Set Theory, Infinite Games, and Strong Axioms

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Use the PgDn or the down arrow to scroll through slides.

Press Esc when done.

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Precisely, $A \approx B$ iff there is a relation connecting elements of A with elements of B, so that each element of A is connected to exactly one element of B and vice versa.

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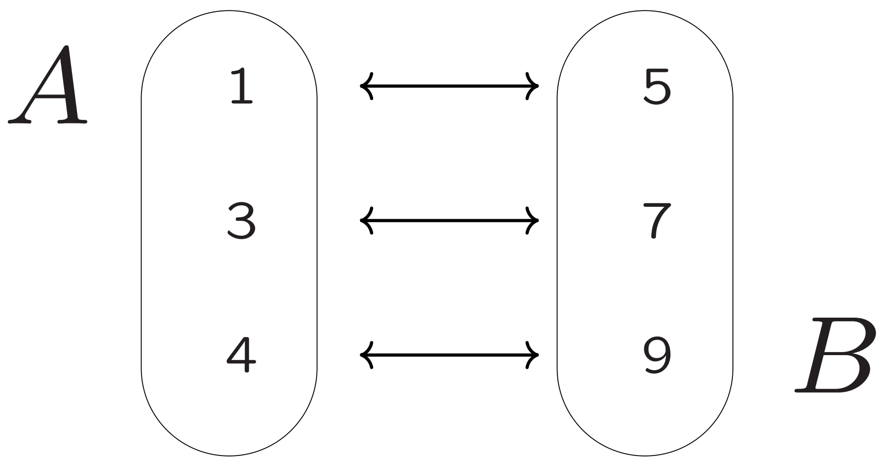

Rephrasing:

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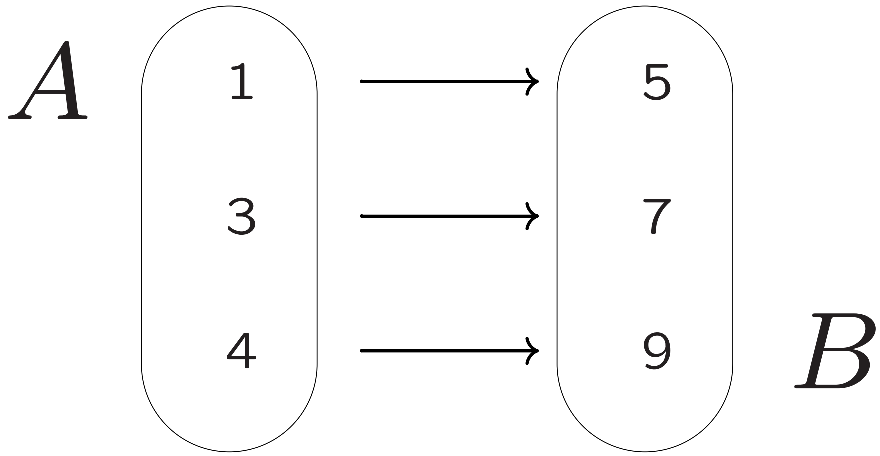

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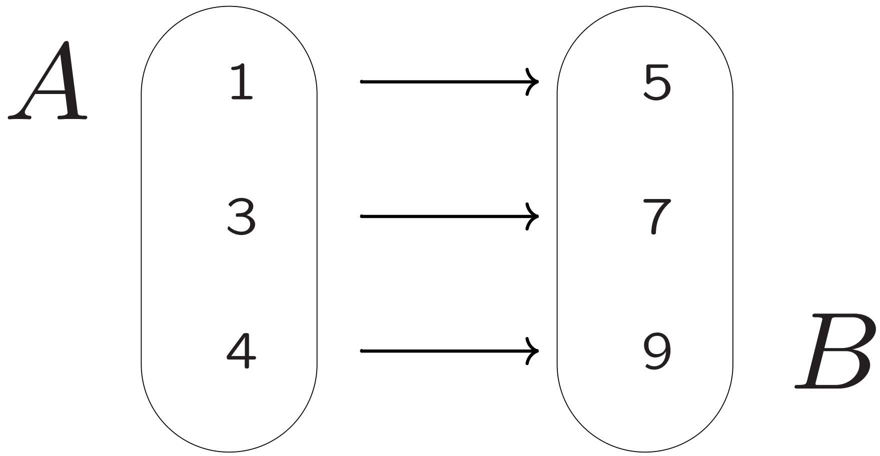

$A \approx B$ just in case that there is a function $f\colon A \to B$ so that:

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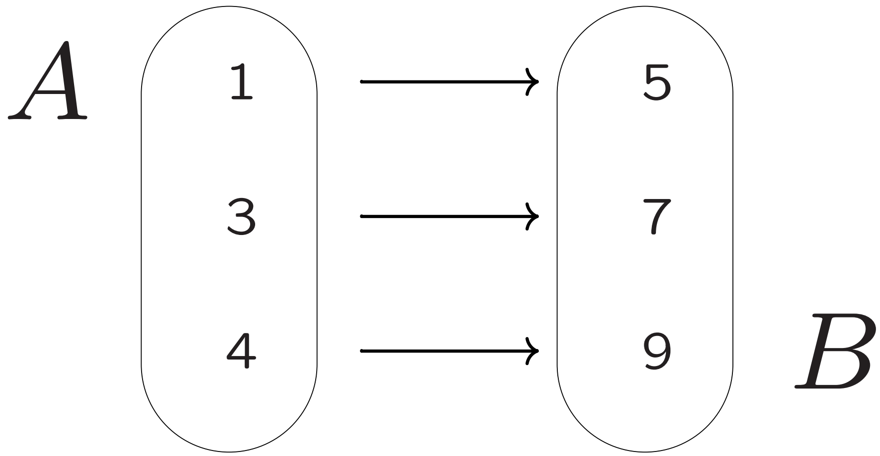

$A \approx B$ just in case that there is a function $f\colon A \to B$ so that:

- f is **one-to-one**.

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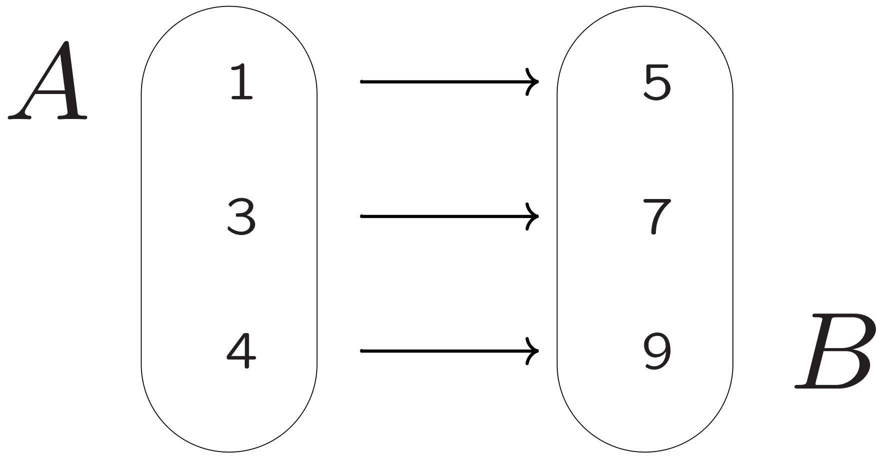

$A \approx B$ just in case that there is a function $f\colon A \to B$ so that:

- f is **one-to-one**. ($x \neq y \Rightarrow f(x) \neq f(y)$.)

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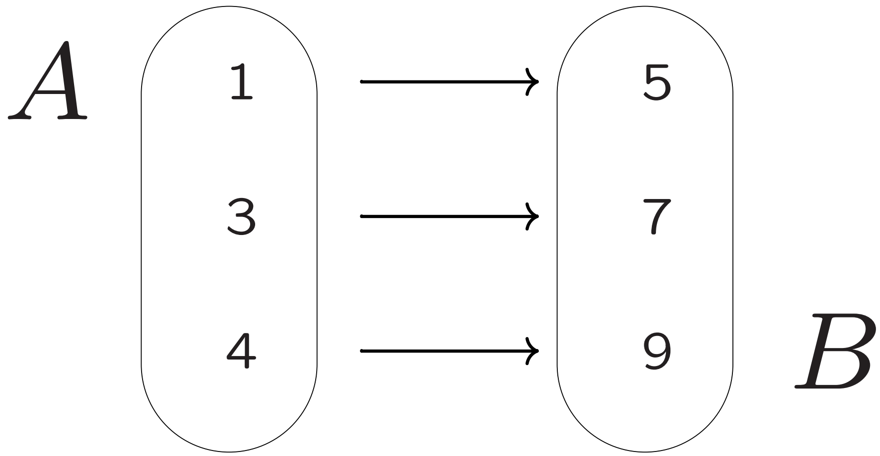

$A \approx B$ just in case that there is a function $f\colon A \to B$ so that:

- f is **one-to-one**. $(x \neq y \Rightarrow f(x) \neq f(y).)$

- f is **onto**.

