

HILBERT MODULAR FORMS WITH PRESCRIBED RAMIFICATION

JARED WEINSTEIN

ABSTRACT. Let K be a totally real field. We present an asymptotic formula for the number of Hilbert modular cusp forms f with given ramification at every place v of K . When v is an infinite place, this means specifying the weight of f at k , and when v is finite, this means specifying the restriction to inertia of the local Weil-Deligne representation attached to f at v . Our formula shows that with essentially finitely many exceptions, the cusp forms of K exhibit every possible sort of ramification behavior.

1. INTRODUCTION AND MAIN THEOREM

Let K be a totally real field. Our investigation is concerned with counting the number of cuspidal automorphic representations $\pi = \otimes_v \pi_v$ of the adèle group $\mathrm{GL}(2, \mathbf{A}_K)$ whose local components π_v have prescribed ramification for all places v of K . We must explain what is meant by “prescribed ramification”: When v is an infinite place, it means that π_v is a prescribed essentially discrete series representation of $\mathrm{GL}(2, \mathbf{R})$. When v is a finite place, it means that the Weil-Deligne representation associated to π_v under the local Langlands correspondence has prescribed restriction to inertia and monodromy operator.

Our problem may be restated in terms of the ℓ -adic Galois representations attached to Hilbert modular eigenforms over K . Suppose f is such a form with coefficients in $\overline{\mathbf{Q}}_\ell$. Then there is a corresponding ℓ -adic Galois representation $\rho_f: \mathrm{Gal}(\overline{K}/K) \rightarrow \mathrm{GL}(2, \overline{\mathbf{Q}}_\ell)$, as in [Tay89]. Fix an ℓ and suppose the following data are given:

- (1) For each infinite place v , a weight $k_v \geq 2$, and
- (2) For each finite place v not dividing ℓ , a representation $\sigma_v: I_{K_v} \rightarrow \mathrm{GL}(2, \overline{\mathbf{Q}}_\ell)$ of the inertia group I_{K_v} which extends to the full Galois group $\mathrm{Gal}(\overline{K}_v/K_v)$. Assume almost all of these are trivial.

We are then concerned with counting the number of Hilbert modular forms f of weights (k_v) and level prime to ℓ for which the restriction of

⁰The author was supported by grant DMS-0803089 from the National Science Foundation.

ρ_f to I_{K_v} is σ_v for each finite place v . Our main Theorem 1.1 gives an asymptotic formula for the number of such forms. Barring a natural obstruction coming from the central character, it shows that such forms always exist unless the data (k_v) and (σ_v) given above belong to a finite set of exceptional data up to twisting.

This theorem affirms the existence, at least for $GL(2)$, of automorphic representations subject to local constraints which are more stringent than those previously considered in the literature. Clozel [Clo86], building on a result of DeGeorge and Wallach [DW78], begins with a real semisimple Lie group G and a discrete series representation δ of G and then uses a trace formula to count the multiplicity of δ in $L^2(\Gamma \backslash G)$ as Γ ranges through a tower of arithmetic subgroups of G : This is akin to counting the number of modular forms of a given weight k whose level is supported on a finite set of primes, though it may be deeply ramified at those primes. Chenevier [Che07] uses the Peter-Weyl theorem to construct automorphic representations of a certain sort of unitary group (which is compact at the infinite places) which has prescribed ramification at a prime but arbitrary behavior at infinity; this representation is then used to construct number fields with given ramification. Our present investigation is limited to the group $GL(2)$, but we control both the weight and the ramification at all finite places. In this sense our main theorem is similar to a theorem of Khare and Prasad [KP96], which shows the existence of classical cuspidal eigenforms of weight 2 for the principal congruence subgroup $\Gamma(p)$ which have a particular sort of ramification behavior at one prime p . In Section 4.3 we offer some very detailed information in the case of $K = \mathbf{Q}$; in particular, we compute the class of the space of cusp forms $S_k(\Gamma(N))$ in the Grothendieck group of $SL(2, \mathbf{Z}/N\mathbf{Z})$, thus generalizing the classical dimension formulas in [Shi71] and [CO77].

To state our main theorem, we need to introduce some notation regarding “inertial types” for $GL(2)$ over a local or global field.

1.1. Local Inertial Types. Let F be a finite extension of \mathbf{Q}_p , with ring of integers \mathcal{O}_F . Let $\mathcal{A}_2(F)$ be the set of isomorphism classes of complex-valued irreducible admissible representations of $GL(2, F)$. By the local Langlands correspondence, there is a bijection $\pi \mapsto \sigma(\pi)$ mapping $\mathcal{A}_2(F)$ onto the set of isomorphism classes of two-dimensional Frobenius-semisimple Weil-Deligne representations of F preserving L -functions and epsilon factors, see for instance [Kut80]. (See [Tat79] for the definition of Weil-Deligne representation and for the construction of its L -function and epsilon factor.)

Let \mathcal{O}_F be the ring of integers of F , so that $\mathrm{GL}(2, \mathcal{O}_F)$ is a maximal compact open subgroup of $\mathrm{GL}(2, F)$. In [Hen02] it is shown that if $\pi \in \mathcal{A}_2(F)$, then $\pi|_{\mathrm{GL}(2, \mathcal{O}_F)}$ contains an irreducible finite-dimensional subspace $\tau(\pi)$ of $\mathrm{GL}(2, \mathcal{O}_F)$ which characterizes the restriction of $\sigma(\pi)$ to the inertia group of F . We shall call $\tau(\pi)$ the inertial type of π . We will leave the precise definition of $\tau(\pi)$ for Section 2, but we remark that when π belongs to the unramified principal series, $\tau(\pi)$ is the trivial representation. Let $\mathrm{Types}(F)$ be the set of isomorphism classes of representations of $\mathrm{GL}(2, \mathcal{O}_F)$ which arise as inertial types for members of $\mathcal{A}_2(F)$. For $\tau \in \mathrm{Types}(F)$ we define the quantity

$$d(\tau) = \begin{cases} q - 1, & \tau \text{ is the type of a special representation,} \\ \dim \tau, & \text{otherwise;} \end{cases}$$

here q is the cardinality of the residue field of \mathcal{O}_F . (A special representation of $\mathrm{GL}(2, F)$ is a twist of the Steinberg representation of this group. Weil-Deligne representations corresponding to special representations are precisely those with nontrivial monodromy operator.)

Now suppose $F = \mathbf{R}$. When $k \geq 2$ and w are two integers of the same parity, let $\mathcal{D}_{k,w}$ be the essentially discrete series representation of $\mathrm{GL}(2, \mathbf{R})$ as in 0.2 of [Car86]. Then the central character of $\mathcal{D}_{k,w}$ is $t \mapsto t^{-w}$. Let $\mathrm{Types}(F)$ denote the set of all such representations $\mathcal{D}_{k,w}$. If π is such a representation we simply define $\tau(\pi) = \pi$. We define the function d on $\mathrm{Types}(F)$ by $d(\mathcal{D}_{k,w}) = k - 1$.

In either case, suppose $\tau \in \mathrm{Types}(F)$. When χ is a (one-dimensional) character of F^* we denote by $\tau \otimes \chi$ the representation $g \mapsto \chi(\det g)\tau(g)$; this also belongs to $\mathrm{Types}(F)$.

1.2. Global Inertial Types. Now suppose K is a totally real field of degree n . Let S (resp., S_f, S_∞) be the set of places (resp., finite places, infinite places) of K . To a cuspidal automorphic representation π of $\mathrm{GL}(2)_K$ arising from a Hilbert modular form, we can associate the representation $\tau(\pi) = \otimes_{v \in S} \tau(\pi_v)$ of $\mathrm{GL}(2, \hat{\mathcal{O}}_K \times (K \otimes \mathbf{R}))$. Loosely speaking, $\tau(\pi)$ measures the ramification of π at the finite places and records the components of π at the infinite places. If the collection of infinite places is denoted $\{v_1, \dots, v_n\}$ and if $\pi_{v_i} \cong \mathcal{D}_{k_i, w_i}$, then $w_1 = \dots = w_n$ and the integers k_i and w_i all have the same parity. Such a representation π arises from a Hilbert modular form of weights (k_1, \dots, k_d) , cf. [Oht83].

We have that $\tau(\pi_v)$ is the trivial representation for all finite places not dividing the level of π . Furthermore, the central character χ of π is an algebraic Hecke character of \mathbf{A}_K^* whose restriction to $\mathcal{O}_{K,v}^*$ (resp., K_v^*) equals the central character of $\tau(\pi_v)$ for all $v \in S_f$ (resp., $v \in S_\infty$).

In light of this we define the set $\text{Types}(K)$ of *global inertial types* to consist of the collections $\tau = (\tau_v)_{v \in S}$ satisfying the conditions:

- (1) For all but finitely many v , τ_v is the trivial representation.
- (2) There exists an algebraic Hecke character of \mathbf{A}_K whose component at each place v agrees with the central character of τ_v . (If one exists, then there are as many as the class number of F .)

We remark that condition (2) is equivalent to the condition that if χ_v is the central character of τ_v , then

$$(1.1) \quad \chi = \prod_v \chi_v \text{ vanishes on } \mathcal{O}_K^*.$$

Indeed, χ is a character of the subgroup $\prod_{v \in S_f} \mathcal{O}_{K_v}^* \times \prod_{v \in S_\infty} K_v^*$ of \mathbf{A}_K^* , and for χ to extend to an algebraic Hecke character it is necessary and sufficient that χ vanish on the intersection of this subgroup with the diagonally embedded group $K^* \subset \mathbf{A}_K^*$. This intersection is exactly the unit group \mathcal{O}_K^* .

If $\tau \in \text{Types}(K)$ is a global inertial type, we shall write $\tau = \tau_f \otimes \tau_\infty$ to denote the decomposition of τ into its finite and infinite components. Note that τ_f is a finite-dimensional representation of the compact group $\text{GL}(2, \hat{\mathcal{O}}_F)$ and τ_∞ is a representation of $\text{GL}(2, K \otimes \mathbf{R})$. If we write $\tau_{v_i} \cong D_{k_i, w_i}$ for the infinite places v_i , then Eq. 1.1 and the Dirichlet Unit Theorem imply that $w_1 = \cdots = w_n$.

Whenever π is a cuspidal automorphic representation of $\text{GL}(2)_K$ arising from a Hilbert modular form, $\tau(\pi)$ belongs to $\text{Types}(K)$. For any $\tau \in \text{Types}(K)$, we define

$$d(\tau) = \prod_v d(\tau_v),$$

the product making sense because all but finitely many factors are 1.

The notion of a global type τ being a twist of another type τ' is evident: this shall mean that for each finite (resp. infinite) place v there exist characters χ_v of \mathcal{O}_v^* (resp. K_v^*) such that $\tau_v = \tau'_v \otimes \chi_v$ for all places. (This can only be so if $\prod_v \chi_v^2$ vanishes on \mathcal{O}_K^* .)

1.3. Main Theorem. Let $S(\tau)$ denote the set of Hilbert modular forms π for which $\tau(\pi) = \tau$. Our main theorem is an estimate for the cardinality of $S(\tau)$.

Theorem 1.1. *We have*

$$\#S(\tau) = 2^{1-n} |\zeta_K(-1)| hd(\tau) + O(2^{\nu(\tau)}),$$

where

$$\begin{aligned} \zeta_K(s) &= \text{the Dedekind zeta function for } K \\ h &= \text{the class number of } K \\ \nu(\tau) &= \text{the number of finite places } v \text{ for which } \dim \tau_v > 1. \end{aligned}$$

The constant in the “O” only depends on the field K .

Note that $\zeta_K(-1)$ is a nonzero (in fact rational) number. By comparing the quantity $d(\tau)$ with the error term $2^{\nu(\tau)}$ (see Section 2.1), we will deduce the following

Corollary 1.2. *Up to twisting by one-dimensional characters, the set of global inertial types $\tau \in \text{Types}(K)$ for which $S(\tau) = \emptyset$ is finite.*

Stated rather loosely, this means that there always exists a Hilbert modular form with prescribed ramification data, so long as: the desired weight is large enough at one of the infinite places, OR the desired inertial representation at a finite place v is ramified deeply enough, OR enough primes are permitted to ramify. See Section 4.3 for an explicit account of the case $K = \mathbf{Q}$.

