0
            break
        endif

        % check to see if we've reached the desired precision
        if((b-a)/2.0 < tol)
            break
        endif

        % find which points are of opposite sign
        if( f(a)*f(p) < 0 )
            b = p;
        else
            a = p;
        endif

    end
```

Then run the script:

```
    octave:1> bisection
    01:  p = 1.50000000
    02:  p = 1.25000000
    03:  p = 1.37500000
    04:  p = 1.43750000
    05:  p = 1.40625000
    06:  p = 1.42187500
    07:  p = 1.41406250
    08:  p = 1.41796875
    09:  p = 1.41601562
    10:  p = 1.41503906
    11:  p = 1.41455078
    12:  p = 1.41430664
    13:  p = 1.41418457
    14:  p = 1.41424561
```

Notice that we reach $|p_n - p| = |1.41424561 - \sqrt{2}| \approx 3.205 \times 10^{-5}$ after 14 iterations, as predicted in part a.