These notes are based on my talk in the participating number theory seminar at UCLA on January 23, 2013. They are based heavily on Mumford’s book *Abelian Varieties*, sections four and five. We assume our audience is familiar with algebraic geometry as presented in [2] chapter II as well as the basic notions of sheaf cohomology presented in section III.

**Definition 1.** An abelian variety $X$ is a complete variety over an algebraically closed field $k$ (i.e. an integral, proper scheme over $S = \text{Spec } k$) together with morphisms

\begin{align*}
m : & X \times X \to X \\
i : & X \to X \\
e : & S \to X
\end{align*}

making $X$ into a group scheme.

The definition of a group scheme can be found, for example, on page 94 of [4]. We fix the algebraically closed field $k$ for the remainder of these notes. Whenever we make a fiber product of schemes and we do not specify the scheme over which the fiber product is taken (as in the definition of $m$ above), we shall always mean over $S$. Similarly, all schemes will implicitly be defined over $S$ unless otherwise specified.

One can keep in mind that elliptic curves are an example of an abelian variety. Recall that we can view $X$ as a functor $X : \text{SCH}/S \to \text{SETS}$ by

$$X(V) = \text{Hom}_{\text{SCH}/S}(V, X).$$

This is called the $V$-valued points of $X$. By the universal property of fiber products, for any $Y/S$ we have a canonical identification

$$\text{Hom}_{\text{SCH}/S}(V, X \times Y) \cong \text{Hom}_{\text{SCH}/S}(V, X) \times \text{Hom}_{\text{SCH}/S}(V, Y).$$

We mention this because we will often want to represent an element of the fiber product $X \times Y$ by an ordered pair $(x, y)$ with $x \in X$ and $y \in Y$. It is well known (e.g. Exercise II.3.9 in [2]) that this does not necessarily make sense. But the above identification tells us that it does make sense if we talk about $V$-valued points of $X \times Y$. In particular, if we write $(x, y) \in X \times Y$ we mean we have morphisms $x : S \to X, y : S \to Y$. Further, for any scheme $V/S$ we compose
these maps with the structure morphism $V \to S$ and by abuse of notation call the resulting maps $x, y$. So we have

\[ x : V \to S \xrightarrow{\approx} X \in \text{Hom}_{\text{SCH}/S}(V, X) \]
\[ y : V \to S \xrightarrow{\approx} Y \in \text{Hom}_{\text{SCH}/S}(V, Y), \]

and this defines $(x, y) \in \text{Hom}_{\text{SCH}/S}(V, X \times Y)$.

Now for our first observations about abelian varieties:

**Lemma 1.** Every abelian variety is non-singular.

**Proof.** Recall that a variety $X$ is non-singular at a point $x \in X$ if $\mathcal{O}_{X,x}$ is a regular local ring. We say $X$ is non-singular if it is non-singular at every point. Localization at a prime ideal preserves regularity ([2] Theorem 8.14A), so it suffices to show that $X$ is non-singular at every closed point of $X$. Now every variety has an open dense subset on which it is non-singular ([2], Corollary II.8.16) so there is a closed point $x_0 \in X$ where $X$ is non-singular. If $x \in X$ is a closed point, define the translation by $x$ map by

\[ T_x : X \to X \in \text{Aut}(X) \]
\[ y \mapsto m(x, y). \]

Clearly $T_x$ is an automorphism of $X$ since its inverse is $T^{-1}$.

Now $T_{m(x_0^{-1}, x)}(x_0) = x$ so $x$ is the image of a non-singular point under an automorphism and is therefore non-singular, as desired. \qed

**Lemma 2.** Any morphism from a complete connected variety to an affine variety is constant.

**Proof.** Let $\gamma : X \to Y = \text{Spec} \ A$ be such a morphism. Since $A$ is a $k$-algebra of finite type, $A = k[x_1, \ldots, x_r]/I$ for some ideal $I$. Thus there is a natural closed immersion

\[ \text{Spec} \ A \to \text{Spec} k[x_1, \ldots, x_r] = \mathbb{A}^r_{/k}. \]