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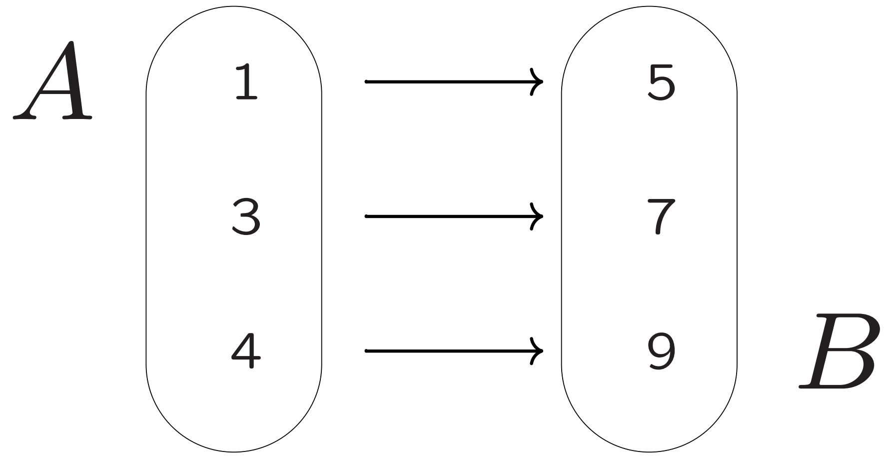

$A \approx B$ just in case that there is a function $f\colon A \to B$ so that:

- f is **one-to-one**. ($x \neq y \Rightarrow f(x) \neq f(y)$.)

- f is **onto**. (All elements of B are in the range of f.)

Some examples:

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$$\mathbb{N} \;=\; \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\}$$

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$$\begin{aligned}\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\ \mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\}\end{aligned}$$

Some examples:

$$
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\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\} \\
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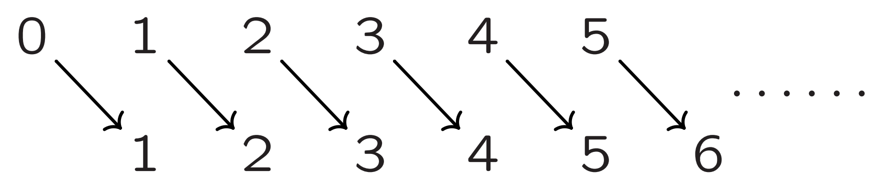
\end{aligned}
$$



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$\mathbb{N} - \{0\}$ and $\mathbb{N}$ have the same size.

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0 1 2 3 4 5 ······

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0

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0 1 2 3 4 5 $\cdots\cdots$

0 -1

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$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \cdots\cdots$$

$$0 \quad -1 \quad 1 \quad -2$$

Some examples:

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$0 \quad -1 \quad 1 \quad -2 \quad 2$

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$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \cdots\cdots$$

$$0 \quad -1 \quad 1 \quad -2 \quad 2 \quad -3$$

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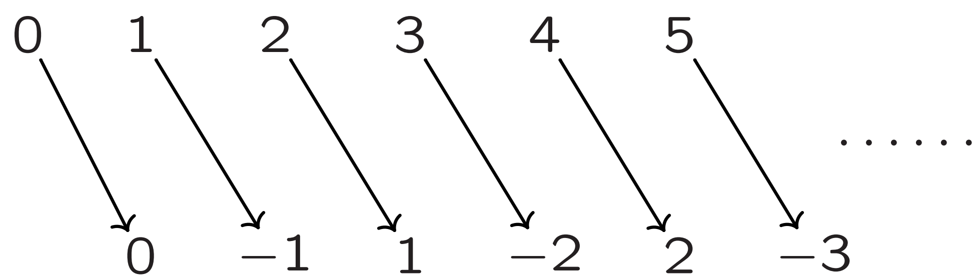

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$\mathbb{Z}$ and $\mathbb{N}$ have the same size.

Some examples:

$$\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2, -1, 0, 1, 2, 3, \ldots\ldots\ldots\} \\
\mathbb{Q}^+ &= \{\frac{m}{p} \mid m, p \in \mathbb{N},\ p \neq 0\}
\end{aligned}$$

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&=
\end{aligned}
$$

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\mathbb{Q}^{+} &= \{\frac{m}{p} \mid m, p \in \mathbb{N},\ p \neq 0\} \\
&= \{0, \frac{1}{2},
\end{aligned}
$$

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\end{aligned}$$

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&= \{0,\frac{1}{2},\frac{1}{3},\frac{12}{17},
\end{aligned}
$$

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&= \{0,\frac{1}{2},\frac{1}{3},\frac{12}{17},\frac{47}{6},
\end{aligned}
$$

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&= \{0, \frac{1}{2}, \frac{1}{3}, \frac{12}{17}, \frac{47}{6}, \frac{103}{58},
\end{aligned}
$$

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&= \{0, \frac{1}{2}, \frac{1}{3}, \frac{12}{17}, \frac{47}{6}, \frac{103}{58}, \frac{34}{7}, \frac{54}{12}, \frac{81}{62},
\end{aligned}
$$

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\end{aligned}
$$

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\end{aligned}$$

List all pairs:

Some examples:

$$\begin{aligned}
\mathbb{N} &= \{0,1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
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\end{aligned}$$

List all pairs:

$$\frac{0}{1}$$

Some examples:

$$
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\end{aligned}
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List all pairs:

$\frac{0}{1}$ $\frac{1}{0}$

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List all pairs:

$\frac{0}{1}$ $\frac{1}{0}$ $\frac{0}{2}$

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List all pairs:

$\frac{0}{1}\ \frac{1}{0}\qquad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0}$

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$$

List all pairs:

$$\frac{0}{1} \quad \frac{1}{0} \qquad \frac{0}{2} \quad \frac{1}{1} \quad \frac{2}{0} \qquad \frac{0}{3}$$

Some examples:

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$$\frac{0}{1} \quad \frac{1}{0} \qquad \frac{0}{2} \quad \frac{1}{1} \quad \frac{2}{0} \qquad \frac{0}{3} \quad \frac{1}{2} \quad \frac{2}{1}$$

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$$
\begin{aligned}
\mathbb{N} &= \{0,1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N}-\{0\} &= \{1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2,-1,0,1,2,3,\ldots\ldots\ldots\} \\
\mathbb{Q}^+ &= \{\frac{\mathsf{m}}{\mathsf{p}} \mid m,p \in \mathbb{N},\ p \neq 0\}
\end{aligned}
$$

List all pairs:

$$
\frac{0}{1}\ \frac{1}{0} \qquad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0} \qquad \frac{0}{3}\ \frac{1}{2}\ \frac{2}{1}\ \frac{3}{0} \qquad \frac{0}{4}
$$

Some examples:

$$
\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2, -1, 0, 1, 2, 3, \ldots\ldots\ldots\} \\
\mathbb{Q}^+ &= \{\frac{m}{p} \mid m, p \in \mathbb{N},\ p \neq 0\}
\end{aligned}
$$

List all pairs:

$$
\frac{0}{1}\ \frac{1}{0} \qquad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0} \qquad \frac{0}{3}\ \frac{1}{2}\ \frac{2}{1}\ \frac{3}{0} \qquad \frac{0}{4}\ \frac{1}{3}
$$

Some examples:

$$
\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2, -1, 0, 1, 2, 3, \ldots\ldots\ldots\} \\
\mathbb{Q}^{+} &= \{\frac{m}{p} \mid m, p \in \mathbb{N},\ p \neq 0\}
\end{aligned}
$$

List all pairs:

$$
\frac{0}{1}\ \frac{1}{0} \qquad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0} \qquad \frac{0}{3}\ \frac{1}{2}\ \frac{2}{1}\ \frac{3}{0} \qquad \frac{0}{4}\ \frac{1}{3}\ \frac{2}{2}
$$

Some examples:

$$\begin{aligned}
\mathbb{N} &= \{0,1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N}-\{0\} &= \{1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2,-1,0,1,2,3,\ldots\ldots\ldots\} \\
\mathbb{Q}^+ &= \{\frac{m}{p} \mid m,p\in\mathbb{N},\ \ p\neq 0\}
\end{aligned}$$

List all pairs:

$$\frac{0}{1}\ \frac{1}{0}\qquad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0}\qquad \frac{0}{3}\ \frac{1}{2}\ \frac{2}{1}\ \frac{3}{0}\qquad \frac{0}{4}\ \frac{1}{3}\ \frac{2}{2}\ \frac{3}{1}$$

Some examples:

$$
\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2, -1, 0, 1, 2, 3, \ldots\ldots\ldots\} \\
\mathbb{Q}^+ &= \{\frac{m}{p} \mid m, p \in \mathbb{N},\ p \neq 0\}
\end{aligned}
$$

List all pairs:

$$\frac{0}{1}\ \frac{1}{0} \qquad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0} \qquad \frac{0}{3}\ \frac{1}{2}\ \frac{2}{1}\ \frac{3}{0} \qquad \frac{0}{4}\ \frac{1}{3}\ \frac{2}{2}\ \frac{3}{1}\ \frac{4}{0}$$

Some examples:

$$\begin{aligned}\mathbb{N} &= \{0,1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\ \mathbb{N}-\{0\} &= \{1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\ \mathbb{Z} &= \{\ldots\ldots\ldots -2,-1,0,1,2,3,\ldots\ldots\ldots\} \\ \mathbb{Q}^+ &= \{\frac{m}{p} \mid m,p\in\mathbb{N},\ p\neq 0\}\end{aligned}$$

List all pairs:

$$\frac{0}{1}\ \frac{1}{0}\qquad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0}\qquad \frac{0}{3}\ \frac{1}{2}\ \frac{2}{1}\ \frac{3}{0}\qquad \frac{0}{4}\ \frac{1}{3}\ \frac{2}{2}\ \frac{3}{1}\ \frac{4}{0}\qquad \frac{0}{5}$$