We have found it most convenient to divide the proof of Theorem 1.1 into two cases, depending on the parity of $n = [K : \mathbf{Q}]$. If n is even, we work with the definite quaternion algebra D ramified exactly at the infinite places. If n is odd, we work with the quaternion algebra D ramified at all but one of the infinite places. In each case, we wish to compute the multiplicity of a global type in a space of automorphic forms on a Shimura variety corresponding to D . The dimension of the Shimura variety will be 0 or 1 as n is even or odd, respectively.

2. TYPES FOR $\text{GL}(2)$

2.1. Definition of Types. Let F be a p -adic field with ring of integers \mathcal{O}_F , maximal ideal \mathfrak{p}_F , and residue field \mathbf{F}_q . In this section we gather the necessary definitions and facts concerning types for $\text{GL}(2, F)$.

We must first define the association $\pi \mapsto \tau(\pi)$ attaching types to objects of $\mathcal{A}_2(F)$. We do this by cases as follows. It will simplify matters to assume in the following discussion that π is of minimal conductor among its twists by characters of F^* . Types for π not satisfying this condition can be defined via the relation $\tau(\pi \otimes \chi) = \tau(\pi) \otimes \chi|_{\mathcal{O}_F^*}$.

- (1) If π belongs to the unramified principal series then $\tau(\pi)$ is the trivial representation of $\text{GL}(2, \mathcal{O}_F)$.
- (2) If π is the principal series representation corresponding to a pair of characters χ_1, χ_2 of F^* with $\chi_1\chi_2^{-1}$ ramified, twist π so as to

assume that χ_2 is trivial. Let \mathfrak{p}^c be the conductor of χ_1 . Then $\tau(\pi)$ is induced from the character $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \chi_1(a)$ of

$$\Gamma_0(\mathfrak{p}^c) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathcal{O}_F) \mid c \equiv 0 \pmod{\mathfrak{p}^c} \right\}.$$

By [Cas73], Prop. 1, $\tau(\pi)$ is an irreducible representation of $\mathrm{GL}(2, \mathcal{O}_F)$. We then have $d(\pi) = \dim \tau(\pi) = q^{c-1}(q+1)$.

- (3) If $\pi = \mathrm{St}$ is the Steinberg representation of $\mathrm{GL}(2, F)$, then $\tau(\pi)$ is pulled back from the unique irreducible q -dimensional representation $\mathrm{St}_{\mathrm{GL}(2, \mathcal{O}_F)}$ of $\mathrm{GL}(2, \mathbf{F}_q)$ contained in the permutation representation on $\mathbf{P}^1(\mathbf{F}_q)$. In this case we have defined $d(\tau(\pi)) = q - 1$.
- (4) If π is supercuspidal, then it is induced from a finite-dimensional irreducible representation λ of a compact-mod-center subgroup $J \subset \mathrm{GL}(2, F)$. These were constructed in [Kut78a] and [Kut78b]; we review the construction in Section 2.2. By replacing J with a conjugate we may assume that the maximal compact subgroup J^0 of J lies in $\mathrm{GL}(2, \mathcal{O}_F)$; then $\tau(\pi) = \mathrm{Ind}_{J^0}^{\mathrm{GL}(2, \mathcal{O}_F)} \lambda|_{J^0}$ is irreducible; see [Hen02], A.3.1. When $\sigma(\pi)$ is induced from a character θ of a quadratic extension E/F of conductor $c \geq 1$, we will see in the next section that the dimension of $\tau(\pi)$ is given by

$$\dim \tau(\pi) = \begin{cases} q^{c-1}(q-1), & E/F \text{ unramified,} \\ q^{c-2}(q^2-1), & E/F \text{ ramified.} \end{cases}$$

A representation π of $\mathrm{GL}(2, F)$ is special if it is a twist of St . We shall call a type $\tau(\pi)$ principal series (resp., special, supercuspidal) when π is principal series (resp., special, supercuspidal). From [Hen02], A.1.5 we deduce the theorem:

Theorem 2.1. *Let $\pi, \pi' \in \mathcal{A}_2(F)$. The following are equivalent:*

- (1) $\pi'|_{\mathrm{GL}(2, \mathcal{O}_F)}$ contains $\tau(\pi)$.
- (2) $\sigma(\pi)|_{I_F} \cong \sigma(\pi')|_{I_F}$ or $\pi = \chi \otimes \mathrm{St}$ and π' is the principal series representation attached to two unequal characters of F^* whose restriction to \mathcal{O}_F^* agrees with $\chi|_{\mathcal{O}_F^*}$.

Therefore if $\pi, \pi' \in \mathcal{A}_2(F)$ and π' contains $\tau(\pi)$, then $\tau(\pi') = \tau(\pi)$, unless we are in the case of the ‘‘or’’ clause above, in which case $\tau(\pi)$ is a one-dimensional character χ and $\tau(\pi') = \chi \otimes \mathrm{St}_{\mathrm{GL}(2, \mathcal{O}_F)}$.

A word on the dimensions of types is in order. Suppose τ runs through a sequence of irreducible representations of $\mathrm{GL}(2, \mathcal{O}_F)$. If we

assume that no member of this sequence is the twist of any other by a one-dimensional character, then we must have $\dim \tau \rightarrow \infty$. Of course this implies $d(\tau) \rightarrow \infty$ as well. Furthermore, an irreducible representation of $\mathrm{GL}(2, \mathcal{O}_F)$ of least dimension other than 1 is one that is inflated from a cuspidal representation of $\mathrm{GL}(2, \mathbf{F}_q)$, and this has dimension $q - 1$. Therefore we have the lower bound $d(\tau) \geq q - 1$ whenever τ is not one-dimensional.

We shall now deduce Cor. 1.2 from Thm. 1.1. Let K be a totally real field and suppose that τ runs through a sequence of global inertial types of K , no two of which are twists of each other. To prove the corollary it will suffice to show that the sequence $d(\tau)/2^{\nu(\tau)}$ tends to infinity. Assume instead that it is bounded. By the previous paragraph we have the lower bound

$$d(\tau)/2^{\nu(\tau)} \geq \prod_v \frac{q_v - 1}{2},$$

where the product runs over the finite places v for which τ_v is not one-dimensional. Since the left hand side of this is bounded, the global types τ can only have higher-dimensional components at a finite set S_0 of finite places. This implies that $\nu(\tau)$ is bounded from above; by the hypothesis that $d(\tau)/2^{\nu(\tau)}$ is bounded we must have that $d(\tau)$ is bounded as well. We claim there must exist a place $v_0 \in S_0$ for which τ_{v_0} assumes infinitely many distinct values up to twisting. The alternative is that for each place v in S_0 , the sequence τ_v comprises only finitely many types along with their twists; by the pigeonhole principle, this would imply that the sequence τ contains two types which are twists of one another, contradicting the hypothesis. But then by the previous paragraph $d(\tau_{v_0}) \rightarrow \infty$, contradicting the boundedness of $d(\tau)$. The corollary is proved.

2.2. Trace bounds for types. Define the *level* of a type τ to be the least integer ℓ for which τ factors through $\mathrm{GL}(2, \mathcal{O}_F/\mathfrak{p}^\ell)$. We shall say that the *essential level* of τ is the least level of all the twists of τ .

We need a lemma concerning the non-archimedean local types.

Lemma 2.2. *Let $g \in \mathrm{GL}(2, \mathcal{O}_F)$ be a matrix whose characteristic polynomial has discriminant in \mathcal{O}_F^* . Then for all $\tau \in \mathrm{Types}(F)$, we have*

$$(2.1) \quad \begin{cases} |\mathrm{Tr}(\tau(g))| = 1, & \tau \text{ one-dimensional or special,} \\ |\mathrm{Tr}(\tau(g))| \leq 2, & \text{all other cases.} \end{cases}$$

Furthermore, if we relax the hypothesis on g and merely assume that g has distinct eigenvalues in \overline{F}^* , then $|\mathrm{Tr} \tau(g)|$ is bounded as τ ranges through $\mathrm{Types}(F)$.

Proof. We proceed by taking cases with respect to the structure of τ . If τ is one-dimensional, the inequality of the lemma is obvious. If τ is special, then up to twisting it is the inflation of the Steinberg representation $\text{St}_{\text{GL}(2, \mathbf{F}_q)}$. If the characteristic polynomial of $g \in \text{GL}(2, \mathcal{O}_F)$ has unit discriminant then its reduction $\bar{g} \in \text{GL}(2, \mathbf{F}_q)$ has distinct eigenvalues. From the formula

$$\text{St}_{\text{GL}(2, \mathbf{F}_q)} = [\text{Ind}_B^{\text{GL}(2, \mathbf{F}_q)} 1] - [1], \quad B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\},$$

we see easily that $\text{Tr}(\text{St}_{\text{GL}(2, \mathbf{F}_q)}(\bar{g}))$ equals 1 if \bar{g} has eigenvalues in \mathbf{F}_q^* and -1 otherwise, thus establishing Eq. 2.1. Of course we have $|\text{Tr} \tau(g)| \leq \dim \tau = q$ no matter what g is.

We now give a more explicit description of τ in the remaining cases. In each of these cases τ is induced from a “small” representation of a subgroup of $\text{GL}(2, \mathcal{O}_F)$. Since we are interested in the trace of τ , the following form of Mackey’s theorem will be useful:

$$(2.2) \quad \text{Tr}(\text{Ind}_H^G \eta)(g) = \sum_{[x] \in (G/H)^g} \text{Tr} \eta(x^{-1}gx),$$

where the sum ranges over cosets $[x] = xH \in G/H$ which are fixed under the left action of g . We will be applying this equation to various finite-index subgroups $J^0 \subset \text{GL}(2, \mathcal{O}_F)$.

Suppose τ is the type of a ramified principal series representation. By replacing τ with a twist we may assume, as in the previous subsection,

that τ is induced from the character $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \chi(a)$ of $\Gamma_0(\mathfrak{p}^c)$. Note

that we can identify the quotient $\text{GL}(2, \mathcal{O}_F)/\Gamma_0(\mathfrak{p}^c)$ with the projective line $\mathbf{P}^1(\mathcal{O}_F/\mathfrak{p}^c)$, together with its natural action of $\text{GL}(2, \mathcal{O}_F)$. If \bar{g} has distinct eigenvalues then g only has zero or two fixed points on $\mathbf{P}^1(\mathcal{O}_F/\mathfrak{p}^c)$ and therefore $|\text{Tr} \tau(g)| \leq 2$ in this case. If we lift this assumption and merely assume that g has distinct eigenvalues in \bar{F}^* ,

then g may have more than two fixed points on $\mathbf{P}^1(\mathcal{O}_F/\mathfrak{p}^c)$. However, we claim that as $c \rightarrow \infty$ then this number of fixed points remains bounded. To establish this, we may pass from F to an extension E containing the eigenvalues of g and then show that g has a bounded number of fixed points on $\mathbf{P}^1(\mathcal{O}_E/\mathfrak{p}_E^c)$ as $c \rightarrow \infty$. Working over E ,

we have up to conjugacy $g = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$, with $\alpha, \beta \in \mathcal{O}_E^*$; let m be the valuation in E of $\alpha - \beta$. Then for $c \geq m$, the fixed points of g on $\mathbf{P}^1(\mathcal{O}_E/\mathfrak{p}_E^c)$ are exactly the points $[x : 1]$ and $[1 : y]$, where x and y range through those elements of $\mathcal{O}_E/\mathfrak{p}_E^c$ of valuation at least $c - m$. Thus g has at most $2(\#\mathcal{O}_E/\mathfrak{p}_E^c)^m$ fixed points on $\mathbf{P}^1(\mathcal{O}_E/\mathfrak{p}_E^c)$, and this

number is bounded as $\mathfrak{c} \rightarrow \infty$, proving the claim. Therefore $|\mathrm{Tr} \tau(g)|$ is bounded as τ ranges through all types of ramified principal series representations.