From here we have several natural ‘projection to coordinate’ morphisms

\[ \mathbb{A}^r_{/k} \to \mathbb{A}^1_{/k} \]

induced by the various inclusions $k[x_i] \to k[x_1, \ldots, x_r]$. Finally, we can include $\mathbb{A}^1_{/k} \to \mathbb{P}^1_{/k}$. We now apply [2] Exercise II.4.4 to the entire composition

\[ X \xrightarrow{\gamma} Y \to \mathbb{A}^r_{/k} \to \mathbb{A}^1_{/k} \to \mathbb{P}^1_{/k}. \]
to conclude that the image is closed in $\mathbb{P}_k^1$. The only closed sets in $\mathbb{P}_k^1$ are $\mathbb{P}_k^1$ and finite sets of points. As $X$ is connected and the image of the above composition lies in $\mathbb{A}_k^1$, the above map must be constant. This implies that the image of $X$ is constant in $Y$ because all coordinates are constant, as desired. 

\textbf{Theorem 1} (Rigidity Lemma). Let $X$ be a complete variety, $Y, Z$ any varieties, $f : X \times Y \to Z$ a morphism such that for some $y_0 \in Y$, $f(X \times \{ y_0 \}) = z_0 \in Z$. Then there is a morphism $g : Y \to Z$ such that $f = g \circ p_2$, where

\[
\begin{array}{ccc}
X \times Y & \to & Z \\
p_1 & & \downarrow \\
X & \downarrow & \quad.
\end{array}
\]

This is saying that if there is some ‘slice’ on which $f$ doesn’t ‘see the $x$-coordinate’, then in fact $f$ never ‘sees the $x$-coordinate’. We note that since $X$ is complete, $X \to S$ is proper and thus any base extension of that morphism is closed. (This is the definition of properness.) In particular, $p_2$ is a closed map.

\textit{Proof.} Fix $x_0 \in X$ and define $g : Y \to Z$ by $g(y) = f(x_0, y)$. It’s enough to show that there is some nonempty open subset $V \subset X \times Y$ such that $f|_V = g \circ p_2|_V$ ([2], Exercise II.4.2).

Take $U$ an open affine neighborhood of $z_0$. Then $z_0 \not\in F := Z \setminus U$ is a closed subset of $Z$, and $G = p_2(f^{-1}(F))$ is a closed subset of $Y$ since $p_2$ is a closed map. Note that if $y_0 \in G$ then for some $x \in X$ we have by hypothesis $z_0 = f(x, y_0) \in F$, a contradiction. Therefore $y_0 \not\in G$. Hence $V = Y \setminus G$ is a nonempty open subset of $Y$. Hence for any $y \in V$

\[f : X \times \{ y \} \to U\]

is a morphism from a complete variety to an affine scheme and must therefore be constant. Hence for all $x \in X$

\[f(x, y) = f(x_0, y) = g(y) = g \circ p_2(x, y),\]

as desired. 

\[\square\]
**Corollary 1.** If $X, Y$ are abelian varieties and $f : X \to Y$ is any morphism, then

$$f(x) = h(x)a$$

where $h : X \to Y$ is a homomorphism and $a \in Y$.

**Proof.** Note that if this is to hold, we must have $f(e_X) = h(e_X)a = e_Y a = a$. Thus by replacing $f$ with $f^{-1}a$ we may assume that $f(e_X) = e_Y$ and show that $f$ is a homomorphism.

Define $\phi : X \times X \to Y$ by $\phi(x, y) = f(xy)f(x)^{-1}f(y)^{-1}$. Clearly $f$ is a homomorphism if and only if $\phi \equiv e_Y$. Note that $\phi(X \times \{e_X\}) = e_Y$, so by the Rigidity Lemma $\phi = g \circ p_2$ for some $g : X \to Y$. That is, $\phi$ depends only on the second coordinate so for any $x, y \in X$ we have

$$\phi(x, y) = \phi(e_X, y) = e_Y.$$

Thus $f$ is a homomorphism. 