Some examples:

$$\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2, -1, 0, 1, 2, 3, \ldots\ldots\ldots\} \\
\mathbb{Q}^+ &= \{\frac{m}{p} \mid m, p \in \mathbb{N},\ p \neq 0\}
\end{aligned}$$

List all pairs:

$$\frac{0}{1}\ \frac{1}{0} \qquad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0} \qquad \frac{0}{3}\ \frac{1}{2}\ \frac{2}{1}\ \frac{3}{0} \qquad \frac{0}{4}\ \frac{1}{3}\ \frac{2}{2}\ \frac{3}{1}\ \frac{4}{0} \qquad \frac{0}{5}\ \frac{1}{4}$$

Some examples:

$$
\begin{aligned}
\mathbb{N} &= \{0,1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N}-\{0\} &= \{1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2,-1,0,1,2,3,\ldots\ldots\ldots\} \\
\mathbb{Q}^{+} &= \{\frac{m}{p} \mid m,p\in\mathbb{N},\ p\neq 0\}
\end{aligned}
$$

List all pairs:

$$
\frac{0}{1}\ \frac{1}{0}\qquad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0}\qquad \frac{0}{3}\ \frac{1}{2}\ \frac{2}{1}\ \frac{3}{0}\qquad \frac{0}{4}\ \frac{1}{3}\ \frac{2}{2}\ \frac{3}{1}\ \frac{4}{0}\qquad \frac{0}{5}\ \frac{1}{4}\ \frac{2}{3}
$$

Some examples:

$$
\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2, -1, 0, 1, 2, 3, \ldots\ldots\ldots\} \\
\mathbb{Q}^{+} &= \{\, \frac{m}{p} \mid m, p \in \mathbb{N},\ p \neq 0\}
\end{aligned}
$$

List all pairs:

$$
\frac{0}{1}\ \frac{1}{0}\quad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0}\quad \frac{0}{3}\ \frac{1}{2}\ \frac{2}{1}\ \frac{3}{0}\quad \frac{0}{4}\ \frac{1}{3}\ \frac{2}{2}\ \frac{3}{1}\ \frac{4}{0}\quad \frac{0}{5}\ \frac{1}{4}\ \frac{2}{3}\ \cdots
$$

Some examples:

$$\begin{aligned}
\mathbb{N} &= \{0,1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N}-\{0\} &= \{1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2,-1,0,1,2,3,\ldots\ldots\ldots\} \\
\mathbb{Q}^+ &= \{\frac{m}{p} \mid m,p \in \mathbb{N},\ p \neq 0\}
\end{aligned}$$

$$\frac{0}{1}\ \frac{1}{0}\qquad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0}\qquad \frac{0}{3}\ \frac{1}{2}\ \frac{2}{1}\ \frac{3}{0}\qquad \frac{0}{4}\ \frac{1}{3}\ \frac{2}{2}\ \frac{3}{1}\ \frac{4}{0}\qquad \frac{0}{5}\ \frac{1}{4}\ \frac{2}{3}\ \ldots$$

Some examples:

$$\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2, -1, 0, 1, 2, 3, \ldots\ldots\ldots\} \\
\mathbb{Q}^+ &= \{\frac{m}{p} \mid m, p \in \mathbb{N},\ p \neq 0\}
\end{aligned}$$

Eliminate divisions by zero and repetitions:

$$\frac{0}{1}\ \frac{1}{0} \quad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0} \quad \frac{0}{3}\ \frac{1}{2}\ \frac{2}{1}\ \frac{3}{0} \quad \frac{0}{4}\ \frac{1}{3}\ \frac{2}{2}\ \frac{3}{1}\ \frac{4}{0} \quad \frac{0}{5}\ \frac{1}{4}\ \frac{2}{3}\ \ldots$$

Some examples:

$$
\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2, -1, 0, 1, 2, 3, \ldots\ldots\ldots\} \\
\mathbb{Q}^+ &= \{\frac{m}{p} \mid m, p \in \mathbb{N},\ \ p \neq 0\}
\end{aligned}
$$

Eliminate divisions by zero and repetitions:

$$\frac{0}{1} \quad \frac{1}{0} \qquad \frac{0}{2} \quad \frac{1}{1} \quad \frac{2}{0} \qquad \frac{0}{3} \quad \frac{1}{2} \quad \frac{2}{1} \quad \frac{3}{0} \qquad \frac{0}{4} \quad \frac{1}{3} \quad \frac{2}{2} \quad \frac{3}{1} \quad \frac{4}{0} \qquad \frac{0}{5} \quad \frac{1}{4} \quad \frac{2}{3} \quad \ldots$$

Some examples:

$$\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2, -1, 0, 1, 2, 3, \ldots\ldots\ldots\} \\
\mathbb{Q}^+ &= \{\frac{m}{p} \mid m, p \in \mathbb{N},\ p \neq 0\}
\end{aligned}$$

Eliminate divisions by zero and repetitions:

$$\frac{0}{1} \quad \frac{1}{0} \qquad \frac{0}{2} \quad \frac{1}{1} \quad \frac{2}{0} \qquad \frac{0}{3} \quad \frac{1}{2} \quad \frac{2}{1} \quad \frac{3}{0} \qquad \frac{0}{4} \quad \frac{1}{3} \quad \frac{2}{2} \quad \frac{3}{1} \quad \frac{4}{0} \qquad \frac{0}{5} \quad \frac{1}{4} \quad \frac{2}{3} \quad \cdots$$

Some examples:

$$\begin{array}{rcl} \mathbb{N} & = & \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\ \mathbb{N} - \{0\} & = & \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\ \mathbb{Z} & = & \{\ldots\ldots\ldots -2, -1, 0, 1, 2, 3, \ldots\ldots\ldots\} \\ \mathbb{Q}^+ & = & \{\frac{m}{p} \mid m, p \in \mathbb{N},\ p \neq 0\} \end{array}$$

Eliminate divisions by zero and repetitions:

$$\frac{0}{1} \quad \frac{1}{0} \qquad \frac{0}{2} \quad \frac{1}{1} \quad \frac{2}{0} \qquad \frac{0}{3} \quad \frac{1}{2} \quad \frac{2}{1} \quad \frac{3}{0} \qquad \frac{0}{4} \quad \frac{1}{3} \quad \frac{2}{2} \quad \frac{3}{1} \quad \frac{4}{0} \qquad \frac{0}{5} \quad \frac{1}{4} \quad \frac{2}{3} \quad \cdots$$

Some examples:

$$\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2, -1, 0, 1, 2, 3, \ldots\ldots\ldots\} \\
\mathbb{Q}^+ &= \{\frac{m}{p} \mid m, p \in \mathbb{N},\ p \neq 0\}
\end{aligned}$$

Eliminate divisions by zero and repetitions:

$$\frac{0}{1}\ \frac{1}{0}\qquad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0}\qquad \frac{0}{3}\ \frac{1}{2}\ \frac{2}{1}\ \frac{3}{0}\qquad \frac{0}{4}\ \frac{1}{3}\ \frac{2}{2}\ \frac{3}{1}\ \frac{4}{0}\qquad \frac{0}{5}\ \frac{1}{4}\ \frac{2}{3}\ \ldots$$

Some examples:

$$\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2, -1, 0, 1, 2, 3, \ldots\ldots\ldots\} \\
\mathbb{Q}^+ &= \{\frac{m}{p} \mid m, p \in \mathbb{N},\ p \neq 0\}
\end{aligned}$$

Eliminate divisions by zero and repetitions:

$$\frac{0}{1}\ \frac{1}{0} \quad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0} \quad \frac{0}{3}\ \frac{1}{2}\ \frac{2}{1}\ \frac{3}{0} \quad \frac{0}{4}\ \frac{1}{3}\ \frac{2}{2}\ \frac{3}{1}\ \frac{4}{0} \quad \frac{0}{5}\ \frac{1}{4}\ \frac{2}{3}\ \ldots$$

Some examples:

$$
\begin{aligned}
\mathbb{N} &= \{0,1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N}-\{0\} &= \{1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2,-1,0,1,2,3,\ldots\ldots\ldots\} \\
\mathbb{Q}^{+} &= \{\frac{m}{p} \mid m,p \in \mathbb{N},\ p \neq 0\}
\end{aligned}
$$

Eliminate divisions by zero and repetitions:

$$\frac{0}{1}\ \frac{1}{0}\qquad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0}\qquad \frac{0}{3}\ \frac{1}{2}\ \frac{2}{1}\ \frac{3}{0}\qquad \frac{0}{4}\ \frac{1}{3}\ \frac{2}{2}\ \frac{3}{1}\ \frac{4}{0}\qquad \frac{0}{5}\ \frac{1}{4}\ \frac{2}{3}\ \ldots$$

Some examples:

$$
\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2, -1, 0, 1, 2, 3, \ldots\ldots\ldots\} \\
\mathbb{Q}^+ &= \{\frac{m}{p} \mid m, p \in \mathbb{N},\ p \neq 0\}
\end{aligned}
$$