Now suppose τ is the type of a minimal supercuspidal representation π . To prove the required inequality, we need to give a more detailed description of τ . Here we follow [Hen02], A.3.2–A.3.8, wherein an exhaustive list of the supercuspidal types is given. Let $k = \mathcal{O}_F/\mathfrak{p}_F$, let ϖ_F be a uniformizer of F , and let ψ be an additive character of F which vanishes on \mathfrak{p}_F but not on \mathcal{O}_F . There are three cases to consider:

- (1) π has conductor \mathfrak{p}^2 . In this case τ is inflated from an irreducible cuspidal representation of $\mathrm{GL}_2(k)$. These in turn are in correspondence with certain characters θ of the multiplicative group of a quadratic extension k_2/k . If θ is a character of k_2^* unequal to its k -conjugate, then there is a unique $(q-1)$ -dimensional representation τ_θ of $\mathrm{GL}(2, k)$ satisfying

$$\mathrm{Tr} \tau_\theta(\bar{g}) = \begin{cases} -(\theta(\alpha_1) + \theta(\alpha_2)), & \bar{g} \text{ has eigenvalues } \alpha_1, \alpha_2 \in k_2^* \setminus k^* \\ 0, & \bar{g} \text{ has distinct eigenvalues in } k^*. \end{cases}$$

Then the type τ is inflated from one of the representations τ_θ . The desired inequality $|\mathrm{Tr} \tau(g)| \leq 2$ is obvious from the above description so long as \bar{g} has distinct eigenvalues.

- (2) π has conductor $\mathfrak{p}_F^\mathfrak{c}$, $\mathfrak{c} \geq 4$ even. Let E/F be the unique unramified quadratic extension field, and let $n = \mathfrak{c}/2$. Let $b \in E^*$ be of the form $\varpi^{-n}u$, where $u \in \mathcal{O}_E^*$ has residue class in $\mathcal{O}_E/\mathfrak{p}_E$ which generates that field over k . Let θ be a character of E^* for which $\theta(1+x) = \psi \circ \mathrm{Tr}_{E/F}(bx)$ for $b \in \mathfrak{p}_E^{\lfloor (n+1)/2 \rfloor}$.

Choose an embedding $E \hookrightarrow M_2(F)$ so that $M_2(\mathcal{O}_F) \cap E = \mathcal{O}_E$. Define the subgroup $J^0 \subset \mathrm{GL}(2, \mathcal{O}_F)$ by

$$J^0 = \mathcal{O}_E^* \left(1 + \mathfrak{p}_F^{\lfloor (n+1)/2 \rfloor} M_2(\mathcal{O}_F) \right).$$

Then $\tau = \mathrm{Ind}_{J^0}^{\mathrm{GL}(2, \mathcal{O}_F)} \eta$ for a representation η of J^0 which we now describe. If n is odd then η is a character of J^0 defined by the conditions $\eta|_{\mathcal{O}_E^*} = \theta|_{\mathcal{O}_E^*}$ and $\theta(1+x) = \psi \circ \mathrm{Tr}(bx)$ for $x \in \mathfrak{p}_F^{\lfloor (n+1)/2 \rfloor} M_2(\mathcal{O}_F)$. If n is even then η is the unique irreducible representation of J^0 of dimension q satisfying $\mathrm{Tr} \eta(\alpha(1+x)) = -\theta(\alpha)$ whenever $x \in \mathfrak{p}_F^{\lfloor (n+1)/2 \rfloor}$ and $\alpha \in \mathcal{O}_E^*$ is such that the image of α in $\mathcal{O}_E/\mathfrak{p}_E$ does not lie in k . We remark that representations π of $\mathrm{GL}(2, F)$ containing the type τ have conductor $\mathfrak{p}_F^{2n} = \mathfrak{p}_F^\mathfrak{c}$.

To apply Mackey's theorem we need to consider the coset space $\mathrm{GL}(2, \mathcal{O}_F)/J^0$ together with its left $\mathrm{GL}(2, \mathcal{O}_F)$ -action. If

we let

$$\mathcal{H}_n = \left\{ \alpha \in \left(\mathcal{O}_E / \mathfrak{p}_E^{\lfloor (n+1)/2 \rfloor} \right)^* \mid \alpha \pmod{\mathfrak{p}_E} \notin k \right\},$$

then there is a natural left action $(g, b) \mapsto g \cdot b$ of $\mathrm{GL}(2, \mathcal{O}_F)$ on \mathcal{H}_c via fractional linear transformations. If $b \in \mathcal{H}_n$ is a fixed point of \mathcal{O}_E^* then $g \mapsto g \cdot b$ gives a $\mathrm{GL}(2, \mathcal{O}_F)$ -equivariant bijection

$$(2.3) \quad \mathrm{GL}(2, \mathcal{O}_F) / J^0 \cong \mathcal{H}_n,$$

much as the coset space $\mathrm{GL}(2, \mathbf{R}) / O(2, \mathbf{R})$ is identified with the upper half plane. If $g \in \mathrm{GL}(2, \mathcal{O}_F)$ is such that \bar{g} has distinct eigenvalues, then g has at most two fixed points in \mathcal{H}_n and we find $|\mathrm{Tr} \tau(g)| \leq 2$. (In fact if g has eigenvalues α, β lying in $\mathcal{O}_{F'}$ then $\mathrm{Tr} \tau(g) = (-1)^{n-1}(\theta(\alpha) + \theta(\beta))$.) On the other hand if g is merely assumed to have distinct eigenvalues then g may have more than two fixed points on \mathcal{H}_n . However, the number of fixed points is bounded as $n \rightarrow \infty$, by the same argument given in the previous paragraph. Therefore $|\mathrm{Tr} \tau(g)|$ is bounded as τ runs through types of supercuspidal representations of this sort.

- (3) π has conductor \mathfrak{p}_F^c , $c \geq 3$ odd. Let E/F be a ramified quadratic extension, and let $n = c - 2$. Let $b \in E^*$ have valuation $-n$. Let θ be a character of E^* satisfying $\theta(1+x) = \psi \circ \mathrm{Tr}_{E/F}(bx)$ whenever $x \in \mathfrak{p}_E^{(n+1)/2}$.

Let $\mathfrak{A} \subset M_2(\mathcal{O}_F)$ be the algebra

$$\mathfrak{A} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \in \mathfrak{p}_F \right\},$$

and choose an embedding $E \hookrightarrow M_2(F)$ in such a way that $\mathfrak{A} \cap E = \mathcal{O}_E^*$. Let $P_{\mathfrak{A}} \subset \mathfrak{A}$ be the double-sided ideal of matrices

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, c, d \in \mathfrak{p}_F$. Our subgroup J^0 is then

$$J^0 = \mathcal{O}_E^* \left(1 + P_{\mathfrak{A}}^{(n+1)/2} \right)$$

and η is the character $\alpha(1+x) \mapsto \theta(\alpha)\psi(\mathrm{Tr}(bx))$ for $\alpha \in \mathcal{O}_E^*$, $x \in P_{\mathfrak{A}}^{(n+1)/2}$. Then $\tau = \mathrm{Ind}_{J^0}^{\mathrm{GL}(2, \mathcal{O}_F)} \eta$ is a type contained in supercuspidal representations π of $\mathrm{GL}(2, F)$ of conductor $n+2$.

Now suppose $g \in \mathrm{GL}(2, \mathcal{O}_F)$ is such that \bar{g} has distinct eigenvalues. If \bar{g} has irreducible characteristic polynomial, then no conjugate of g can possibly lie in \mathfrak{A}^* , let alone in J^0 , so that $\mathrm{Tr} \tau(g) = 0$. The alternative is that up to conjugacy $g \in \mathfrak{A}^*$

equals the diagonal matrix with eigenvalues $\alpha, \beta \in \mathcal{O}_F^*$ whose residue classes are unequal. Let \mathcal{H}_n be the quotient of the set $\mathfrak{p}_E^{-1} \setminus \mathcal{O}_E$ by the group $1 + \mathfrak{p}_E^n$. The analogue of Eq. 2.3 is the \mathfrak{A}^* -equivariant bijection $\mathfrak{A}^*/J^0 \cong \mathcal{H}_n$. Since g has no fixed points on \mathcal{H}_n we have $\text{Tr } \tau(g) = 0$ as well.

Now assume only that g has distinct eigenvalues. Let $\lambda = \text{Ind}_{J^0}^{\mathfrak{A}^*} \eta$. Then by Mackey's Theorem $\text{Tr } \tau(g)$ is a sum over at most $\# \text{GL}(2, \mathcal{O}_F)/\mathfrak{A}^* = q + 1$ terms of the trace $\text{Tr } \lambda$ evaluated on conjugates of g . The same argument from the previous paragraph shows that an element $h \in \mathfrak{A}^*$ with distinct eigenvalues has a bounded number of fixed points on \mathcal{H}_n as $n \rightarrow \infty$. Thus $|\text{Tr } \tau(g)|$ is bounded as τ runs through types of supercuspidal representations of this sort as well. □

Lemma 2.2 has a global consequence which we will need in the sequel. Let K be a totally real field and suppose B/K is a quaternion algebra (possibly $M_2(K)$) which is split at all finite places. Suppose $\mathcal{O}_B \subset B$ is a maximal order. Then for all finite places v we may identify $\mathcal{O}_B \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v}$ with $\text{GL}(2, \mathcal{O}_{F_v})$. Let g_v be the image of $g \otimes 1$ under this isomorphism. If $\tau = \tau_f \otimes \tau_\infty \in \text{Types}(K)$, let τ_f be the representation of \mathcal{O}_B^* defined by $g \mapsto \prod_{v \text{ finite}} \tau_v(g_v)$.

Lemma 2.3. *Let $g \in \mathcal{O}_B$ be an element whose (reduced) characteristic polynomial has distinct eigenvalues in \overline{K}^* . There is a constant C depending only on g (and of course K) such that for all types $\tau = \tau_f \otimes \tau_\infty \in \text{Types}(K)$, $|\text{Tr } \tau_f(g)| \leq C 2^{\nu(\tau) - n_{\text{sp}}(\tau)}$, where $n_{\text{sp}}(\tau)$ is the number of finite places of K at which τ is special.*

Proof. Indeed, if such a g is given then only finitely many finite places of K will divide the discriminant of the characteristic polynomial of g . Let S_g be the set of such places. For each $v \in S_g$, the preceding lemma shows that there exists a bound C_v so that for all $\tau_v \in \text{Types}(K_v)$ we have $|\text{Tr } \tau_v(g)| \leq C_v$. Let $C = \prod_{v \in S_g} C_v$. For every finite place $v \notin S_g$, Lemma 2.1 shows that $|\text{Tr } \tau_v(g_v)| \leq 1$ if τ_v is special or one-dimensional and $|\text{Tr } \tau_v(g_v)| \leq 2$ otherwise. Therefore if τ_f is the finite part of a global type, we have the inequality

$$|\text{Tr } \tau_f(g)| = \prod_{v \text{ finite}} |\tau_v(g_v)| \leq \left(\prod_{v \in S_g} C_v \right) 2^{\nu(\tau) - n_{\text{sp}}(\tau)}$$

as required. □

3. PROOF OF THEOREM 1.1 IN THE CASE OF $[K : \mathbf{Q}]$ EVEN

Assume that $n = [K : \mathbf{Q}]$ is even. Let S_f and S_∞ denote the finite and infinite places of K , respectively. Let \mathbf{A} and \mathbf{A}_f denote the adèles and finite adèles of K . Finally, let D/K be the quaternion algebra ramified exactly at S_∞ , and let $G = D^*$ be the inner form of $\mathrm{GL}(2)$ corresponding to D . The Jacquet-Langlands correspondence $\mathrm{JL} : \pi' \mapsto \pi$ puts automorphic representations of $G(\mathbf{A})$ in bijection with those automorphic representations of $\mathrm{GL}(2, \mathbf{A})$ which are discrete series at the infinite places, see [JL70]. We will also use the symbol JL to mean the local Jacquet-Langlands correspondence between representations of D_v and those of $\mathrm{GL}(2, K_v)$ for any particular v .

Now suppose $\tau \in \mathrm{Types}(K)$ is a global type for K . Choose a maximal order \mathcal{O} of D . For each finite place $v \in S_0$ there exists an isomorphism $\mathcal{O}_v \cong \mathrm{GL}(2, \mathcal{O}_{K_v})$, unique up to conjugacy. Let τ'_v be the pull-back of τ_v through any such isomorphism; this is well-defined up to isomorphism. Let $\tau'_f = \bigotimes_{v \in S_f} \tau'_v$; this is a finite-dimensional irreducible representation of $\hat{\mathcal{O}}^*$. Let also $\tau'_\infty = \bigotimes_{v \in S_\infty} \mathrm{JL}^{-1}(\tau_v)$; this is a finite-dimensional representation of $G(K \otimes \mathbf{R}) = \prod_{v \in S_\infty} G(K_v)$. Finally, let $\tau' = \tau'_f \otimes \tau'_\infty$; this is a representation of $G(\hat{\mathcal{O}}_K \times (K \otimes \mathbf{R}))$. We record the relationship

$$(3.1) \quad d(\tau) = \dim \tau' \prod_{v \in S_{\mathrm{sp}}} \left(1 - \frac{1}{q_v}\right),$$

where S_{sp} is the set of places at which τ is special and q_v is the cardinality of the residue field of a finite place v .