**Corollary 2.** If $X$ is an abelian variety the $m(x, y) = m(y, x)$ for all $x, y \in X$.

**Proof.** Consider the morphism $\iota : X \to X$ given by $\iota(x) = x^{-1}$. Clearly $\iota(e) = e$, so $\iota$ is a homomorphism by Corollary 1. A group is commutative if and only if inversion is a homomorphism, so $X$ is abelian, as desired.

Due to time constraints, we now move on to the material in section five of [4] even though there is material in section four that we have not covered. This section is concerned with results on the cohomology of fibers of a family of varieties.

For now, assume the following theorem:

**Theorem 2.** Let $f : X \to Y = \text{Spec} A$ be a proper morphism of noetherian schemes, $\mathcal{F}$ a coherent sheaf on $X$, flat over $Y$. There exists a finite complex

$$K^\bullet : 0 \to K^0 \xrightarrow{d^0} K^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} K^n \to 0$$

of finitely generated projective $A$-modules such that for any $A$-algebra $B$ and $p \geq 0$,

$$H^p(X \times_Y \text{Spec} B, \mathcal{F} \otimes_A B) \cong H^p(K^\bullet \otimes_A B).$$

We will return to the proof of this theorem in the next lecture. For now, we derive some corollaries. Perhaps we should note that many of these corollaries are
proven in [2] with different techniques. Here we give proofs that rely only on the above theorem.

Given \( y \in Y \), we denote its residue field by \( k(y) \). That is, \( k(y) = \mathcal{O}_{Y,y}/m_y \).

It will be useful to keep in mind the following easy fact from commutative algebra.

**Lemma 3.** Let \( f : M \rightarrow N \) be a homomorphism of \( A \)-modules. Let \( B \) be an \( A \)-algebra.

1. If \( \ker(f) \oplus M' = M \) for some \( A \)-submodule \( M' \) of \( M \), then
   \[
   \ker(f) \otimes_A B = \ker(f \otimes_A 1_B).
   \]

2. If \( \text{Im}(f) \oplus N' = N \) for some \( A \)-submodule \( N' \) of \( N \), then
   \[
   \text{Im}(f) \otimes_A B \cong \text{Im}(f \otimes_A 1_B).
   \]

3. If \( B \) is a flat \( A \)-module, then \( \ker(f) \otimes_A B = \ker(f \otimes_A 1_B) \) and \( \text{Im}(f) \otimes_A B \cong \text{Im}(f \otimes_A 1_B) \).

**Proof.**

1. Without any hypothesis on \( f \) we always have \( \ker(f) \otimes_A B \subseteq \ker(f \otimes_A 1_B) \). Recall that all functors preserve isomorphisms. Note that \( f|_{M'} : M' \rightarrow \text{Im}(f) \) is an isomorphism, so \( (f|_{M'}) \otimes_A 1_B : M' \otimes_A B \rightarrow (\text{Im}(f)) \otimes_A B \) is an isomorphism. Clearly \( (f|_{M'}) \otimes_A 1_B = (f \otimes_A 1_B)|_{M' \otimes_A B} \) since they are equal on every element of \( M' \otimes_A B \). Hence \( (f \otimes_A 1_B)|_{M' \otimes_A B} \) is injective. Since tensor products break over direct sums we have \( M \otimes_A B = (\ker(f) \otimes_A B) \oplus (M' \otimes_A B) \). Since we have \( M' \otimes_A B \cap \ker(f \otimes_A 1_B) = \{0\} \) it follows that \( \ker(f \otimes_A 1_B) \subseteq \ker(f) \otimes_A B \) so they are equal, as desired.