Eliminate divisions by zero and repetitions:

$$\frac{0}{1} \quad \frac{1}{0} \qquad \frac{0}{2} \quad \frac{1}{1} \quad \frac{2}{0} \qquad \frac{0}{3} \quad \frac{1}{2} \quad \frac{2}{1} \quad \frac{3}{0} \qquad \frac{0}{4} \quad \frac{1}{3} \quad \frac{2}{2} \quad \frac{3}{1} \quad \frac{4}{0} \qquad \frac{0}{5} \quad \frac{1}{4} \quad \frac{2}{3} \quad \ldots$$

Some examples:

$$
\begin{aligned}
\mathbb{N} &= \{0,1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N}-\{0\} &= \{1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2,-1,0,1,2,3,\ldots\ldots\ldots\} \\
\mathbb{Q}^+ &= \{\frac{m}{p} \mid m,p \in \mathbb{N},\ \ p \neq 0\}
\end{aligned}
$$

Eliminate divisions by zero and repetitions:

$$
\frac{0}{1}\ \frac{1}{0}\quad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0}\quad \frac{0}{3}\ \frac{1}{2}\ \frac{2}{1}\ \frac{3}{0}\quad \frac{0}{4}\ \frac{1}{3}\ \frac{2}{2}\ \frac{3}{1}\ \frac{4}{0}\quad \frac{0}{5}\ \frac{1}{4}\ \frac{2}{3}\ \ldots
$$

Some examples:

$$\begin{aligned}
\mathbb{N} &= \{0,1,2,3,4,\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N}-\{0\} &= \{1,2,3,4,\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2,-1,0,1,2,3,\ldots\ldots\ldots\} \\
\mathbb{Q}^+ &= \{\frac{m}{p} \mid m,p \in \mathbb{N},\ p \neq 0\}
\end{aligned}$$

Eliminate divisions by zero and repetitions:

$$\frac{0}{1} \quad \frac{1}{0} \qquad \frac{0}{2} \quad \frac{1}{1} \quad \frac{2}{0} \qquad \frac{0}{3} \quad \frac{1}{2} \quad \frac{2}{1} \quad \frac{3}{0} \qquad \frac{0}{4} \quad \frac{1}{3} \quad \frac{2}{2} \quad \frac{3}{1} \quad \frac{4}{0} \qquad \frac{0}{5} \quad \frac{1}{4} \quad \frac{2}{3} \quad \ldots$$

Some examples:

$$\begin{aligned}
\mathbb{N} &= \{0,1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N}-\{0\} &= \{1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2,-1,0,1,2,3,\ldots\ldots\ldots\} \\
\mathbb{Q}^+ &= \{\frac{m}{p} \mid m,p \in \mathbb{N},\ p \neq 0\}
\end{aligned}$$

$$\frac{0}{1}\ \frac{1}{0} \qquad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0} \qquad \frac{0}{3}\ \frac{1}{2}\ \frac{2}{1}\ \frac{3}{0} \qquad \frac{0}{4}\ \frac{1}{3}\ \frac{2}{2}\ \frac{3}{1}\ \frac{4}{0} \qquad \frac{0}{5}\ \frac{1}{4}\ \frac{2}{3}\ \ldots$$

Some examples:

$$
\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2, -1, 0, 1, 2, 3, \ldots\ldots\ldots\} \\
\mathbb{Q}^+ &= \{\frac{m}{p} \mid m, p \in \mathbb{N},\ p \neq 0\}
\end{aligned}
$$

Count remaining pairs:

$$\frac{0}{1}\ \frac{1}{0} \quad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0} \quad \frac{0}{3}\ \frac{1}{2}\ \frac{2}{1}\ \frac{3}{0} \quad \frac{0}{4}\ \frac{1}{3}\ \frac{2}{2}\ \frac{3}{1}\ \frac{4}{0} \quad \frac{0}{5}\ \frac{1}{4}\ \frac{2}{3}\ \cdots$$

Some examples:

$$\begin{aligned}
\mathbb{N} &= \{0,1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N}-\{0\} &= \{1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2,-1,0,1,2,3,\ldots\ldots\ldots\} \\
\mathbb{Q}^{+} &= \{\,\frac{m}{p}\ \mid\ m,p\in\mathbb{N},\ \ p\neq 0\}
\end{aligned}$$

$$\frac{0}{1}\ \frac{1}{0}\qquad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0}\qquad \frac{0}{3}\ \frac{1}{2}\ \frac{2}{1}\ \frac{3}{0}\qquad \frac{0}{4}\ \frac{1}{3}\ \frac{2}{2}\ \frac{3}{1}\ \frac{4}{0}\qquad \frac{0}{5}\ \frac{1}{4}\ \frac{2}{3}\ \ldots$$

Some examples:

$$
\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2, -1, 0, 1, 2, 3, \ldots\ldots\ldots\} \\
\mathbb{Q}^{+} &= \{\, \frac{\mathsf{m}}{\mathsf{p}} \mid m, p \in \mathbb{N},\ p \neq 0\}
\end{aligned}
$$

0

$$\frac{0}{1}\ \frac{1}{0}\qquad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0}\qquad \frac{0}{3}\ \frac{1}{2}\ \frac{2}{1}\ \frac{3}{0}\qquad \frac{0}{4}\ \frac{1}{3}\ \frac{2}{2}\ \frac{3}{1}\ \frac{4}{0}\qquad \frac{0}{5}\ \frac{1}{4}\ \frac{2}{3}\ \ldots$$

Some examples:

$$\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\} \\
\mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots -2, -1, 0, 1, 2, 3, \ldots\ldots\} \\
\mathbb{Q}^+ &= \{\frac{\mathsf{m}}{\mathsf{p}} \mid m, p \in \mathbb{N},\ p \neq 0\}
\end{aligned}$$

0 1

$$\frac{0}{1} \quad \frac{1}{0} \qquad \frac{0}{2} \quad \frac{1}{1} \quad \frac{2}{0} \qquad \frac{0}{3} \quad \frac{1}{2} \quad \frac{2}{1} \quad \frac{3}{0} \qquad \frac{0}{4} \quad \frac{1}{3} \quad \frac{2}{2} \quad \frac{3}{1} \quad \frac{4}{0} \qquad \frac{0}{5} \quad \frac{1}{4} \quad \frac{2}{3} \quad \ldots$$

Some examples:

$$
\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2, -1, 0, 1, 2, 3, \ldots\ldots\ldots\} \\
\mathbb{Q}^+ &= \{\, \frac{m}{p} \mid m, p \in \mathbb{N},\ \ p \neq 0\}
\end{aligned}
$$

$$0 \qquad\qquad 1 \qquad\qquad 2$$

$$\frac{0}{1}\ \frac{1}{0} \qquad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0} \qquad \frac{0}{3}\ \frac{1}{2}\ \frac{2}{1}\ \frac{3}{0} \qquad \frac{0}{4}\ \frac{1}{3}\ \frac{2}{2}\ \frac{3}{1}\ \frac{4}{0} \qquad \frac{0}{5}\ \frac{1}{4}\ \frac{2}{3}\ \ldots$$

Some examples:

$$
\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2, -1, 0, 1, 2, 3, \ldots\ldots\ldots\} \\
\mathbb{Q}^+ &= \{\frac{m}{p} \mid m, p \in \mathbb{N},\ p \neq 0\}
\end{aligned}
$$

$0 \quad 1 \quad 2 \quad 3$

$\frac{0}{1} \ \frac{1}{0} \quad \frac{0}{2} \ \frac{1}{1} \ \frac{2}{0} \quad \frac{0}{3} \ \frac{1}{2} \ \frac{2}{1} \ \frac{3}{0} \quad \frac{0}{4} \ \frac{1}{3} \ \frac{2}{2} \ \frac{3}{1} \ \frac{4}{0} \quad \frac{0}{5} \ \frac{1}{4} \ \frac{2}{3} \ \ldots$

Some examples:

$$
\begin{aligned}
\mathbb{N} &= \{0,1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N}-\{0\} &= \{1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots-2,-1,0,1,2,3,\ldots\ldots\ldots\} \\
\mathbb{Q}^{+} &= \{\frac{m}{p} \mid m,p \in \mathbb{N},\ p \neq 0\}
\end{aligned}
$$

0			1			2	3			4									
$\frac{0}{1}$	$\frac{1}{0}$	$\frac{0}{2}$	$\frac{1}{1}$	$\frac{2}{0}$	$\frac{0}{3}$	$\frac{1}{2}$	$\frac{2}{1}$	$\frac{3}{0}$	$\frac{0}{4}$	$\frac{1}{3}$	$\frac{2}{2}$	$\frac{3}{1}$	$\frac{4}{0}$	$\frac{0}{5}$	$\frac{1}{4}$	$\frac{2}{3}$	$\ldots$		

Some examples:

$$\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
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\end{aligned}$$

0 1 2 3 4 5

$$\frac{0}{1}\ \frac{1}{0}\quad \frac{0}{2}\ \frac{1}{1}\ \frac{2}{0}\quad \frac{0}{3}\ \frac{1}{2}\ \frac{2}{1}\ \frac{3}{0}\quad \frac{0}{4}\ \frac{1}{3}\ \frac{2}{2}\ \frac{3}{1}\ \frac{4}{0}\quad \frac{0}{5}\ \frac{1}{4}\ \frac{2}{3}\ \cdots$$

Some examples:

$$\begin{aligned} \mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\} \\ \mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\} \\ \mathbb{Z} &= \{\ldots\ldots -2, -1, 0, 1, 2, 3, \ldots\ldots\} \\ \mathbb{Q}^+ &= \{ \frac{m}{p} \mid m, p \in \mathbb{N},\ p \neq 0\} \end{aligned}$$