Because multiplicity one holds for $\mathrm{GL}(2)$ and for G , we see that counting the automorphic representations of $\mathrm{GL}(2, \mathbf{A}_K)$ whose restriction to $\mathrm{GL}(2, \hat{\mathcal{O}}_K \times (K \otimes \mathbf{R}))$ contains τ is the same as counting the automorphic representations of $G(\mathbf{A}_K)$ whose restriction to $G(\hat{\mathcal{O}}_K \times (K \otimes \mathbf{R}))$ contains τ' . Write $\mu(\tau)$ for this number of automorphic representations. It is not necessarily the case that $\mu(\tau)$ is the cardinality of $S(\tau)$, the set of automorphic representations of type τ , due to the possibility of special components. We will compute $\#S(\tau)$ in terms of $\mu(\tau)$ at the end of the section.

To compute $\mu(\tau)$, we first realize it as the multiplicity of τ'_f inside a space of automorphic forms of “weight τ'_∞ ”, namely

$$(3.2) \quad M(\tau'_\infty) = \mathcal{C}_{G(K \otimes \mathbf{R})}(G(K) \backslash G(\mathbf{A}), \tau'_\infty),$$

the space of functions f on $G(\mathbf{A})$ taking values in the vector space underlying τ'_∞ which are left- $G(K)$ -invariant and which satisfy $f(xg_{\mathbf{R}}^{-1}) =$

$\tau'_\infty(g_{\mathbf{R}})f(x)$ for $x \in G(\mathbf{A})$, $g_{\mathbf{R}} \in G(K \otimes \mathbf{R})$. This space has a left action of $G(\mathbf{A}_f)$ via $(gf)(x) = f(xg)$; the automorphic representations of G which are τ'_∞ at the infinite places are exactly the irreducible $G(\mathbf{A}_f)$ -stable subrepresentations of $M(\tau'_\infty)$. Because an element of $M(\tau'_\infty)$ is determined by its restriction to $G(\mathbf{A}_f)$, we may rewrite Eq (3.2) as

$$(3.3) \quad M(\tau'_\infty) = \mathcal{C}_{G(K)}(G(\mathbf{A}_f), \tau'_\infty);$$

that is, the space of functions f on $G(\mathbf{A}_f)$ with values in $V(\tau'_\infty)$ satisfying $f(kg) = \tau'_\infty(k)f(g)$ for $k \in G(K)$ and $g \in G(\mathbf{A}_f)$.

Let T be a set of double coset representatives for $G(K) \backslash G(\mathbf{A}_f) / \hat{\mathcal{O}}^*$. Then T is finite by [Vig80], Théorème 5.4. For each $t \in T$, there is a corresponding maximal order $\mathcal{O}_t = t\hat{\mathcal{O}}t^{-1} \cap D$. Then the right-hand side of Eq (3.3) decomposes into a sum of $\hat{\mathcal{O}}^*$ -stable spaces indexed by T :

$$M(\tau'_\infty) = \bigoplus_{t \in T} \mathcal{C}_{G(K)}(G(K)t\hat{\mathcal{O}}^*, \tau'_\infty);$$

with some thought each summand on the right is seen to be isomorphic as a left $\hat{\mathcal{O}}^*$ -module to $\text{Ind}_{\mathcal{O}_t^*}^{\hat{\mathcal{O}}^*}(\tau'_\infty)^\vee$, where \mathcal{O}_t^* is to be regarded as a subgroup of $\hat{\mathcal{O}}^*$ via conjugation by t , and $(\tau'_\infty)^\vee$ is the contragredient of τ'_∞ .

By Frobenius reciprocity, the desired multiplicity $\mu(\tau)$ is therefore a sum of terms

$$\mu(\tau) = \sum_{t \in T} \mu_t,$$

where μ_t is the inner product of the \mathcal{O}_t^* -modules $\tau'_f|_{\mathcal{O}_t^*}$ and $(\tau'_\infty)^\vee|_{\mathcal{O}_t^*}$. That is, μ_t is the multiplicity of the trivial character in the restriction of $\tau' = \tau'_f \otimes \tau'_\infty$ to \mathcal{O}_t^* . By Eq. 1.1, τ is trivial on \mathcal{O}_K^* ; it therefore factors through the finite group $W_t = \mathcal{O}_t^* / \mathcal{O}_K^*$. Let e_t be the order of W_t . Then

$$(3.4) \quad \mu_t = \frac{1}{e_t} \sum_{\omega \in W_t} \text{Tr } \tau'(\omega)$$

We claim that the term with $\omega = 1$ dominates the sum in Eq. (3.4). Indeed, suppose $\omega \in \mathcal{O}_t^*$ is *outside* of \mathcal{O}_K^* . Since ω^{e_t} belongs to the center of $G(K)$ but ω itself does not, ω is semisimple. Therefore by Lemma 2.3,

$$(3.5) \quad |\tau'_f(\omega)| \leq C_1 2^{\nu(\tau) - n_{\text{sp}}(\tau)}$$

for a constant C_1 . *A priori* this C_1 depends on ω , but since there are only finitely many ω under consideration, we may take C_1 to depend only on K .

We now turn to the infinite places. For each infinite place v , let $\iota_v: K_v \cong \mathbf{R}$ be the corresponding isomorphism. We have that $G(K_v)$ is isomorphic to the group of quaternions $\alpha + \beta j$, with α, β complex numbers which are not both zero. Let $\rho_v: G(K_v) \rightarrow \mathrm{GL}(2, \mathbf{C})$ be the representation

$$(3.6) \quad \alpha + \beta j \mapsto \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix},$$

so that the central character of ρ_v is ι_v . Suppose that τ_v is the discrete series representation $\mathcal{D}_{k_v, w}$ of $\mathrm{GL}(2, \mathbf{R})$. (Note that the integer w will not depend on the place v .) Then

$$\tau'_v = (\iota_v \circ N_{D_v/K_v})^{1 - \frac{k+w}{2}} \otimes \mathrm{Sym}^{k_v-2} \rho_v,$$

where N_{D_v/K_v} is the reduced norm from D_v to K_v ; see [Car86], 0.10.

Now suppose that ω is an element of $\mathcal{O}_t^* \setminus \mathcal{O}_K^*$. Let ω_v be the image of ω under ρ_v . Since $\omega_v^{e_t}$ is a scalar but ω_v is not, the eigenvalues of ω_v are $\zeta\alpha$ and $\zeta^{-1}\alpha$ for $\zeta \neq \pm 1$ satisfying $\zeta^{e_t} = 1$ and $\alpha^2 = \det w_v = \iota(N_{D_v/K_v} \omega)$. The trace of $\tau'_v(\omega)$ is then

$$\mathrm{Tr} \tau'_v(\omega) = (\zeta^k + \zeta^{k-2} + \cdots + \zeta^{-k}) \iota_v(N_{D_v/K_v} \omega)^{-w/2}.$$

The sum of roots of unity is bounded by a constant C_2 which depends only on e_t , and since t runs over a finite set, this constant may be taken to depend only on K . Therefore for each $v|\infty$ we have

$$(3.7) \quad |\mathrm{Tr} \tau'_v(\omega)| \leq C_2 \iota_v(N_{D_v/K_v} \omega)^{-w/2}.$$

Multiplying Eq. 3.7 for $v \in S_\infty$ gives

$$(3.8) \quad |\mathrm{Tr} \tau'_\infty(\omega)| \leq C_2^n \left(\prod_{v|\infty} N_{D_v/K_v} \omega \right)^{-w/2} = C_2^n,$$

where in the last step we have used the product formula together with the fact that the reduced norm of ω belongs to \mathcal{O}_K^* . Putting together Eqs. 3.5 and 3.8 we find the bound

$$(3.9) \quad |\mathrm{Tr} \tau'(\omega)| = |\mathrm{Tr} \tau_f(\omega) \mathrm{Tr} \tau'_\infty(\omega)| \leq C_1 C_2^n 2^{\nu(\tau) - n_{\mathrm{sp}}(\tau)}.$$

Applying Eq. (3.9) to the non-identity elements of the sum in Eq. (3.4) gives

$$(3.10) \quad \mu_t = \frac{1}{e_t} \dim \tau' + O(2^{\nu(\tau) - n_{\mathrm{sp}}(\tau)});$$

summing this over $t \in T$ gives

$$(3.11) \quad \mu(\tau) = \sum_{t \in T} \mu_t = \sum_{t \in T} \frac{1}{e_t} \dim \tau' + O(2^{\nu(\tau) - n_{\text{sp}}(\tau)})$$

Here we apply the ‘‘mass formula’’ (see [Vig80], p. 142, Corollaire 2.3):

$$\sum_{t \in T} \frac{1}{e_t} = 2^{1-n} |\zeta_K(-1)| h.$$

Applying this to Eq. (3.11) gives

$$(3.12) \quad \mu(\tau) = 2^{1-n} |\zeta_K(-1)| h \dim \tau' + O(2^{\nu(\tau) - n_{\text{sp}}(\tau)}).$$

As mentioned earlier, it is not necessarily true that $\mu(\tau) = \#S(\tau)$. Indeed, if v is a place at which τ_v is special, then $\mu(\tau)$ counts automorphic representations which are principal series as well as special, owing to the exceptional clause of Theorem 2.1: Types inside of special representations are also contained in principal series representations. We adjust for this possibility using the inclusion-exclusion principle. Let S_{sp} be the set of places at which τ is special. For $v \in S_{\text{sp}}$, suppose $\tau_v^{\text{ps}} \in \text{Types}(K_v)$ is the type of any principal series representation whose restriction to $\text{GL}(2, \mathcal{O}_{K,v})$ contains τ_v . Then τ_v^{ps} is nothing but the central character of τ_v .

For each subset Y of S_{sp} , let $\tau^Y \in \text{Types}(K)$ have the same components as τ , but with τ_v^{ps} in place of τ_v for each $v \in Y$. Note that for $v \in S_{\text{sp}}$, we have $\dim \tau_v = q_v$, so that

$$(3.13) \quad \dim(\tau^Y)' = \dim \tau' \prod_{v \in Y} q_v^{-1}.$$

Letting $\kappa = 2^{1-n} \zeta_K(-1) h$, we have the following expression for $S(\tau)$:

$$\begin{aligned} S(\tau) &= \sum_{Y \subseteq S_{\text{sp}}} (-1)^{\#Y} \mu(\tau^Y) \\ &= \sum_{Y \subseteq S_{\text{sp}}} (-1)^{\#Y} (\kappa \dim(\tau^Y)' + O(2^{\nu(\tau) - \#S_{\text{sp}}})) \quad \text{by Eq. (3.12)} \\ &= \sum_{Y \subseteq S_{\text{sp}}} (-1)^{\#Y} \kappa \dim \tau' \prod_{v \in Y} q_v^{-1} + O(2^{\nu(\tau)}) \quad \text{by Eq. (3.13)} \\ &= \kappa \dim \tau' \prod_{v \in S_{\text{sp}}} \left(1 - \frac{1}{q_v}\right) + O(2^{\nu(\tau)}) \\ &= \kappa d(\tau) + O(2^{\nu(\tau)}) \quad \text{by Eq. (3.1),} \end{aligned}$$

thus completing the proof of Theorem 1.1 when $[K : \mathbf{Q}]$ is even.

4. PROOF IN THE CASE OF $[K : \mathbf{Q}]$ ODD

4.1. Shimura Curves. In this section $n = [K : \mathbf{Q}]$ is odd. Label the infinite places of K as v_0, \dots, v_{n-1} . Let D/K be a quaternion algebra ramified exactly at $S_\infty \setminus \{v_0\}$, let G be the inner form of $\mathrm{GL}(2)_K$ corresponding to D , and let \mathcal{O} be a maximal order of D . Keep the notations \mathbf{A} and \mathbf{A}_f from the preceding section. For simplicity we will assume at first that $K \neq \mathbf{Q}$, and then explain how to modify the proof in the case of $K = \mathbf{Q}$ at the end of the section.