2. We derive this from the previous fact and the fact that tensor products are right exact. From the exact sequence
   \[
   M \xrightarrow{f} N \rightarrow \text{Coker}(f) \rightarrow 0
   \]
   we get an exact sequence
   \[
   M \otimes_A B \xrightarrow{f \otimes_A 1_B} N \otimes_A B \rightarrow (\text{Coker}(f)) \otimes_A B \rightarrow 0.
   \]
   This shows that \( \text{Coker}(f \otimes_A 1_B) \cong (\text{Coker}(f)) \otimes_A B \). Let \( \pi : N \rightarrow \text{Coker}(f) \) be the natural map, so \( \text{Im}(f) = \ker(\pi) \). By the previous part of
the lemma we see that $\text{Im}(f) \otimes_A B = \ker(\pi) \otimes_A B = \ker(\pi \otimes_A 1_B)$. Thus we only need to show that $\ker(\pi \otimes_A 1_B) \cong \text{Im}(f \otimes_A 1_B)$. This follows from what we just established about cokernels and the five lemma applied to the following diagram:

$$
\begin{array}{cccc}
0 & \longrightarrow & \ker(\pi \otimes_A 1_B) & \longrightarrow & N \otimes_A B & \longrightarrow & (\text{Coker}(f)) \otimes_A B \\
\downarrow & & \downarrow & & \downarrow & & \cong \\
0 & \longrightarrow & \text{Im}(f \otimes_A 1_B) & \longrightarrow & N \otimes_A B & \longrightarrow & \text{Coker}(f \otimes_A 1_B)
\end{array}
$$

3. We have an exact sequence

$$0 \to \ker(f) \to M \xrightarrow{f} N.$$

Since $B$ is flat, the following sequence is also exact

$$0 \to \ker(f) \otimes_A B \to M \otimes_A B \xrightarrow{f \otimes_A 1_B} N \otimes_A B.$$

This exactly means that $\ker(f) \otimes_A B = \ker(f \otimes_A 1_B)$. Now we can use the first isomorphism theorem to get

$$\text{Im}(f) \otimes_A B \cong (M/\ker(f)) \otimes_A B \cong (M \otimes_A B)/(\ker(f \otimes_A B))$$

$$= (M \otimes_A B)/\ker(f \otimes_A 1_B) \cong \text{Im}(f \otimes_A 1_B),$$

as desired.

**Corollary 3.** [Semicontinuity theorem] Let $X, Y, f, F$ be as in the theorem, though $Y$ need not be affine.

1. For $p \geq 0$ the function $\rho_p : y \mapsto \dim_{k(y)} H^p(X_y, \mathcal{F}_y)$ is upper semicontinuous (i.e. for all $y \in Y$ there is an open neighborhood $U$ of $y$ such that $\rho_p(y') \leq \rho_p(y)$ for all $y' \in U$).

2. The function

$$y \mapsto \chi(y) = \sum_{n=0}^{\infty} (-1)^n \dim_{k(y)} H^n(X_y, \mathcal{F}_y)$$

is locally constant on $Y$. 

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Proof. Note that all of the statements are local so we may assume that \( Y = \text{Spec} \, A \). Let \( K^\bullet \) be the complex from the theorem. Since the \( K^i \) are finitely generated and projective and \( A \) is noetherian, the \( K^i \)'s are locally free. Thus by replacing \( Y \) with an open subset, we may assume that the \( K^i \)'s are free.

We prove the second statement first. Note that
\[
\dim_{k(y)} H^p(X_y, F_y) = \dim_{k(y)} \ker(d^p \otimes_A 1_{k(y)}) - \dim_{k(y)} \text{Im}(d^{p-1} \otimes_A 1_{k(y)})
\]
\[
= \dim_{k(y)} K^p \otimes_A k(y) - \dim_{k(y)} \text{Im}(d^p \otimes 1_{k(y)}) - \dim_{k(y)} \text{Im}(d^{p-1} \otimes 1_{k(y)}).
\]
Note that \( \dim_{k(y)} K^p \otimes_A k(y) = \text{rk}_A K^p \) is independent of \( y \). Hence when we take the alternating sum the latter two terms all cancel out and we are left with a sum of terms that is independent of \( y \). I should mention that the sum is finite because the complex \( K^\bullet \) is finite. Furthermore, we get that \( \chi \) is locally constant (rather than just constant) because we had to localize to make the \( K^i \)'s free.