0			1			2	3			4		5				6		
$\frac{0}{1}$	$\frac{1}{0}$	$\frac{0}{2}$	$\frac{1}{1}$	$\frac{2}{0}$	$\frac{0}{3}$	$\frac{1}{2}$	$\frac{2}{1}$	$\frac{3}{0}$	$\frac{0}{4}$	$\frac{1}{3}$	$\frac{2}{2}$	$\frac{3}{1}$	$\frac{4}{0}$	$\frac{0}{5}$	$\frac{1}{4}$	$\frac{2}{3}$	$\ldots$	

Some examples:

$$
\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
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\end{aligned}
$$

0			1			2	3			4		5			6	7	
$\frac{0}{1}$	$\frac{1}{0}$	$\frac{0}{2}$	$\frac{1}{1}$	$\frac{2}{0}$	$\frac{0}{3}$	$\frac{1}{2}$	$\frac{2}{1}$	$\frac{3}{0}$	$\frac{0}{4}$	$\frac{1}{3}$	$\frac{2}{2}$	$\frac{3}{1}$	$\frac{4}{0}$	$\frac{0}{5}$	$\frac{1}{4}$	$\frac{2}{3}$	$\ldots$

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$$

$$
0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \cdots
$$

$$
\frac{0}{1} \ \frac{1}{0} \quad \frac{0}{2} \ \frac{1}{1} \ \frac{2}{0} \quad \frac{0}{3} \ \frac{1}{2} \ \frac{2}{1} \ \frac{3}{0} \quad \frac{0}{4} \ \frac{1}{3} \ \frac{2}{2} \ \frac{3}{1} \ \frac{4}{0} \quad \frac{0}{5} \ \frac{1}{4} \ \frac{2}{3} \ \cdots
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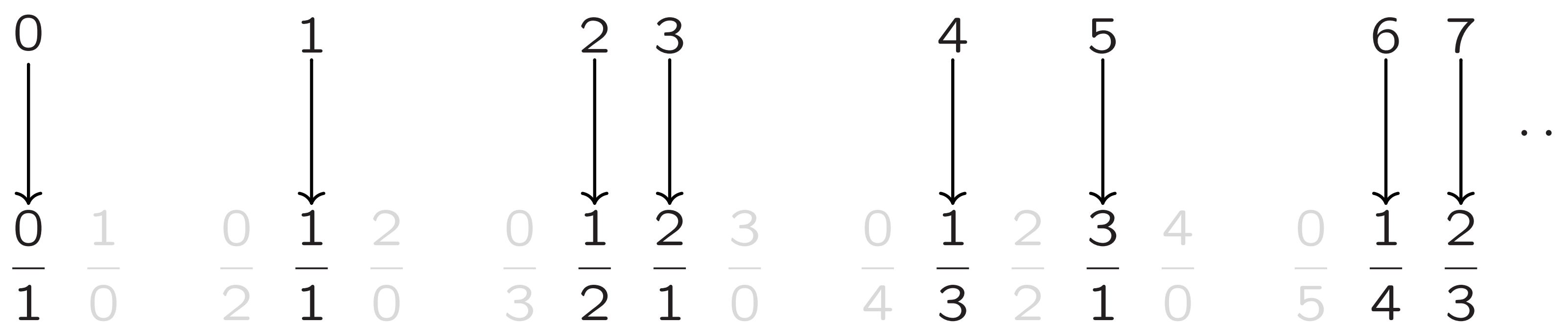
$$



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$$
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\end{aligned}
$$

$\mathbb{Q}^+$ and $\mathbb{N}$ have the same size.

Some examples:

$$
\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{N} - \{0\} &= \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
\mathbb{Z} &= \{\ldots\ldots\ldots -2, -1, 0, 1, 2, 3, \ldots\ldots\ldots\} \\
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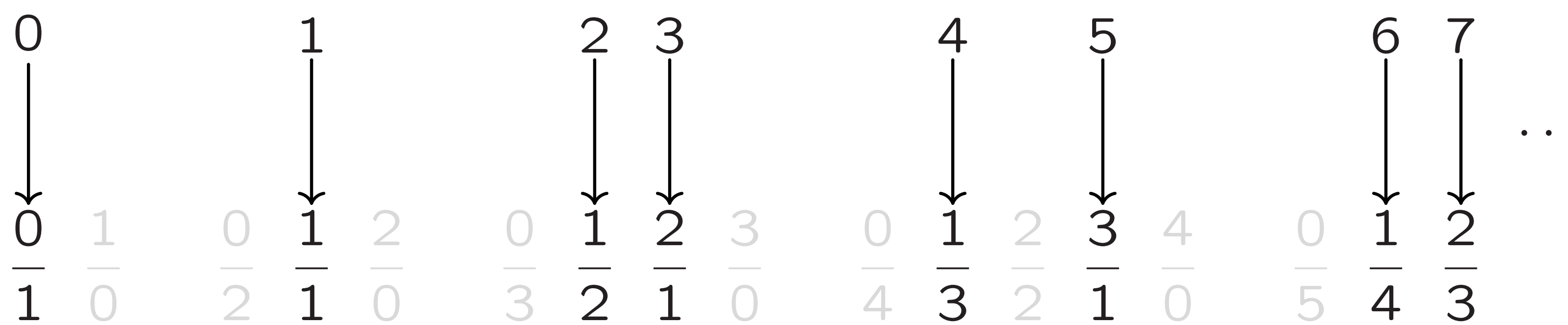

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\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \\
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\mathbb{R} &= \{\text{all real numbers}\}
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$\mathbb{N} - \{0\}$, $\mathbb{Z}$, $\mathbb{Q}^{+}$, and $\mathbb{Q}$ are all equinumerous with $\mathbb{N}$.

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\mathbb{Q}^+ & = & \{\frac{\mathsf{m}}{\mathsf{p}} \mid m, p \in \mathbb{N},\ p \neq 0\} \\
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\end{array}$$

$\mathbb{N} - \{0\}$, $\mathbb{Z}$, $\mathbb{Q}^+$, and $\mathbb{Q}$ are all equinumerous with $\mathbb{N}$.
But ....

Theorem (Cantor, 1873). $\mathbb{R}$ and $\mathbb{N}$ are *not* of the same size.

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Let $[0, 1]$ denote the interval $\{x \mid 0 \leq x \leq 1\}$.

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(For example, if $f(x) = 79.121212\ldots$ then $g(x) = 0.121212\ldots$.)

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Note that $g \colon \mathbb{N} \to [0, 1]$ is *onto*.

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Let $[0, 1]$ denote the interval $\{x \mid 0 \le x \le 1\}$.

Define $g(x) = f(x) - \lfloor f(x) \rfloor$.

Note that $g\colon \mathbb{N} \to [0, 1]$ is *onto*.

Consider the following table:

$$
\begin{aligned}
g(0) &= \\
g(1) &= \\
g(2) &= \\
g(3) &= \\
g(4) &= \\
g(5) &= \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
g(0) &= 0 \\
g(1) &= \\
g(2) &= \\
g(3) &= \\
g(4) &= \\
g(5) &= \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
g(0) &= 0 \quad . \\
g(1) &= \\
g(2) &= \\
g(3) &= \\
g(4) &= \\
g(5) &= \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
g(0) &= 0 \;\; . \;\; a_0^0 \\
g(1) &= \\
g(2) &= \\
g(3) &= \\
g(4) &= \\
g(5) &= \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
g(0) &= 0 \;\; . \;\; a_0^0 \;\; a_1^0 \\
g(1) &= \\
g(2) &= \\
g(3) &= \\
g(4) &= \\
g(5) &= \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
g(0) &= 0 \;.\; a_0^0 \; a_1^0 \; a_2^0 \\
g(1) &= \\
g(2) &= \\
g(3) &= \\
g(4) &= \\
g(5) &= \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
g(0) &= 0 \; . \; a_0^0 \;\; a_1^0 \;\; a_2^0 \;\; a_3^0 \\
g(1) &= \\
g(2) &= \\
g(3) &= \\
g(4) &= \\
g(5) &= \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
g(0) &= 0\ .\ a_0^0\ a_1^0\ a_2^0\ a_3^0\ a_4^0 \\
g(1) &= \\
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g(5) &= \\
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$$
\begin{aligned}
g(0) &= 0 \;.\; a_0^0 \;\; a_1^0 \;\; a_2^0 \;\; a_3^0 \;\; a_4^0 \;\; a_5^0 \;\; \cdots \\
g(1) &= \\
g(2) &= \\
g(3) &= \\
g(4) &= \\
g(5) &= \\
&\vdots \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
g(0) &= 0 \;.\; a_0^0 \; a_1^0 \; a_2^0 \; a_3^0 \; a_4^0 \; a_5^0 \; \cdots \\
g(1) &= \\
g(2) &= \\
g(3) &= \\
g(4) &= \\
g(5) &= \\
&\vdots
\end{aligned}
$$

(Each a_n^i is a digit between 0 and 9.)