Let $X = \mathbf{C} \setminus \mathbf{R}$. For an ideal N of \mathcal{O}_K , the Shimura curve $X_D(N)$ of (full) level N is the one whose complex points are

$$X_D(N)(\mathbf{C}) = G(K) \backslash (X \times G(\mathbf{A}_f)) / U_N,$$

where $U_N \subset \hat{\mathcal{O}}^*$ is the compact open subgroup consisting of elements congruent to the identity modulo N . Then $X_D(N)$ admits an action of $\hat{\mathcal{O}}^*/U_N \cong \mathrm{GL}(2, \mathcal{O}_K/N)$.

Now suppose $\tau \in \mathrm{Types}(K)$ is a global type. For $0 \leq i \leq n-1$, suppose that $\tau_{v_i} \cong \mathcal{D}_{k_v, w}$, and let $\mathbf{k} = (k_0, \dots, k_{n-1})$. As in the case of n even, we define a representation τ'_f of $\hat{\mathcal{O}}^*$ as well as a representation τ'_∞ of $G(K \otimes \mathbf{R})$, the only difference being that τ'_{v_0} is essentially the same as τ_{v_0} because G is split at that place. Let also N be the level of τ'_f . As in the previous section, let $\mu(\tau)$ be the number of cuspidal automorphic representations π of G containing $\tau = \tau_f \otimes \tau_\infty$. The strategy is to determine $\mu(\tau)$ by computing the multiplicity of τ in the appropriate cohomology group of the Shimura curve $X_D(N)$.

We proceed as in [Car83], 2.1.2, by defining a complex vector bundle $\mathcal{L}/X_D(N)$ analytically by

$$\mathcal{L} = G(K) \backslash (X \times G(\mathbf{A}_f) \times V) / U_N.$$

Here $V = \bigotimes_{i=0}^{n-1} V_i$ is a certain representation of $\prod_{i=0}^{n-1} G(K_{v_i})$; for $i > 0$, V_i is τ'_{v_i} , while for $i = 0$, V_i is a twist of the $(k-2)$ nd symmetric power of the tautological representation of $G(K_{v_0}) = \mathrm{GL}(2, \mathbf{R})$ on \mathbf{C}^2 . Note that the vector bundle \mathcal{L} is equivariant under the action of $\mathrm{GL}(2, \mathcal{O}/N)$, so that the cohomology $H^1(X_D(N), \mathcal{L})$ admits an action of this group. We now appeal to [BW00], VII, 3.2 (or for our particular application, see [Car83], 2.2.4) to compute the de Rham cohomology of \mathcal{L} :

$$(4.1) \quad H^1(X_D(N), \mathcal{L}) \cong \bigoplus_{\pi} H^1(\mathfrak{g}, K_\infty, \pi_\infty \otimes V) \otimes (\pi_f)^{U_N}.$$

Here \mathfrak{g} is the Lie algebra of $G(K \otimes \mathbf{R})$ and $K_\infty \subset G(K \otimes \mathbf{R})$ is the stabilizer of a point of X . The sum runs over cuspidal automorphic representations π of G for which the summand is nonzero. It follows

from [RT83], Prop. 1.5, that $H^1(\mathfrak{g}, K_\infty, \pi_\infty \otimes V)$ is zero unless $\pi_\infty \cong \tau_\infty$, in which case it has dimension 2. The isomorphism in Eq. 4.1 being $\mathrm{GL}(2, \mathcal{O}_K/N)$ -equivariant, we find that

$$(4.2) \quad \mu(\tau) = \frac{1}{2} \langle \tau_f, H^1(X_D(N), \mathcal{L}) \rangle_{\mathrm{GL}(2, \mathcal{O}_K/N)}.$$

We now decompose the curve $X_D(N)$ into pieces which are stable under the action of $(\mathcal{O}/N\mathcal{O})^*$. Let $T = G(K) \backslash G(\mathbf{A}_f) / \hat{\mathcal{O}}^*$. There is a bijection between T and the class group $K^* \backslash \mathbf{A}_f^* / \hat{\mathcal{O}}_K^*$ of K , see [Vig80], Corollaire 5.7, part (i). We find that

$$\begin{aligned} X_D(N)(\mathbf{C}) &= \coprod_{t \in T} G(K) \backslash (X \times G(K)t\hat{\mathcal{O}}^*) / U_N \\ &= \coprod_{t \in T} \mathcal{O}_t^* \backslash (X \times \hat{\mathcal{O}}^*) / U_N, \end{aligned}$$

where $\mathcal{O}_t = t\hat{\mathcal{O}}t^{-1} \cap D$ is to be considered a subgroup of $\hat{\mathcal{O}}^*$ via conjugation by t . Write $X_{D,t}(N)$ for $\mathcal{O}_t^* \backslash (X \times \hat{\mathcal{O}}^*) / U_N$; it is a Riemann surface with an action of $\hat{\mathcal{O}}^* / U_N \cong \mathrm{GL}(2, \mathcal{O}_K/N)$. Suppose $\Gamma_t(1)$ is the image in $\mathrm{GL}(2, \mathbf{R})$ of the group of units of \mathcal{O}_t^* and that $\Gamma_t(N) \subset \Gamma_t(1)$ arises from the subgroup of elements congruent to 1 modulo N . It is easy to check that

$$(4.3) \quad X_{D,t}(N) = \hat{\mathcal{O}}^* / U_N \times_{\Gamma_t(1)/\Gamma_t(N)} \Gamma_t(N) \backslash X.$$

From Eq. 4.3 we have the isomorphism of $\hat{\mathcal{O}}^* / U_N$ -modules

$$H^1(X_{D,t}(N), \mathcal{L}) = \mathrm{Ind}_{\Gamma_t(1)/\Gamma_t(N)}^{\hat{\mathcal{O}}^*/U_N} H^1(\Gamma_t(N) \backslash X, \mathcal{L});$$

by abuse of notation we have written \mathcal{L} for the restriction of that vector bundle to both $X_{D,t}(N)$ and $\Gamma_t(N) \backslash X$. Let μ_t be the multiplicity of τ_f' inside $H^1(X_{D,t}, \mathcal{L})$ as an $\hat{\mathcal{O}}^* / U_N$ -module, so that by Eq. 4.2, $2\mu(\tau) = \sum_{t \in T} \mu_t$. By Frobenius reciprocity,

$$(4.4) \quad \mu_t = \langle \tau_f' |_{\Gamma_t(1)/\Gamma_t(N)}, H^1(\Gamma_t(N) \backslash X, \mathcal{L}) \rangle_{\Gamma_t(1)/\Gamma_t(N)}.$$

4.2. Cohomology of discrete groups. It therefore remains to compute the structure of $H^1(\Gamma_t(N) \backslash X, \mathcal{L})$ as a $\Gamma_t(1)/\Gamma_t(N)$ -module. Assume that N is large enough so that every nonscalar element of $\Gamma_t(N)$ acts without fixed points on X . As $\Gamma_t(N) \backslash X$ is compact, we have an isomorphism $H^1(\Gamma_t(N) \backslash X, \mathcal{L}) \cong H^1(\Gamma_t(N), V)$. It will suffice to compute the Euler characteristic

$$\chi(\Gamma_t(N), V) = \sum_{i=0}^2 (-1)^i [H^i(\Gamma_t(N), V)]$$

in the Grothendieck group of $\Gamma_t(1)/\Gamma_t(N)$. For this we have the:

Lemma 4.1. *Let $\Gamma \subset \mathrm{GL}(2, \mathbf{R})$ be a discrete subgroup acting on X with compact quotient. Let $x_1, \dots, x_r \in X$ be a complete set of Γ -inequivalent fixed points having fixed subgroups $\Gamma_1, \dots, \Gamma_r$. Suppose $\Gamma' \subset \Gamma$ is a normal subgroup acting without fixed points on X . Let $G = \Gamma/\Gamma'$ and let $Z \subset \Gamma$ be the group of elements of Γ which are scalar. Suppose V is a finite-dimensional complex vector space admitting an action of Γ such that scalar matrices in Γ' act trivially on V . Then we have the equality:*

$$(4.5) \quad \chi(\Gamma', V) = (\chi(\Gamma \backslash X) - r) \mathrm{Ind}_Z^G V|_Z + \sum_{i=1}^r \mathrm{Ind}_{\Gamma_i}^G V|_{\Gamma_i}$$

in the Grothendieck group of G .

Proof. We construct a simplicial complex \mathcal{K} with underlying topological space X in such a way that Γ acts on \mathcal{K} and such that each elliptic fixed point x_j is a vertex of \mathcal{K} . For $i = 0, 1, 2$ let $C_i(\mathcal{K})$ be the free \mathbf{C} -vector space with basis the i -cells of \mathcal{K} , and let $N_i = \dim C_i(\mathcal{K})$. Let $C^i(\mathcal{K}, V)$ be the space of $\mathbf{C}[\Gamma']$ -module homomorphisms $C_i(\mathcal{K}) \rightarrow V$. Then $C^i(\mathcal{K}, V)$ carries a left Γ -module structure: if $\gamma \in \Gamma$ and $f \in C^i(\mathcal{K}, V)$, then $(\gamma f)(x) = \gamma(f(\gamma^{-1}(y)))$. The action factors through an action of G . It is standard (see [Shi71], Prop. 8.1) that the cohomology of the complex

$$0 \rightarrow C^0(\mathcal{K}, V) \rightarrow C^1(\mathcal{K}, V) \rightarrow C^2(\mathcal{K}, V) \rightarrow 0$$

agrees with $H^i(\Gamma', V)$. Therefore $\chi(\Gamma', V)$ is the alternating sum of the $C^i(\mathcal{K}, V)$ in the Grothendieck ring of G .

We wish to compute the structure of $C^0(\mathcal{K}, V)$ as an $\mathbf{C}[G]$ -module. Let x_1, \dots, x_{N_0} be a complete set of Γ -inequivalent 0-cells of \mathcal{K} , with the fixed points x_1, \dots, x_r as in the hypothesis. Writing W_i for the \mathbf{C} -span of the Γ -orbit of x_i , we have a decomposition of $\mathbf{C}[\Gamma]$ -modules $C_0(\mathcal{K}) = \bigoplus_{i=1}^{N_0} W_i$ of $C_0(\mathcal{K})$, and therefore

$$C^0(\mathcal{K}, V) = \bigoplus_{i=1}^{N_0} \mathrm{Hom}_{\Gamma'}(W_i, V).$$

Note that $W_i \approx \mathrm{Ind}_{\Gamma_i}^{\Gamma} 1$ for $i \leq r$, and that $\Gamma_i = Z$ for $i \geq r + 1$. For each i , the $\mathbf{C}[G]$ -module $\mathrm{Hom}_{\Gamma'}(W_i, V)$ can be modeled on the space of functions $\Phi: \Gamma \rightarrow V$ satisfying $\Phi(\gamma'\gamma k) = \rho(\gamma')\Phi(\gamma)$ whenever $\gamma' \in \Gamma'$, $\gamma \in \Gamma$, $k \in \Gamma_i$. The action of G is given by $\bar{g}(\Phi)(\gamma) = \rho(g)\Phi(g^{-1}\gamma)$ whenever $\bar{g} = \Gamma'g \in G$.

We claim that there is an isomorphism of G -modules

$$\mathrm{Hom}_{\Gamma'}(W_i, V) \xrightarrow{\sim} \mathrm{Ind}_{\Gamma_i}^G V|_{\Gamma_i}.$$

The space on the right is modeled on the space of functions $\Psi: G \rightarrow V$ satisfying $\Psi(kg\Gamma') = \rho(k)\Psi(g\Gamma')$ for all $k \in \Gamma_i$, $g\Gamma \in G$. Choose a set of representatives g_1, \dots, g_s for $G/\Gamma_i = \Gamma' \backslash \Gamma/\Gamma_i$; it is easily checked that an isomorphism is given by $\Phi \mapsto \Psi$, where $\Psi(\Gamma'g_j) = g_j\Phi(g_j^{-1})$.