For the proof of the first statement it suffices to show that \( \phi_p(y) = \dim_{k(y)} \text{Im}(d^p \otimes 1_{k(y)}) \) is lower semicontinuous. For \( r \geq 0 \) define
\[
S_r = \{ y \in Y \mid \phi_p(y) < r \}.
\]
Suppose that these \( S_r \) are closed in \( Y \), and take \( y_0 \in Y \). Then for \( r = \phi_p(y_0) \) it is easy to see that \( Y \setminus S_r \) is an open neighborhood of \( y_0 \) satisfying the necessary property in the definition of lower semicontinuity. Hence it suffices to show that these \( S_r \) are closed.

Note that
\[
y \in S_r \iff \phi_p(y) < r \iff \dim_{k(y)} \text{Im}(d^p \otimes 1_{k(y)}) < r
\]
\[
\iff 0 = \Lambda^r(\text{Im}(d^p \otimes_A 1_{k(y)})) \cong \text{Im}(\Lambda^r(d^p \otimes_A 1_{k(y)})).
\]
The last isomorphism follows from the fact that exterior products commute with base extension ([3], page 284). Now \( \Lambda^r(d^p \otimes_A 1_{k(y)}) \) is a morphism of finite dimensional \( k(y) \)-vector spaces, so we can represent it as a matrix \( \Lambda^r(d^p \otimes_A 1_{k(y)}) = (a_{i,j}(y))_{i,j} \). Clearly \( \text{Im}(\Lambda^r(d^p \otimes_A 1_{k(y)})) = 0 \) if and only if \( a_{i,j}(y) = 0 \) for all \( i, j \).

Thus
\[
S_r = \{ y \in Y \mid a_{i,j}(y) = 0 \} = V(\{a_{i,j}\}_{i,j})
\]
is closed in \( Y \), as desired.

We now present two lemmas that are needed in the proof of the next corollary.
Lemma 4. Let $Y$ be a reduced noetherian scheme and $F$ a coherent sheaf on $Y$ such that

$$\dim_{k(y)} F \otimes_{\mathcal{O}_Y} k(y) = r$$

for all $y \in Y$. Then $F$ is locally free of rank $r$.

Proof. For $y \in Y$ let $\sigma^1_y, \ldots, \sigma^r_y \in F_y$ such that $\sigma^1_y \otimes 1, \ldots, \sigma^r_y \otimes 1$ are a basis for $F_y \otimes_{\mathcal{O}_{Y,y}} k(y)$. This is possible because by assumption $F_y \otimes_{\mathcal{O}_{Y,y}} k(y)$ is a $k(y)$-vector space of dimension $r$, so we just lift a basis to $F_y$. Let $V$ be an open neighborhood of $y$ on which the sections $\sigma^1, \ldots, \sigma^r$ are all defined. By applying NAK ([4] Theorem 2.2) to $\text{Coker}(\sigma_y)$ we find that $\sigma_y$ is surjective. As surjectivity is a local property, by shrinking $V$ if necessary we may assume that $\sigma$ is surjective. Therefore for all $y' \in V$ we have

$$\sigma_{y'} \otimes 1_{k(y')} : k(y')^r \to F_{y'} \otimes_{\mathcal{O}_{Y,y'}} k(y')$$

is an isomorphism since it is a surjective morphism of $r$-dimensional $k(y')$-vector spaces according to our hypothesis.

Let $D = \ker \sigma$. Then it follows that $D_{y'} \subseteq m_{y'} \mathcal{O}_{Y,y'}$ for all $y' \in V$. Since $Y$ is reduced, this implies that $D = 0$. (This is easiest to see when $r = 1$, because then $D$ is contained in the intersection of all prime ideals of a ring $A$ - say $\text{Spec} A \subset V$ - and $Y$ being reduced means that the nilradical of $A$ is equal to zero. Hence $D = 0$.)

Lemma 5. Let $Y = \text{Spec} A$ be reduced and noetherian and $\phi : F \to D$ a morphism of coherent locally free $\mathcal{O}_Y$-modules. If $\dim_{k(y)} \text{Im}(\phi \otimes 1_{k(y)})$ is locally constant, then $F = F_1 \oplus F_2$ and $D = D_1 \oplus D_2$ with $F_1 \subset \ker \phi, D_1 \subset \text{Im} \phi$ and $\phi|_{F_2} : F_2 \to D_1$ is an isomorphism.