$$
\begin{aligned}
g(0) &= 0 \;.\; a_0^0 \; a_1^0 \; a_2^0 \; a_3^0 \; a_4^0 \; a_5^0 \; \cdots \\
g(1) &= \\
g(2) &= \\
g(3) &= \\
g(4) &= \\
g(5) &= \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
g(0) &= 0\ .\ a_0^0\ a_1^0\ a_2^0\ a_3^0\ a_4^0\ a_5^0\ \cdots \\
g(1) &= 0\ .\ a_0^1\ a_1^1\ a_2^1\ a_3^1\ a_4^1\ a_5^1\ \cdots \\
g(2) &= \\
g(3) &= \\
g(4) &= \\
g(5) &= \\
&\vdots
\end{aligned}
$$

$$
\begin{array}{lclllllllll}
g(0) & = & 0 & . & a_0^0 & a_1^0 & a_2^0 & a_3^0 & a_4^0 & a_5^0 & \cdots \\
g(1) & = & 0 & . & a_0^1 & a_1^1 & a_2^1 & a_3^1 & a_4^1 & a_5^1 & \cdots \\
g(2) & = & 0 & . & a_0^2 & a_1^2 & a_2^2 & a_3^2 & a_4^2 & a_5^2 & \cdots \\
g(3) & = & & & & & & & & & \\
g(4) & = & & & & & & & & & \\
g(5) & = & & & & & & & & & \\
 & \vdots & & & & & & & & &
\end{array}
$$

$$
\begin{aligned}
g(0) &= 0\ .\ a_0^0\ a_1^0\ a_2^0\ a_3^0\ a_4^0\ a_5^0\ \cdots \\
g(1) &= 0\ .\ a_0^1\ a_1^1\ a_2^1\ a_3^1\ a_4^1\ a_5^1\ \cdots \\
g(2) &= 0\ .\ a_0^2\ a_1^2\ a_2^2\ a_3^2\ a_4^2\ a_5^2\ \cdots \\
g(3) &= 0\ .\ a_0^3\ a_1^3\ a_2^3\ a_3^3\ a_4^3\ a_5^3\ \cdots \\
g(4) &= \\
g(5) &= \\
&\vdots
\end{aligned}
$$

$$
\begin{array}{lcl}
g(0) & = & 0 \;.\; a_0^0 \; a_1^0 \; a_2^0 \; a_3^0 \; a_4^0 \; a_5^0 \; \cdots \\
g(1) & = & 0 \;.\; a_0^1 \; a_1^1 \; a_2^1 \; a_3^1 \; a_4^1 \; a_5^1 \; \cdots \\
g(2) & = & 0 \;.\; a_0^2 \; a_1^2 \; a_2^2 \; a_3^2 \; a_4^2 \; a_5^2 \; \cdots \\
g(3) & = & 0 \;.\; a_0^3 \; a_1^3 \; a_2^3 \; a_3^3 \; a_4^3 \; a_5^3 \; \cdots \\
g(4) & = & 0 \;.\; a_0^4 \; a_1^4 \; a_2^4 \; a_3^4 \; a_4^4 \; a_5^4 \; \cdots \\
g(5) & = & \\
 & \vdots &
\end{array}
$$

$$
\begin{array}{rclllllll}
g(0) & = & 0 & . & a_0^0 & a_1^0 & a_2^0 & a_3^0 & a_4^0 & a_5^0 & \cdots \\
g(1) & = & 0 & . & a_0^1 & a_1^1 & a_2^1 & a_3^1 & a_4^1 & a_5^1 & \cdots \\
g(2) & = & 0 & . & a_0^2 & a_1^2 & a_2^2 & a_3^2 & a_4^2 & a_5^2 & \cdots \\
g(3) & = & 0 & . & a_0^3 & a_1^3 & a_2^3 & a_3^3 & a_4^3 & a_5^3 & \cdots \\
g(4) & = & 0 & . & a_0^4 & a_1^4 & a_2^4 & a_3^4 & a_4^4 & a_5^4 & \cdots \\
g(5) & = & 0 & . & a_0^5 & a_1^5 & a_2^5 & a_3^5 & a_4^5 & a_5^5 & \cdots \\
 & \vdots & & & & & & & & &
\end{array}
$$

$$
\begin{array}{lcccccccccc}
g(0) & = & 0 & . & \boldsymbol{a_0^0} & a_1^0 & a_2^0 & a_3^0 & a_4^0 & a_5^0 & \cdots \\
g(1) & = & 0 & . & a_0^1 & a_1^1 & a_2^1 & a_3^1 & a_4^1 & a_5^1 & \cdots \\
g(2) & = & 0 & . & a_0^2 & a_1^2 & a_2^2 & a_3^2 & a_4^2 & a_5^2 & \cdots \\
g(3) & = & 0 & . & a_0^3 & a_1^3 & a_2^3 & a_3^3 & a_4^3 & a_5^3 & \cdots \\
g(4) & = & 0 & . & a_0^4 & a_1^4 & a_2^4 & a_3^4 & a_4^4 & a_5^4 & \cdots \\
g(5) & = & 0 & . & a_0^5 & a_1^5 & a_2^5 & a_3^5 & a_4^5 & a_5^5 & \cdots \\
 & \vdots & & & & & & & & &
\end{array}
$$

$$
\begin{array}{lcllllllll}
g(0) & = & 0 & . & \mathbf{a_0^0} & a_1^0 & a_2^0 & a_3^0 & a_4^0 & a_5^0 & \cdots \\
g(1) & = & 0 & . & a_0^1 & \mathbf{a_1^1} & a_2^1 & a_3^1 & a_4^1 & a_5^1 & \cdots \\
g(2) & = & 0 & . & a_0^2 & a_1^2 & a_2^2 & a_3^2 & a_4^2 & a_5^2 & \cdots \\
g(3) & = & 0 & . & a_0^3 & a_1^3 & a_2^3 & a_3^3 & a_4^3 & a_5^3 & \cdots \\
g(4) & = & 0 & . & a_0^4 & a_1^4 & a_2^4 & a_3^4 & a_4^4 & a_5^4 & \cdots \\
g(5) & = & 0 & . & a_0^5 & a_1^5 & a_2^5 & a_3^5 & a_4^5 & a_5^5 & \cdots \\
 & \vdots & & & & & & & & &
\end{array}
$$

$$
\begin{array}{lcllllllll}
g(0) & = & 0 & . & \mathbf{a_0^0} & a_1^0 & a_2^0 & a_3^0 & a_4^0 & a_5^0 & \cdots \\
g(1) & = & 0 & . & a_0^1 & \mathbf{a_1^1} & a_2^1 & a_3^1 & a_4^1 & a_5^1 & \cdots \\
g(2) & = & 0 & . & a_0^2 & a_1^2 & \mathbf{a_2^2} & a_3^2 & a_4^2 & a_5^2 & \cdots \\
g(3) & = & 0 & . & a_0^3 & a_1^3 & a_2^3 & a_3^3 & a_4^3 & a_5^3 & \cdots \\
g(4) & = & 0 & . & a_0^4 & a_1^4 & a_2^4 & a_3^4 & a_4^4 & a_5^4 & \cdots \\
g(5) & = & 0 & . & a_0^5 & a_1^5 & a_2^5 & a_3^5 & a_4^5 & a_5^5 & \cdots \\
 & \vdots & & & & & & & & & \\
 & \vdots & & & & & & & & &
\end{array}
$$

$$
\begin{array}{lcccccccccc}
g(0) & = & 0 & . & \boldsymbol{a_0^0} & a_1^0 & a_2^0 & a_3^0 & a_4^0 & a_5^0 & \cdots \\
g(1) & = & 0 & . & a_0^1 & \boldsymbol{a_1^1} & a_2^1 & a_3^1 & a_4^1 & a_5^1 & \cdots \\
g(2) & = & 0 & . & a_0^2 & a_1^2 & \boldsymbol{a_2^2} & a_3^2 & a_4^2 & a_5^2 & \cdots \\
g(3) & = & 0 & . & a_0^3 & a_1^3 & a_2^3 & \boldsymbol{a_3^3} & a_4^3 & a_5^3 & \cdots \\
g(4) & = & 0 & . & a_0^4 & a_1^4 & a_2^4 & a_3^4 & a_4^4 & a_5^4 & \cdots \\
g(5) & = & 0 & . & a_0^5 & a_1^5 & a_2^5 & a_3^5 & a_4^5 & a_5^5 & \cdots \\
 & \vdots & & & & & & & & &
\end{array}
$$

$$
\begin{array}{lcccccccccc}
g(0) & = & 0 & . & \boldsymbol{a_0^0} & a_1^0 & a_2^0 & a_3^0 & a_4^0 & a_5^0 & \cdots \\
g(1) & = & 0 & . & a_0^1 & \boldsymbol{a_1^1} & a_2^1 & a_3^1 & a_4^1 & a_5^1 & \cdots \\
g(2) & = & 0 & . & a_0^2 & a_1^2 & \boldsymbol{a_2^2} & a_3^2 & a_4^2 & a_5^2 & \cdots \\
g(3) & = & 0 & . & a_0^3 & a_1^3 & a_2^3 & \boldsymbol{a_3^3} & a_4^3 & a_5^3 & \cdots \\
g(4) & = & 0 & . & a_0^4 & a_1^4 & a_2^4 & a_3^4 & \boldsymbol{a_4^4} & a_5^4 & \cdots \\
g(5) & = & 0 & . & a_0^5 & a_1^5 & a_2^5 & a_3^5 & a_4^5 & a_5^5 & \cdots \\
 & \vdots & & & & & & & & &
\end{array}
$$