Thus in the Grothendieck group of G we have

$$\begin{aligned} C^0(\mathcal{K}, V) &= \sum_{i=1}^{N_0} \mathrm{Ind}_{\Gamma_i}^G V|_{\Gamma_i} \\ &= (N_0 - r) \mathrm{Ind}_Z^G V|_Z + \sum_{i=1}^r \mathrm{Ind}_{\Gamma_i}^G V|_{\Gamma_i}. \end{aligned}$$

A similar calculation holds for $C^1(\mathcal{K}, V)$ and $C^2(\mathcal{K}, V)$, except that there are no cells fixed by elements of Γ . Therefore:

$$\begin{aligned} C^1(\mathcal{K}, V) &= N_1 \mathrm{Ind}_Z^G V \\ C^2(\mathcal{K}, V) &= N_2 \mathrm{Ind}_Z^G V \end{aligned}$$

Taking the alternating sum of the $C^i(\mathcal{K}, V)$ gives the expression in the Lemma. \square

We apply Lemma 4.1 to the groups $\Gamma_t(N) \subset \Gamma_t(1)$ and to the representation V of $\Gamma_t(1)$. Keep the notations Γ_i , x_i , and r from the lemma. Since the multiplicity of τ in H^0 and H^2 is bounded, the multiplicity of τ in $\bigoplus_{t \in T} \chi(\Gamma_t(N), V)$ is $-\sum_{t \in T} \mu_t = -2\mu(\tau)$ to within an error term depending only on K . From Eq. (4.4) we have

$$(4.6) \quad -\mu_t = (\chi(X_t(1)) - r) \dim V \dim \tau'_f + \sum_{i=1}^r \langle \tau'_f|_{\Gamma_i}, V|_{\Gamma_i} \rangle + O(1)$$

Let $W_i = \Gamma_i/Z$ have order e_i . Since τ'_f and V have the same values on $Z = \mathcal{O}_K^*$, the summand on the right is

$$\langle \tau'_f|_{\Gamma_i}, V|_{\Gamma_i} \rangle = \frac{1}{e_i} \sum_{w \in W_i} \mathrm{Tr} \tau'_f(w) \overline{\mathrm{Tr}(w|V)}.$$

By an argument of the same sort as in the previous section, this quantity is $\frac{1}{e_i} \dim \tau'_f \dim V + O(2^{\nu(\tau) - n_{\mathrm{sp}}(\tau)})$. Therefore

$$-2\mu_t = \left[\chi(X_h(1)) - r + \sum_{i=1}^r \frac{1}{e_i} \right] \dim \tau'_f \dim V + O(2^{\nu(\tau) - n_{\mathrm{sp}}(\tau)}).$$

The expression in square brackets is

$$\begin{aligned} \chi(X_t(1)) - \sum_{i=1}^r \left(1 - \frac{1}{e_i}\right) &= -\frac{1}{2\pi} \operatorname{vol}(\Gamma_t(1) \backslash X) \\ &= -|\zeta_K(-1)| 2^{2-n} \end{aligned}$$

from [Vig80], p. 109, Exemple 5, together with p. 119, Prop. 2.10. Since $2\mu(\tau) = \sum_{t \in T} \mu_t$ and $\#T = h$, the argument continues exactly as in the previous section.

4.3. The case of $K = \mathbf{Q}$. Now suppose $K = \mathbf{Q}$. Let $\tau = \tau_f \otimes \tau_\infty \in \operatorname{Types}(\mathbf{Q})$. Let N be the level of τ , considered as a rational integer, and suppose the infinite component of τ is $\tau_\infty = D_{k,w}$. Assume $N > 2$. We note that condition (2) in the definition of global types reduces to the condition that the central character of τ_f take the value $(-1)^k$ at -1 .

Let $\mu(\tau)$ be the number of cuspidal automorphic representations π of $\operatorname{GL}(2, \mathbf{A}_{\mathbf{Q}})$ containing $\tau = \tau_f \otimes \tau_\infty$. The analogue of Eq. 4.1 in the case of $K = \mathbf{Q}$ is found in [Lan73], Thm. 2.10. The result is the same except that parabolic cohomology must be used. Let $Y_D(N)$ be the (non-complete) Shimura curve for the split algebra $D = M_2(\mathbf{Q})$. Let $\mathcal{L}/Y_D(N)$ be the vector bundle corresponding to τ_∞ ; then

$$(4.7) \quad \mu(\tau) = \frac{1}{2} \langle \tau_f, H_P^1(X_D(N), \mathcal{L}) \rangle_{\operatorname{GL}(2, \mathbf{Z}/N\mathbf{Z})}.$$

The curve $Y_D(N)$ has connected components, each of which is the classical modular curve $Y(N)$:

$$Y_D(N) = \operatorname{GL}(2, \mathbf{Z}/N\mathbf{Z}) \times_{\operatorname{SL}(2, \mathbf{Z}/N\mathbf{Z})} Y(N).$$

Therefore

$$\mu(\tau) = \frac{1}{2} \langle \tau_f|_{\operatorname{SL}(2, \mathbf{Z}/N\mathbf{Z})}, H_P^1(Y(N), \mathcal{L}) \rangle.$$

Our goal is therefore to determine the class of $H_P^1(Y(N), \mathcal{L})$ in the Grothendieck group of $\operatorname{SL}(2, \mathbf{Z}/N\mathbf{Z})$. Let $S_N = \operatorname{SL}(2, \mathbf{Z}/N\mathbf{Z})$. Since $N > 2$, the group $\Gamma(N)$ acts on the upper half-plane \mathcal{H} without fixed points and we have the S_N -equivariant isomorphism

$$(4.8) \quad H_P^1(Y(N), \mathcal{L}) \cong H_P^1(\Gamma(N), V_k),$$

where $V_k = \operatorname{Sym}^{k-2} V_3$ is the $(k-2)$ nd symmetric power of the tautological representation V_3 of $\operatorname{SL}(2, \mathbf{Z})$ on \mathbf{C}^2 . (For the definition of parabolic cohomology of a Fuchsian group, see [Shi71], 8.1.)

We now compute the S_N -module $H_P^1(\Gamma(N), V_k)$. The calculation hinges on the geometry of the Galois cover $X(N) \rightarrow X(1)$ of (complete) modular curves. The Galois group of this cover is $S_N/\{\pm I\}$. The

cover is branched over three points in $X(1)$, namely the images in $X(1)$ of the points $\rho = e^{2\pi i/3}$, i , and ∞ of the upper half plane \mathcal{H} . Those points have the following stabilizers in $\mathrm{SL}(2, \mathbf{Z})$:

$$\begin{aligned}\Gamma_\rho &= \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\rangle \\ \Gamma_i &= \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle \\ \Gamma_\infty &= \left\langle \pm \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \right\rangle.\end{aligned}$$

For $j \in \{\rho, i, \infty\}$, let $\bar{\Gamma}_j$ be the image of Γ_j in S_N . The center of $\mathrm{SL}(2, \mathbf{Z})$ is $Z = \{\pm 1\}$; let \bar{Z} be its image in S_N . Since $N > 2$, the reduction map $Z \rightarrow \bar{Z}$ is an isomorphism, and therefore any Z -module can be considered an \bar{Z} -module. The same is true for the groups Γ_j and $\bar{\Gamma}_j$.

To compute the structure of $H_P^1(\Gamma(N), V_k)$ as an S_N -module, we must modify Lemma 4.1 to include a term coming from the unique cusp of $X(1)$. Let $\chi_P(\Gamma(N), V_k)$ be the alternating sum of the parabolic cohomology groups $H_P^i(\Gamma(N), V_k)$ in the Grothendieck group of S_N . Let sgn denote the obvious character of \bar{Z} and of $\bar{\Gamma}_\infty$.

Lemma 4.2.

$$\chi_P(\Gamma(N), V_k) = - \left[\mathrm{Ind}_{\bar{Z}}^{S_N} V_k|_Z \right] + \sum_{j \in \{\rho, i\}} \left[\mathrm{Ind}_{\bar{\Gamma}_j}^{S_N} V_k|_{\Gamma_j} \right] + \left[\mathrm{Ind}_{\bar{\Gamma}_\infty}^{S_N} \mathrm{sgn}^k \right].$$

Proof. Remove a small open disc containing ∞ from the projective line $X(1)$, and let \mathcal{H}^o be the preimage of the result. This can be accomplished by removing from \mathcal{H} all the $\mathrm{SL}(2, \mathbf{Z})$ -translates of the region $\{x + iy | y \geq y_0\}$ for y_0 large enough. Construct an $\mathrm{SL}(2, \mathbf{Z})$ -stable simplicial complex \mathcal{K} whose underlying topological space is \mathcal{H}^o such that i and ρ are vertices of \mathcal{K} . Let $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, so that $\Gamma_\infty \cap \Gamma(N)$ is generated by u^N . Assume there is a 1-cell t_1 in \mathcal{K} for which ∂t_1 is of the form $(u - 1)z$ for a vertex z of \mathcal{K} . Then the 1-chain $\gamma := \sum_{j=0}^{N-1} u^j t_1$ has boundary $(u^N - 1)z$. The boundary of the quotient complex $\Gamma(N) \backslash \mathcal{K}$ is exactly the set of translates $g\gamma$, where g runs over a set of coset representatives for $\Gamma(N) \backslash \mathrm{SL}(2, \mathbf{Z})$. Note that $\partial(g\gamma) = (gu^N g^{-1} - 1)gz$. Let $C_P^1(\mathcal{K}, V_k) \subset C^1(\mathcal{K}, V_k)$ be the space of $\mathbf{C}[\Gamma(N)]$ -module homomorphisms $C_1(\mathcal{K}) \rightarrow V_k$ mapping the 1-cycle $g\gamma$ into $(gu^N g^{-1} - 1)V_k$ for each $g \in \mathrm{SL}(2, \mathbf{Z})$. Then the complex

$$1 \rightarrow C^0(\mathcal{K}, V_k) \rightarrow C_P^1(\mathcal{K}, V_k) \rightarrow C^2(\mathcal{K}, V_k) \rightarrow 1$$

has cohomology $H_P^*(\Gamma(N), V_k)$ (*loc. cit.*, Prop. 8.1). We remark that $(u^N - 1)V_k = (u - 1)V_k$ has codimension 1 in V_k .

Let t_1, t_2, \dots, t_{N_1} be a complete set of $\mathrm{SL}(2, \mathbf{Z})$ -inequivalent 1-cells of \mathcal{K} . Writing W_j for the \mathbf{C} -span of the $\mathrm{SL}(2, \mathbf{Z})$ -orbit of t_j , we have an isomorphism of $\mathrm{SL}(2, \mathbf{Z})$ -modules $W_j \cong \mathrm{Ind}_{\mathbf{Z}}^{\mathrm{SL}(2, \mathbf{Z})} 1$. Let $C_P^1(W_j, V_k) \subset C_P^1(\mathcal{K}, V_k)$ be the $\mathrm{SL}(2, \mathbf{Z})$ -submodule of cocycles supported on W_j , so that $C_P^1(\mathcal{K}, V_k) = \bigoplus_{j=1}^{N_1} C_P^1(W_j, V_k)$. Then for $j > 1$ we have

$$[C^P(W_j, V_k)] = [\mathrm{Hom}_{\Gamma(N)}(W_j, V_k)] = \left[\mathrm{Ind}_{\mathbf{Z}}^{S_N} V_k|_{\mathbf{Z}} \right]$$

as classes in the Grothendieck group of S_N . On the other hand for $j = 1$ we have the exact sequence of S_N -modules

$$(4.9) \quad 0 \rightarrow C_P^1(W_1, V_k) \rightarrow \mathrm{Hom}_{\Gamma(N)}(W_1, V_k) \xrightarrow{\xi} \mathrm{Hom}_{\bar{\Gamma}_\infty}(\mathbf{C}[S_N], V_k/(u-1)V_k) \rightarrow 0$$

where $\xi(f): \mathbf{C}[S_N] \rightarrow V_k/(u-1)V_k$ is defined by $\xi(f)(g) = g^{-1}f(g\gamma) + (u-1)V_k$. The homomorphism $\xi(f)$ is $\bar{\Gamma}_\infty$ -equivariant because

$$\begin{aligned} \xi(f)(gu) &= u^{-1}g^{-1}f(gu\gamma) \\ &= u^{-1}g^{-1}f(g(u^N - 1)t_1 + g\gamma) \\ &= u^{-1}g^{-1}(u^N - 1)f(gt_1) + u^{-1}g^{-1}f(g\gamma) \text{ because } u^N - 1 \in \mathbf{C}[\Gamma(N)] \\ &\equiv g^{-1}f(g\gamma) \equiv \xi(f)(g) \pmod{(u-1)V_k}. \end{aligned}$$

Since $[V_k/(u-1)V_k] = [\mathrm{sgn}^k]$ in the Grothendieck group of $\bar{\Gamma}_\infty$, Eq. 4.9 implies that

$$[C_P^1(W_1, V_k)] = \left[\mathrm{Ind}_{\mathbf{Z}}^{S_N} V_k|_{\mathbf{Z}} \right] - \left[\mathrm{Ind}_{\bar{\Gamma}_\infty}^{S_N} \mathrm{sgn}^k \right]$$

and therefore that

$$[C_P^1(\mathcal{K}, V_k)] = \bigoplus_{j=1}^{N_1} [C_P^1(W_j, V_k)] = N_1 \left[\mathrm{Ind}_{\mathbf{Z}}^{S_N} V_k|_{\mathbf{Z}} \right] - \left[\mathrm{Ind}_{\bar{\Gamma}_\infty}^{S_N} \mathrm{sgn}^k \right]$$

in the Grothendieck group of S_N .