Proof. Since $Y$ is affine, this is really just a statement in commutative algebra. In particular, let $M = \Gamma(Y, F)$ and $N = \Gamma(Y, D)$. Then $F = M_{\sim}$ and $D = N_{\sim}$ and $\phi$ comes from the map of $A$-modules $\Gamma(Y, \phi) = f : M \to N$. (Here we are using $\sim$ to denote the sheaf associated to an $A$-module.) The key point here is to note that

$$\dim_{k(y)} (\text{Coker}(f) \otimes_A 1_{k(y)}) = \dim_{k(y)} (\text{Coker}(f \otimes_A 1_{k(y)}))$$

$$= \dim_{k(y)} N \otimes_A k(y) - \dim_{k(y)} \text{Im}(f \otimes_A 1_{k(y)}).$$

Since $N$ is locally free, the first term is locally constant and the second is locally constant by hypothesis. Hence this dimension is locally constant, so by Lemma
4 Coker($f$) is locally free. (Here we are really applying the lemma to some neighborhood of $y$ on which $\dim_{k(y')} \text{Coker}(f) \otimes_A k(y')$ is constant.) Recall that over a noetherian ring $A$, a finitely generated $A$-module is locally free if and only if it is projective ([1] Exercise 4.11(b)). Thus $N/f(M)$ is projective. This implies that the exact sequence

$$0 \rightarrow f(M) \rightarrow N \rightarrow N/f(M) \rightarrow 0$$

splits, so $N = f(M) \oplus N'$. Now $N$ is locally free hence projective and $f(M)$ is a direct summand of $N$, therefore projective. Hence the exact sequence

$$0 \rightarrow \ker(f) \rightarrow M \rightarrow f(M) \rightarrow 0$$

splits, say $M = \ker(f) \oplus M'$. Then we can take $F_1 = \ker(f) \sim, F_2 = (M') \sim, D_1 = f(M) \sim, D_2 = (N') \sim$.

**Corollary 4.** Let $X,Y,f,\mathcal{F}$ be as in Corollary 3 with $Y$ reduced and connected. For each $p \geq 0$, the following are equivalent:

1. The function $y \mapsto \dim_{k(y)} H^p(X_y, \mathcal{F}_y)$ is constant.
2. $R^p f_* (\mathcal{F})$ is locally free on $Y$, and for all $y \in Y$

   $$R^p f_* (\mathcal{F}) \otimes_{O_Y} k(y) \cong H^p(X_y, \mathcal{F}_y).$$

**Proof.** The fact that (2) implies (1) is clear, so assume (1). Take $K^\bullet$ as in Theorem 2. Recall from the proof of Corollary 3 that $\dim_{k(y)} \text{Im}(d^j \otimes 1_{k(y)})$ is lower semicontinuous for all $j$. It is easy to see that if $\phi, \rho$ are two lower semicontinuous functions on $Y$ such that $\phi + \rho$ is constant then $\rho, \phi$ must each be constant. From the proof of Corollary 3 we have

$$-\dim_{k(y)} H^p(X_y, \mathcal{F}_y) + \dim_{k(y)} K^p \otimes_A k(y) = \dim_{k(y)} \text{Im}(d^p \otimes 1_{k(y)}) + \dim_{k(y)} \text{Im}(d^{p-1} \otimes 1_{k(y)}).$$

By assumption the left hand side of this equation is locally constant, so each of $\dim_{k(y)} \text{Im}(d^p \otimes 1_{k(y)}), \dim_{k(y)} \text{Im}(d^{p-1} \otimes 1_{k(y)})$ must be locally constant.