$$
\begin{array}{lclllllllll}
g(0) & = & 0 & . & \color{red}{\boldsymbol{a_0^0}} & a_1^0 & a_2^0 & a_3^0 & a_4^0 & a_5^0 & \cdots \\
g(1) & = & 0 & . & a_0^1 & \color{red}{\boldsymbol{a_1^1}} & a_2^1 & a_3^1 & a_4^1 & a_5^1 & \cdots \\
g(2) & = & 0 & . & a_0^2 & a_1^2 & \color{red}{\boldsymbol{a_2^2}} & a_3^2 & a_4^2 & a_5^2 & \cdots \\
g(3) & = & 0 & . & a_0^3 & a_1^3 & a_2^3 & \color{red}{\boldsymbol{a_3^3}} & a_4^3 & a_5^3 & \cdots \\
g(4) & = & 0 & . & a_0^4 & a_1^4 & a_2^4 & a_3^4 & \color{red}{\boldsymbol{a_4^4}} & a_5^4 & \cdots \\
g(5) & = & 0 & . & a_0^5 & a_1^5 & a_2^5 & a_3^5 & a_4^5 & \color{red}{\boldsymbol{a_5^5}} & \cdots \\
 & \vdots & & & & & & & & &
\end{array}
$$

$$
\begin{array}{lcccccccccc}
g(0) & = & 0 & . & \mathbf{a_0^0} & a_1^0 & a_2^0 & a_3^0 & a_4^0 & a_5^0 & \cdots \\
g(1) & = & 0 & . & a_0^1 & \mathbf{a_1^1} & a_2^1 & a_3^1 & a_4^1 & a_5^1 & \cdots \\
g(2) & = & 0 & . & a_0^2 & a_1^2 & \mathbf{a_2^2} & a_3^2 & a_4^2 & a_5^2 & \cdots \\
g(3) & = & 0 & . & a_0^3 & a_1^3 & a_2^3 & \mathbf{a_3^3} & a_4^3 & a_5^3 & \cdots \\
g(4) & = & 0 & . & a_0^4 & a_1^4 & a_2^4 & a_3^4 & \mathbf{a_4^4} & a_5^4 & \cdots \\
g(5) & = & 0 & . & a_0^5 & a_1^5 & a_2^5 & a_3^5 & a_4^5 & \mathbf{a_5^5} & \cdots \\
 & \vdots & & & & & & & & &
\end{array}
$$

$$
\begin{array}{lcllllllll}
g(0) & = & 0 & . & \boldsymbol{a_0^0} & a_1^0 & a_2^0 & a_3^0 & a_4^0 & a_5^0 & \cdots \\
g(1) & = & 0 & . & a_0^1 & \boldsymbol{a_1^1} & a_2^1 & a_3^1 & a_4^1 & a_5^1 & \cdots \\
g(2) & = & 0 & . & a_0^2 & a_1^2 & \boldsymbol{a_2^2} & a_3^2 & a_4^2 & a_5^2 & \cdots \\
g(3) & = & 0 & . & a_0^3 & a_1^3 & a_2^3 & \boldsymbol{a_3^3} & a_4^3 & a_5^3 & \cdots \\
g(4) & = & 0 & . & a_0^4 & a_1^4 & a_2^4 & a_3^4 & \boldsymbol{a_4^4} & a_5^4 & \cdots \\
g(5) & = & 0 & . & a_0^5 & a_1^5 & a_2^5 & a_3^5 & a_4^5 & \boldsymbol{a_5^5} & \cdots \\
 & \vdots & & & & & & & & & \\
\text{Diagonal} & & & & \boldsymbol{a_0^0} & \boldsymbol{a_1^1} & \boldsymbol{a_2^2} & \boldsymbol{a_3^3} & \boldsymbol{a_4^4} & \boldsymbol{a_5^5} & \boldsymbol{\cdots}
\end{array}
$$

$$
\begin{array}{lcl}
g(0) & = & 0\,.\,\mathbf{a_0^0}\ a_1^0\ a_2^0\ a_3^0\ a_4^0\ a_5^0\ \cdots \\
g(1) & = & 0\,.\,a_0^1\ \mathbf{a_1^1}\ a_2^1\ a_3^1\ a_4^1\ a_5^1\ \cdots \\
g(2) & = & 0\,.\,a_0^2\ a_1^2\ \mathbf{a_2^2}\ a_3^2\ a_4^2\ a_5^2\ \cdots \\
g(3) & = & 0\,.\,a_0^3\ a_1^3\ a_2^3\ \mathbf{a_3^3}\ a_4^3\ a_5^3\ \cdots \\
g(4) & = & 0\,.\,a_0^4\ a_1^4\ a_2^4\ a_3^4\ \mathbf{a_4^4}\ a_5^4\ \cdots \\
g(5) & = & 0\,.\,a_0^5\ a_1^5\ a_2^5\ a_3^5\ a_4^5\ \mathbf{a_5^5}\ \cdots \\
 & \vdots & \\
\text{Diagonal} & & \quad \mathbf{a_0^0}\ \mathbf{a_1^1}\ \mathbf{a_2^2}\ \mathbf{a_3^3}\ \mathbf{a_4^4}\ \mathbf{a_5^5}\ \cdots
\end{array}
$$

For a digit $\boldsymbol{a}$ set $\bar{\boldsymbol{a}} = \begin{cases} 4 & \text{if } \boldsymbol{a} = 5 \\ 5 & \text{if } \boldsymbol{a} \neq 5 \end{cases}$.

$$
\begin{array}{llllllllllll}
g(0) & = & 0 & . & \mathbf{a_0^0} & a_1^0 & a_2^0 & a_3^0 & a_4^0 & a_5^0 & \cdots \\
g(1) & = & 0 & . & a_0^1 & \mathbf{a_1^1} & a_2^1 & a_3^1 & a_4^1 & a_5^1 & \cdots \\
g(2) & = & 0 & . & a_0^2 & a_1^2 & \mathbf{a_2^2} & a_3^2 & a_4^2 & a_5^2 & \cdots \\
g(3) & = & 0 & . & a_0^3 & a_1^3 & a_2^3 & \mathbf{a_3^3} & a_4^3 & a_5^3 & \cdots \\
g(4) & = & 0 & . & a_0^4 & a_1^4 & a_2^4 & a_3^4 & \mathbf{a_4^4} & a_5^4 & \cdots \\
g(5) & = & 0 & . & a_0^5 & a_1^5 & a_2^5 & a_3^5 & a_4^5 & \mathbf{a_5^5} & \cdots \\
 & \vdots & & & & & & & & & \\
\text{Diagonal} & & & & \mathbf{a_0^0} & \mathbf{a_1^1} & \mathbf{a_2^2} & \mathbf{a_3^3} & \mathbf{a_4^4} & \mathbf{a_5^5} & \cdots
\end{array}
$$

For a digit $\mathbf{a}$ set $\bar{a} = \begin{cases} 4 & \text{if } \mathbf{a} = 5 \\ 5 & \text{if } \mathbf{a} \neq 5 \end{cases}$. Either way $\bar{a} \neq \mathbf{a}$.

$$
\begin{array}{llllllllll}
g(0) & = & 0 & . & \boldsymbol{a_0^0} & a_1^0 & a_2^0 & a_3^0 & a_4^0 & a_5^0 & \cdots \\
g(1) & = & 0 & . & a_0^1 & \boldsymbol{a_1^1} & a_2^1 & a_3^1 & a_4^1 & a_5^1 & \cdots \\
g(2) & = & 0 & . & a_0^2 & a_1^2 & \boldsymbol{a_2^2} & a_3^2 & a_4^2 & a_5^2 & \cdots \\
g(3) & = & 0 & . & a_0^3 & a_1^3 & a_2^3 & \boldsymbol{a_3^3} & a_4^3 & a_5^3 & \cdots \\
g(4) & = & 0 & . & a_0^4 & a_1^4 & a_2^4 & a_3^4 & \boldsymbol{a_4^4} & a_5^4 & \cdots \\
g(5) & = & 0 & . & a_0^5 & a_1^5 & a_2^5 & a_3^5 & a_4^5 & \boldsymbol{a_5^5} & \cdots \\
 & \vdots & & & & & & & & & \\
\text{Diagonal} & & & & \boldsymbol{a_0^0} & \boldsymbol{a_1^1} & \boldsymbol{a_2^2} & \boldsymbol{a_3^3} & \boldsymbol{a_4^4} & \boldsymbol{a_5^5} & \cdots
\end{array}
$$

$$g(0) = 0\,.\,\mathbf{a_0^0}\; a_1^0\; a_2^0\; a_3^0\; a_4^0\; a_5^0\; \cdots$$

$$g(1) = 0\,.\,a_0^1\; \mathbf{a_1^1}\; a_2^1\; a_3^1\; a_4^1\; a_5^1\; \cdots$$

$$g(2) = 0\,.\,a_0^2\; a_1^2\; \mathbf{a_2^2}\; a_3^2\; a_4^2\; a_5^2\; \cdots$$

$$g(3) = 0\,.\,a_0^3\; a_1^3\; a_2^3\; \mathbf{a_3^3}\; a_4^3\; a_5^3\; \cdots$$

$$g(4) = 0\,.\,a_0^4\; a_1^4\; a_2^4\; a_3^4\; \mathbf{a_4^4}\; a_5^4\; \cdots$$

$$g(5) = 0\,.\,a_0^5\; a_1^5\; a_2^5\; a_3^5\; a_4^5\; \mathbf{a_5^5}\; \cdots$$

$$\vdots$$

Diagonal $\mathbf{a_0^0}\; \mathbf{a_1^1}\; \mathbf{a_2^2}\; \mathbf{a_3^3}\; \mathbf{a_4^4}\; \mathbf{a_5^5}\; \cdots$