Therefore the calculation of $\chi(\Gamma(N), V_k)$ proceeds as in Lemma 4.1 with the only change being that there is a contribution of $\left[\mathrm{Ind}_{\bar{\Gamma}_\infty}^{S_N} \mathrm{sgn}^k \right]$ coming from the space of 1-cochains. Since $\chi(\mathrm{SL}(2, \mathbf{Z}) \backslash \mathcal{H}^o) = 1$ and there are two $\mathrm{SL}(2, \mathbf{Z})$ -orbits of elliptic fixed points, the appropriate modification of Eq. 4.5 is

$$\chi_P(\Gamma(N), V_k) = - \left[\mathrm{Ind}_{\mathbf{Z}}^{S_N} V_k|_{\mathbf{Z}} \right] + \sum_{j \in \{\rho, i\}} \left[\mathrm{Ind}_{\Gamma_j}^{S_N} V_k|_{\Gamma_j} \right] + \left[\mathrm{Ind}_{\bar{\Gamma}_\infty}^{S_N} \mathrm{sgn}^k \right]$$

as required. \square

The relationship between $H_P^1(\Gamma(N), V_k)$ and $\chi_P(\Gamma(N), V_k)$ is given by

$$(4.10) \quad [H_P^1(\Gamma(N), V_k)] = \begin{cases} -\chi_P(\Gamma(N), V_k) - 2[1], & k = 2, \\ -\chi_P(\Gamma(N), V_k), & k > 2, \end{cases}$$

the reason being that both $H_P^0(\Gamma(N), V_k)$ and $H_P^2(\Gamma(N), V_k)$ are one-dimensional if $k = 2$ and vanish if $k > 2$. We analyze each of the three terms appearing on the right-hand side of Eq. 4.2. For the first term, note that $V_k|_Z$ is simply $k - 1$ copies of the sign character sgn^k of Z , so that

$$(4.11) \quad \left[\text{Ind}_{\mathbb{Z}}^{S_N} V_k|_Z \right] = (k - 1) \left[\text{Ind}_{\mathbb{Z}}^{S_N} \text{sgn}^k \right].$$

For the second term of Eq. 4.2, we have $V_3|_{\Gamma_\rho} = \chi_\rho \oplus \chi_\rho^{-1}$, where $\chi_\rho: \Gamma_\rho \rightarrow \mathbf{C}^*$ is the character $\begin{pmatrix} & -1 \\ 1 & 1 \end{pmatrix} \mapsto e^{2\pi i/6}$. Then

$$(4.12) \quad V_k|_{\Gamma_\rho} = \text{Sym}^{k-2} V_3|_{\Gamma_\rho} = \chi_\rho^{k-2} \oplus \chi_\rho^{k-4} \oplus \cdots \oplus \chi_\rho^{-k+2}$$

Each character of Γ_ρ of the same parity as k appears in the sum about $k/3$ times, so we expect the right side of Eq. 4.12 to contain about $k/3$ copies of the sum $\chi_\rho^k \oplus \chi_\rho^{k+2} \oplus \chi_\rho^{k+4} = \text{Ind}_Z^{\Gamma_\rho} \text{sgn}^k$. More precisely, we have the following relation in the Grothendieck group of Γ_ρ :

$$(4.13) \quad V_k|_{\Gamma_\rho} = \left\lfloor \frac{k}{3} \right\rfloor \left[\text{Ind}_Z^{\Gamma_\rho} \text{sgn}^k \right] + \varepsilon_\rho,$$

where the error term is a virtual representation of Γ_ρ given by

$$(4.14) \quad \varepsilon_\rho = \begin{cases} -[1], & k \equiv 0 \pmod{6} \\ 0, & k \equiv 1 \pmod{6} \\ [1], & k \equiv 2 \pmod{6} \\ -[\chi_\rho^3], & k \equiv 3 \pmod{6} \\ 0, & k \equiv 4 \pmod{6} \\ [\chi_\rho^3], & k \equiv 5 \pmod{6}. \end{cases}$$

For the third term of Eq. 4.2, let $\chi_i: \Gamma_i \rightarrow \mathbf{C}^*$ be the character defined by $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \mapsto i$. The analysis is similar to the case of Γ_ρ . We have

$$(4.15) \quad V_k|_{\Gamma_i} = \left\lfloor \frac{k}{2} \right\rfloor \left[\text{Ind}_Z^{\Gamma_i} \text{sgn}^k \right] + \varepsilon_i,$$

where the error term is

$$\varepsilon_i = \begin{cases} -[1], & k \equiv 0 \pmod{4} \\ 0, & k \equiv 1 \pmod{4} \\ -[\chi_i^2], & k \equiv 2 \pmod{4} \\ 0, & k \equiv 3 \pmod{4}. \end{cases}$$

Let $f(k) = (k-1) - \lfloor \frac{k}{3} \rfloor - \lfloor \frac{k}{2} \rfloor$. Substituting Eqs. 4.11, 4.13, and 4.15 into Eq. 4.2 gives:

Theorem 4.3. *For $k = 2$ we have*

(4.16)

$$[H_P^1(\Gamma(N), V_2)] - 2[1] = \left[\text{Ind}_{\mathbb{Z}}^{S_N} 1 \right] - \left[\text{Ind}_{\Gamma_\rho}^{S_N} 1 \right] - \left[\text{Ind}_{\Gamma_i}^{S_N} 1 \right] - \left[\text{Ind}_{\Gamma_\infty}^{S_N} 1 \right].$$

For weight $k > 2$,

(4.17)

$$[H_P^1(\Gamma(N), V_k)] = f(k) \left[\text{Ind}_{\mathbb{Z}}^{S_N} \text{sgn}^k \right] - \left[\text{Ind}_{\Gamma_\rho}^{S_N} \varepsilon_\rho \right] - \left[\text{Ind}_{\Gamma_i}^{S_N} \varepsilon_i \right] - \left[\text{Ind}_{\Gamma_\infty}^{S_N} \text{sgn}^k \right].$$

We may now complete the proof of Theorem 1.1. Let $\tau \in \text{Types}(\mathbf{Q})$ be a type of level N and weight k and let $\mu(\tau)$ be its multiplicity in $H_P^1(\Gamma(N), V_k)$. We claim that $\mu(\tau) = \frac{1}{6}(k-1) \dim \tau_f + O(2^{\nu(\tau) - n_{\text{sp}}(\tau)})$. (Note that $f(k) \sim (k-1)/6 = 2\zeta_{\mathbf{Q}}(-1)(k-1)$.) The only feature separating this calculation from that of the previous section is the appearance of the term $\text{Ind}_{\Gamma_\infty}^{S_N} \text{sgn}^k$. By the Chinese remainder theorem the multiplicity of τ_f in this term is a product of local multiplicities $\langle \tau_p|_{U_p}, 1 \rangle$, where $U_p \subset \text{GL}(2, \mathbf{Z}_p)$ is the unipotent subgroup. A case-by-case analysis shows that the local multiplicity at p is 1 if τ_p is special and at most 2 in any case. (In fact it is 0 if τ_p is supercuspidal.) Therefore the multiplicity of τ_f in $\text{Ind}_{\Gamma_\infty}^{S_N} \text{sgn}^k$ is $O(2^{\nu(\tau) - n_{\text{sp}}(\tau)})$. The claim follows and the calculation proceeds exactly as before.

We remark without proof that $[H_P^1(\Gamma(N), V_k)] = 2[S_k(\Gamma(N))]$, where $S_k(\Gamma(N))$ is the space of cusp forms of weight k for $\Gamma(N)$.

We conclude the section with a table of types $\tau \in \text{Types}(\mathbf{Q})$ for which $S(\tau) = \emptyset$. Table 1 lists configurations of finite components τ_p together with those weights k of the appropriate parity for which $\tau = \left(\bigotimes_p \tau_p \right) \otimes \mathcal{D}_{k,(-1)^k}$ is a global type without any matching cusp-forms. Unfortunately, the list omits local 2-adic types of conductor 2^n , where $n \geq 5$ is odd; the presence of extraordinary (non-dihedral) Galois representations of \mathbf{Q}_2 in that case complicates matters considerably. The list is complete in the sense that any type $\tau \in \text{Types}(\mathbf{Q})$ with $S(\tau) = \emptyset$ is a twist of one of the listed types, unless that type

should include one of the aforementioned 2-adic types. The notations for local components are given in Table 2.

5. THE FIELD OVER WHICH $J_1(p^n)$ IS SEMI-STABLE

The existence of modular forms with prescribed ramification behavior has arithmetic consequences for the Jacobians of Shimura curves. For simplicity we restrict our attention to the case of $K = \mathbf{Q}$; we examine the modular Jacobians $J = J_1(p^n)$ for $p \geq 3$ prime.

In [Kri96] an explicit extension M of \mathbf{Q}_p^{nr} is constructed over which $J_0(p^n)$ becomes semi-stable. The result of this section is a converse to this sort of theorem, whereby we construct an explicit extension of \mathbf{Q}_p^{nr} which contains any other field over which $J_1(p^n)$ becomes semi-stable.

Following the notations of [Kri96], let Ω_i/\mathbf{Q}_p , $i = 1, 2, 3$ be the three quadratic extensions of \mathbf{Q}_p , with Ω_1/\mathbf{Q}_p unramified. One realization of this scenario is $\Omega_1 = \mathbf{Q}_p(\sqrt{D})$, $\Omega_2 = \mathbf{Q}_p(\sqrt{p})$, $\Omega_3 = \mathbf{Q}_p(\sqrt{Dp})$, where $D \in \mathbf{Z}_p^*$ is a quadratic nonresidue. For each i let \mathfrak{p}_i be the maximal ideal of Ω_i and let M_i/Ω_i^{nr} be the class field with norm subgroup U_i defined by

$$U_i = \begin{cases} \pm(1 + \mathfrak{p}_i^{\lfloor n/2 \rfloor}) & i = 1 \\ 1 + \mathfrak{p}_i^{n-1}, & i = 2, 3. \end{cases}$$

Finally let $M = M_1 M_2 M_3 \mathbf{Q}_p^{\text{nr}}(\zeta_p^n)$.

Let A_n denote the set of two-dimensional Weil-Deligne representations ρ_p of \mathbf{Q}_p of conductor dividing p^n and satisfying $\det \rho_p(-1) = 1$. Then M has the following interpretation:

$$(5.1) \quad \bigcap_{\rho \in A_n} \ker \rho_p|_{I_{\mathbf{Q}_p}} \text{ has fixed field precisely } M.$$

Indeed, any $\rho_p \in A_n$ has one of the following forms:

- (1) decomposable as $\varepsilon_1 \oplus \varepsilon_2$, where the ε_i have conductor dividing p^n ,
- (2) $\varepsilon \otimes \text{Sp}(2)$, where ε has conductor dividing p^n , or
- (3) $\text{Ind}_{\Omega_i/\mathbf{Q}_p} \theta$, where $i \in \{1, 2, 3\}$.

In the last case, the condition that ρ_p has conductor dividing p^n translates into the condition that θ has conductor $\lfloor n/2 \rfloor$ if $i = 1$ and $n - 1$ if $i = 2, 3$, as can be determined from the classification in Section 2.2. The condition that $\det \rho_p(-1) = 1$ means that $\theta(-1) = 1$ if $i = 1$, and $\theta(-1) = (-1)^{(p-1)/2}$ if $i = 2, 3$. For a given i , the fixed field of $\rho_p|_{I_{\mathbf{Q}_p}}$ for ρ_p arising from such a character θ of Ω_i^* is exactly M_i . The claim in Eq. 5.1 follows.