Hence we can apply Lemma 5 to $d^p : K^p \rightarrow K^{p+1}$ and $d^{p-1} : K^{p-1} \rightarrow K^p$:
where $Z_i = \ker(d^i), B_i = \text{Im}(d^i)$ and $d^i : (K^i)' \to B_i$ is an isomorphism. This allows us to apply Lemma 3! Thus for any $A$-algebra $B$ we have

$$H^p(K^\bullet \otimes_A B) = \ker(d^p \otimes_A 1_B) / \text{Im}(d^p \otimes_A 1_B) = (Z_p \otimes_A B) / (B_{p-1} \otimes_A B) = H^p(K^\bullet) \otimes_A B.$$  

Taking $B = k(y)$ and using Theorem 2 and Proposition III.8.5 in [2] we see that

$$H^p(X, F_y) \cong H^p(K^\bullet \otimes_A k(y)) = H^p(K^\bullet) \otimes_A k(y) = R^p f_*(F) \otimes_{O_Y} k(y),$$

as desired.

**Definition 2.** A line bundle (or invertible sheaf) $L$ on $X$ is a locally free $O_X$-module of rank one. We say $L$ is trivial if $L \cong O_X$.

Recall that for any line bundle $L$ on $X$ we have $L^{-1} = \mathcal{HOM}(O_X, L)$, where $\mathcal{HOM}$ denotes the sheaf hom ([2] Exercise II.1.15).

Furthermore, for any complete variety $X$ we have

$$H^0(X, O_X) = \Gamma(X, O_X) = k$$

([2] Exercise II.4.5).

**Lemma 6.** Let $X$ be a complete variety and $L$ a line bundle on $X$. Then $L$ is trivial if and only if $\dim_k H^0(X, L) > 0$ and $\dim_k H^0(X, L^{-1}) > 0$.

**Proof.** If $L \cong O_X$ then $L^{-1} = O_X$ as well. By the remark prior to this lemma, we have that both of the dimensions in question are equal to one, hence positive.

Conversely, suppose that $\dim_k H^0(X, L) > 0$ and $\dim_k H^0(X, L^{-1}) > 0$. Recall that for any $O_X$-module $F$ we have $H^0(X, F) \cong \text{Hom}(O_X, F)$ ([2] Proposition III.6.3). Thus $\dim_k H^0(X, L) > 0$ implies that there is some nonzero $\sigma : O_X \to L$ and $\dim_k H^0(X, L^{-1}) > 0$ gives some nonzero $f : O_X \to L^{-1}$. Applying $\mathcal{HOM}(O_X, -)$ to $f$ yields a nonzero map $\tau : L \to O_X$. Then $0 \neq \tau \circ \sigma : O_X \to O_X \in H^0(X, O_X) = k$. Thus $\sigma, \tau$ are isomorphisms, so $L \cong O_X$.

**Corollary 5** (Seesaw Theorem). Let $X$ be a complete variety, $T$ any variety and $L$ a line bundle on $X \times T$. Then

$$T_1 = \{t \in T | L|_{X \times \{t\}} \text{ is trivial on } X \times \{t\} \}$$

is closed in $T$. Further,

$$L|_{X \times \{t\}} \cong p_2^* M$$

for some line bundle $M$ on $T_1$. 

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Proof. By Lemma 6

\[ T_1 = \{ t \in T \mid \dim_k H^0(X \times \{t\}, \mathcal{L}|_{X \times \{t\}}) > 0 \text{ and } \dim_k H^0(X \times \{t\}, \mathcal{L}^{-1}|_{X \times \{t\}}) > 0 \} . \]

By Corollary 3 the functions \( \dim_k H^0(X, \mathcal{L}) \) and \( \dim_k H^0(X, \mathcal{L}^{-1}) \) are upper semicontinuous. Hence \( T_1 \) is closed in \( T \).

Note that

\[ \dim_k H^0(X \times \{t\}, \mathcal{L}|_{X \times \{t\}}) = 1 \]

for all \( t \in T_1 \) since \( X \) is a complete variety and \( \mathcal{L}|_{X \times \{t\}} \cong \mathcal{O}_{X \times \{t\}} \). Hence by Corollary 4 we have that \( p_2_* \mathcal{L}|_{X \times T_1} \) is locally free of rank one and

\[ p_2_* \mathcal{L}|_{X \times T_1} \otimes_{\mathcal{O}_{T_1}} k(t) \cong H^0(X \times \{t\}, \mathcal{L}|_{X \times \{t\}}) = k. \]

\qed

References