TABLE 1. Global inertial types over \mathbf{Q} lacking representation by a cusp form. For explanation of notation, see Table 2.

Cond.	Local Components	k	Cond.	Local Components	k
1		2,4,6,8,10,14	25	$SC_5(12)$	2
2	St_2	2, 4, 6	26	$St_2, PS_{13}(6)$	2
3	St_3	2, 4	27	$SC_{27}(\sqrt{3}, 1)$	2
	$PS_3(2)$	3, 5	28	$SC_2(3), St_7$	2
4	$SC_2(3)$	2, 4, 8	36	$SC_2(3), SC_3(4)$	4
	PS_4	3		$SC_2(3), SC_3(8)$	3
5	St_5	2	45	$SC_3(4), St_5$	2
	$PS_5(2)$	2, 4	49	$SC_7(8)$	2
	$PS_5(4)$	3		$SC_7(24)$	2
6	St_2, St_3	2	50	$St_2, SC_5(8)$	5
7	St_7	2		$St_2, SC_5(24)$	3
	$PS_7(3)$	2	52	$SC_2(3), PS_{13}(2)$	2
	$PS_7(6)$	3	54	$St_2, SC_{27}(\sqrt{-3}, 1)$	3
8	PS_8	2	60	$SC_2(3), St_3, St_5$	2
	SC_8	2	64	$SC_{64}(3)$	2
9	$SC_3(4)$	2, 6		$SC_{64}(2)$	3
	$SC_3(8)$	3	72	$SC_8, SC_3(4)$	2
	PS_9	2	90	$St_2, SC_3(4), PS_5(2)$	2
10	St_2, St_5	2		$St_2, SC_3(8), St_5$	3
	$St_2, PS_5(2)$	2	98	$St_2, SC_7(4)$	2
11	$PS_{11}(5)$	2		$St_2, SC_7(12)$	2
12	$SC_2(3), St_3$	2, 6	100	$SC_2(3), SC_5(3)$	2
13	St_{13}	2		$SC_2(3), SC_5(6)$	2
	$PS_{13}(2)$	2		$SC_2(3), SC_5(8)$	3
	$PS_{13}(3)$	2		$SC_2(3), SC_5(12)$	2
14	$St_2, PS_7(3)$	2	108	$SC_2(3), SC_{27}(\sqrt{3}, 1)$	2
15	$St_3, PS_5(2)$	2	121	$SC_{11}(12)$	2
17	$PS_{17}(2)$	2		$SC_{11}(60)$	2
	$PS_{17}(4)$	2	126	$St_2, SC_3(4), St_7$	2
18	$St_2, SC_3(4)$	2, 4, 8		$St_2, SC_3(8), PS_7(2)$	2
	$St_2, SC_3(8)$	5	135	$SC_{27}(\sqrt{-3}, 1), St_5$	2
19	$PS_{19}(3)$	2	147	$St_3, SC_7(4)$	2
20	$SC_2(3), PS_5(2)$	2		$St_3, SC_7(12)$	2
22	St_2, St_{11}	2	150	$St_2, St_3, SC_5(3)$	2
25	$SC_5(3)$	2		$St_2, St_3, SC_5(6)$	2
	$SC_5(6)$	2		$St_2, St_3, SC_5(8)$	3
	$SC_5(8)$	3		$St_2, St_3, SC_5(12)$	2

Cond.	Local Components	k	Cond.	Local Components	k
162	St_2, SC_{81}	2	578	$St_2, SC_{17}(3)$	2
180	$SC_2(3), SC_3(4), St_5$	2		$St_2, SC_{17}(6)$	2
	$SC_2(3), SC_3(4), PS_5(2)$	2		$St_2, SC_{17}(12)$	2
192	$SC_{64}(1), St_3$	2		$St_2, SC_{17}(24)$	2
196	$SC_2(3), SC_7(4)$	2		$St_2, SC_{17}(48)$	2
	$SC_2(3), SC_7(12)$	2	588	$SC_2(3), St_3, SC_7(8)$	2
225	$SC_3(8), SC_5(8)$	2		$SC_2(3), St_3, SC_7(24)$	2
234	$St_2, SC_3(4), PS_{13}(2)$	2	675	$SC_{27}(\sqrt{-3}, -1), SC_5(8)$	2
242	$St_2, SC_{11}(4)$	2	726	$St_2, St_3, SC_{11}(12)$	2
	$St_2, SC_{11}(20)$	2		$St_2, St_3, SC_{11}(60)$	2
252	$SC_2(3), SC_3(4), St_7$	2	882	$St_2, SC_3(4), SC_7(4)$	2
256	$SC_{256}(0)$	2		$St_2, SC_3(4), SC_7(12)$	2
270	$St_2, SC_{27}(\sqrt{3}, 1), St_5$	2	900	$SC_2(3), SC_3(8), SC_5(8)$	2
294	$St_2, St_3, SC_7(8)$	2	1058	$St_2, SC_{23}(4)$	2
	$St_2, St_3, SC_7(24)$	2		$St_2, SC_{23}(44)$	2
320	$SC_{64}(0), St_5$	2	1089	$SC_3(4), SC_{11}(4)$	2
324	$SC_2(3), SC_{81}$	2		$SC_3(4), SC_{11}(20)$	2
350	$St_2, SC_5(8), PS_7(2)$	2	1350	$St_2, SC_{27}(\sqrt{-3}, 1), SC_5(3)$	2
378	$St_2, SC_{27}(\sqrt{-3}, -1), PS_7(2)$	2		$St_2, SC_{27}(\sqrt{-3}, 1), SC_5(6)$	2
396	$SC_2(3), SC_3(4), St_{11}$	2		$St_2, SC_{27}(\sqrt{-3}, 1), SC_5(12)$	2
441	$SC_3(4), SC_7(8)$	2		$St_2, SC_{27}(\sqrt{-3}, -1), SC_5(24)$	2
	$SC_3(4), SC_7(24)$	2		$St_2, SC_{27}(\sqrt{-3}, 1), SC_5(8)$	2
450	$St_2, SC_3(4), SC_5(3)$	2	1452	$SC_2(3), St_3, SC_{11}(4)$	2
	$St_2, SC_3(4), SC_5(6)$	2		$SC_2(3), St_3, SC_{11}(20)$	2
	$St_2, SC_3(4), SC_5(8)$	3	1600	$SC_{64}(1), SC_5(8)$	2
	$St_2, SC_3(4), SC_5(12)$	2	1728	$SC_{64}(0), SC_{27}(\sqrt{-3}, 1)$	2
	$St_2, SC_3(8), SC_5(24)$	2		$SC_{64}(1), SC_{27}(\sqrt{-3}, -1)$	2
484	$SC_2(3), SC_{11}(3)$	2	1764	$SC_2(3), SC_3(4), SC_7(4)$	2
	$SC_2(3), SC_{11}(6)$	2		$SC_2(3), SC_3(4), SC_7(12)$	2
	$SC_2(3), SC_{11}(15)$	2	2178	$St_2, SC_3(4), SC_{11}(3)$	2
	$SC_2(3), SC_{11}(30)$	2		$St_2, SC_3(4), SC_{11}(6)$	2
540	$SC_2(3), SC_{27}(\sqrt{-3}, 1), St_5$	2		$St_2, SC_3(4), SC_{11}(15)$	2
576	$SC_{64}(1), SC_3(4)$	2		$St_2, SC_3(4), SC_{11}(30)$	2
	$SC_{64}(2), SC_3(8)$	2			

TABLE 2. Explanation of symbols appearing as local types for \mathbf{Q}_p in Table 1, listed with dimension and conductor.

Symbol	Definition	Dimension	Conductor
St_p	Steinberg representation of $\text{GL}(2, \mathbf{Z}/p\mathbf{Z})$.	p	p
$\text{PS}_p(n)$	Principal Series rep. of $\text{GL}(2, \mathbf{Z}/p\mathbf{Z})$ corresponding to the characters ε and 1 of $(\mathbf{Z}/p\mathbf{Z})^*$, where ε has order n .	$p + 1$	p
$\text{SC}_p(n)$	Cuspidal rep. of $\text{GL}(2, \mathbf{Z}/p\mathbf{Z})$ corresponding to a multiplicative character θ of order n of a quadratic field extension of $\mathbf{Z}/p\mathbf{Z}$.	$p - 1$	p^2
PS_4	Principal Series rep. of $\text{GL}(2, \mathbf{Z}/4\mathbf{Z})$ corresponding to the unique primitive character of conductor 4.	6	4
PS_8	Principal Series rep. of $\text{GL}(2, \mathbf{Z}/8\mathbf{Z})$ corresponding to the unique even primitive character of conductor 8.	12	8
PS_9	Principal Series rep. of $\text{GL}(2, \mathbf{Z}/9\mathbf{Z})$ corresponding to any even primitive character of conductor 9.	12	9
SC_8	Type belonging to a supercusp. rep. of $\text{GL}(2, \mathbf{Q}_2)$ of conductor 8 with even central character.	3	8
$\text{SC}_{64}(n)$	Type belonging to a supercusp. rep. of $\text{GL}(2, \mathbf{Q}_2)$ attached to any mult. character θ of conductor 8 of the unramified extension $\mathbf{Q}_2(\rho)$, where ρ is a primitive 6th root of 1; the value of $\theta(\rho)$ is a primitive n th root of 1.	4	64
$\text{SC}_{256}(n)$	Type belonging to a supercusp. rep. of $\text{GL}(2, \mathbf{Q}_2)$ attached to a mult. character θ of conductor 16 of the unramified extension $\mathbf{Q}_2(\rho)$; the value of $\theta(\rho)$ is a primitive n th root of 1.	8	256
$\text{SC}_{27}(\sqrt{\pm 3}, \iota)$	Type belonging to a supercusp. rep. π of $\text{GL}(2, \mathbf{Q}_3)$ attached to a character θ of conductor 9 of $\mathbf{Q}_3(\sqrt{\pm 3})^*$, assuming that the central character of π has sign ι . This translates to the condition $\theta(-1) = -\iota$.	8	27
SC_{81}	Type belonging to any supercusp. rep. π of conductor 81 with even central character.	54	81

Theorem 5.1. *J is semi-stable over M . Conversely, if p^n is any odd prime power other than $3, 5, 7, 9, 11, 13, 17, 19, 27, 49$, or 121 , then M is the minimal extension of \mathbf{Q}_p^{nr} over which J becomes semi-stable.*

Proof. The variety J is isogenous to $\prod_f J_f$, where f runs over Galois orbits of newforms of conductor dividing p^n , and where the ℓ -adic Galois representation corresponding to f arises from the ℓ -adic Tate module of J_f . The abelian variety J_f becomes semi-stable over an extension $L/\mathbf{Q}_p^{\text{nr}}$ if and only if the local Weil-Deligne representation $\rho_{f,p}$ attached to f at p becomes unipotent when restricted to $\text{Gal}(\overline{\mathbf{Q}}_p/L)$; see [Gro72], exp. IX. Since $\rho_{f,p}$ lies in A_n , Eq. 5.1 implies that J_f must become semi-stable over the field M .

For the converse statement, suppose J is semi-stable over $L \supset K^{\text{nr}}$. Let $\rho_p \in A_n$ and assume that no twist of ρ_p is unramified. Let π_p be the admissible representation of $\text{GL}(2, \mathbf{Q}_p)$ corresponding to ρ_p , and let $\tau_p = \tau(\pi_p)$ be its inertial type. Let $\tau \in \text{Types}(\mathbf{Q})$ be a global type of weight 2 whose only nontrivial local component is τ_p . The prime powers listed in the theorem are the only ones which appear as conductors in Table 1. Since p^n is not among these, $S(\tau)$ contains a cusp form f . The assumption on L then implies $W_L \subset \ker \rho_p$. Since ρ_p was arbitrary, we may apply Eq. 5.1 to conclude that $L \supset M$, whence the theorem. \square

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E-mail address: jared@math.ucla.edu

UCLA DEPARTMENT OF MATHEMATICS, BOX 951555 LOS ANGELES, CA 90095-1555 USA