Set: $z = 0\,.$

$$
\begin{array}{llllllllll}
g(0) & = & 0 & . & \boldsymbol{a_0^0} & a_1^0 & a_2^0 & a_3^0 & a_4^0 & a_5^0 \cdots \\
g(1) & = & 0 & . & a_0^1 & \boldsymbol{a_1^1} & a_2^1 & a_3^1 & a_4^1 & a_5^1 \cdots \\
g(2) & = & 0 & . & a_0^2 & a_1^2 & \boldsymbol{a_2^2} & a_3^2 & a_4^2 & a_5^2 \cdots \\
g(3) & = & 0 & . & a_0^3 & a_1^3 & a_2^3 & \boldsymbol{a_3^3} & a_4^3 & a_5^3 \cdots \\
g(4) & = & 0 & . & a_0^4 & a_1^4 & a_2^4 & a_3^4 & \boldsymbol{a_4^4} & a_5^4 \cdots \\
g(5) & = & 0 & . & a_0^5 & a_1^5 & a_2^5 & a_3^5 & a_4^5 & \boldsymbol{a_5^5} \cdots \\
 & \vdots & & & & & & & & \\
\text{Diagonal} & & & & \boldsymbol{a_0^0} & \boldsymbol{a_1^1} & \boldsymbol{a_2^2} & \boldsymbol{a_3^3} & \boldsymbol{a_4^4} & \boldsymbol{a_5^5} \cdots
\end{array}
$$

Set: $z = 0\,.\,\boldsymbol{\bar{a}_0^0}$

$$
\begin{array}{lcllllllll}
g(0) & = & 0\,. & \boldsymbol{a_0^0} & a_1^0 & a_2^0 & a_3^0 & a_4^0 & a_5^0 & \cdots \\
g(1) & = & 0\,. & a_0^1 & \boldsymbol{a_1^1} & a_2^1 & a_3^1 & a_4^1 & a_5^1 & \cdots \\
g(2) & = & 0\,. & a_0^2 & a_1^2 & \boldsymbol{a_2^2} & a_3^2 & a_4^2 & a_5^2 & \cdots \\
g(3) & = & 0\,. & a_0^3 & a_1^3 & a_2^3 & \boldsymbol{a_3^3} & a_4^3 & a_5^3 & \cdots \\
g(4) & = & 0\,. & a_0^4 & a_1^4 & a_2^4 & a_3^4 & \boldsymbol{a_4^4} & a_5^4 & \cdots \\
g(5) & = & 0\,. & a_0^5 & a_1^5 & a_2^5 & a_3^5 & a_4^5 & \boldsymbol{a_5^5} & \cdots \\
 & \vdots & & & & & & & &
\end{array}
$$

Diagonal $\boldsymbol{a_0^0 \; a_1^1 \; a_2^2 \; a_3^3 \; a_4^4 \; a_5^5 \; \cdots}$

Set: $z = 0\,.\,\boldsymbol{\bar{a}_0^0 \; \bar{a}_1^1}$

$$
\begin{array}{lcccccccccc}
g(0) & = & 0 & . & \mathbf{a_0^0} & a_1^0 & a_2^0 & a_3^0 & a_4^0 & a_5^0 & \cdots \\
g(1) & = & 0 & . & a_0^1 & \mathbf{a_1^1} & a_2^1 & a_3^1 & a_4^1 & a_5^1 & \cdots \\
g(2) & = & 0 & . & a_0^2 & a_1^2 & \mathbf{a_2^2} & a_3^2 & a_4^2 & a_5^2 & \cdots \\
g(3) & = & 0 & . & a_0^3 & a_1^3 & a_2^3 & \mathbf{a_3^3} & a_4^3 & a_5^3 & \cdots \\
g(4) & = & 0 & . & a_0^4 & a_1^4 & a_2^4 & a_3^4 & \mathbf{a_4^4} & a_5^4 & \cdots \\
g(5) & = & 0 & . & a_0^5 & a_1^5 & a_2^5 & a_3^5 & a_4^5 & \mathbf{a_5^5} & \cdots \\
 & \vdots & & & & & & & & & \\
\text{Diagonal} & & & & \mathbf{a_0^0} & \mathbf{a_1^1} & \mathbf{a_2^2} & \mathbf{a_3^3} & \mathbf{a_4^4} & \mathbf{a_5^5} & \cdots
\end{array}
$$

Set: $z = 0 \, . \, \bar{\mathbf{a}}_0^0 \; \bar{\mathbf{a}}_1^1 \; \bar{\mathbf{a}}_2^2$

$$
\begin{array}{lcl}
g(0) & = & 0\,.\,\mathbf{a_0^0}\ a_1^0\ a_2^0\ a_3^0\ a_4^0\ a_5^0\ \cdots \\
g(1) & = & 0\,.\,a_0^1\ \mathbf{a_1^1}\ a_2^1\ a_3^1\ a_4^1\ a_5^1\ \cdots \\
g(2) & = & 0\,.\,a_0^2\ a_1^2\ \mathbf{a_2^2}\ a_3^2\ a_4^2\ a_5^2\ \cdots \\
g(3) & = & 0\,.\,a_0^3\ a_1^3\ a_2^3\ \mathbf{a_3^3}\ a_4^3\ a_5^3\ \cdots \\
g(4) & = & 0\,.\,a_0^4\ a_1^4\ a_2^4\ a_3^4\ \mathbf{a_4^4}\ a_5^4\ \cdots \\
g(5) & = & 0\,.\,a_0^5\ a_1^5\ a_2^5\ a_3^5\ a_4^5\ \mathbf{a_5^5}\ \cdots \\
 & \vdots &
\end{array}
$$

Diagonal $\mathbf{a_0^0\ a_1^1\ a_2^2\ a_3^3\ a_4^4\ a_5^5\ \cdots}$

Set: $z = 0\,.\,\bar{a}_0^0\ \bar{a}_1^1\ \bar{a}_2^2\ \bar{a}_3^3$

$$
\begin{array}{lcl}
g(0) & = & 0\ .\ \mathbf{a_0^0}\ a_1^0\ a_2^0\ a_3^0\ a_4^0\ a_5^0\ \cdots \\
g(1) & = & 0\ .\ a_0^1\ \mathbf{a_1^1}\ a_2^1\ a_3^1\ a_4^1\ a_5^1\ \cdots \\
g(2) & = & 0\ .\ a_0^2\ a_1^2\ \mathbf{a_2^2}\ a_3^2\ a_4^2\ a_5^2\ \cdots \\
g(3) & = & 0\ .\ a_0^3\ a_1^3\ a_2^3\ \mathbf{a_3^3}\ a_4^3\ a_5^3\ \cdots \\
g(4) & = & 0\ .\ a_0^4\ a_1^4\ a_2^4\ a_3^4\ \mathbf{a_4^4}\ a_5^4\ \cdots \\
g(5) & = & 0\ .\ a_0^5\ a_1^5\ a_2^5\ a_3^5\ a_4^5\ \mathbf{a_5^5}\ \cdots \\
 & \vdots & \\
\text{Diagonal} & & \mathbf{a_0^0}\ \mathbf{a_1^1}\ \mathbf{a_2^2}\ \mathbf{a_3^3}\ \mathbf{a_4^4}\ \mathbf{a_5^5}\ \cdots
\end{array}
$$

Set: $z = 0\ .\ \bar{\mathbf{a}}_0^0\ \bar{\mathbf{a}}_1^1\ \bar{\mathbf{a}}_2^2\ \bar{\mathbf{a}}_3^3\ \bar{\mathbf{a}}_4^4$

$$
\begin{array}{lcl}
g(0) & = & 0\,.\,\mathbf{a_0^0}\; a_1^0\; a_2^0\; a_3^0\; a_4^0\; a_5^0\; \cdots \\
g(1) & = & 0\,.\,a_0^1\; \mathbf{a_1^1}\; a_2^1\; a_3^1\; a_4^1\; a_5^1\; \cdots \\
g(2) & = & 0\,.\,a_0^2\; a_1^2\; \mathbf{a_2^2}\; a_3^2\; a_4^2\; a_5^2\; \cdots \\
g(3) & = & 0\,.\,a_0^3\; a_1^3\; a_2^3\; \mathbf{a_3^3}\; a_4^3\; a_5^3\; \cdots \\
g(4) & = & 0\,.\,a_0^4\; a_1^4\; a_2^4\; a_3^4\; \mathbf{a_4^4}\; a_5^4\; \cdots \\
g(5) & = & 0\,.\,a_0^5\; a_1^5\; a_2^5\; a_3^5\; a_4^5\; \mathbf{a_5^5}\; \cdots \\
 & \vdots &
\end{array}
$$

Diagonal $\mathbf{a_0^0}\; \mathbf{a_1^1}\; \mathbf{a_2^2}\; \mathbf{a_3^3}\; \mathbf{a_4^4}\; \mathbf{a_5^5}\; \cdots$

Set: $z = 0\,.\,\bar{\mathbf{a}}_0^0\; \bar{\mathbf{a}}_1^1\; \bar{\mathbf{a}}_2^2\; \bar{\mathbf{a}}_3^3\; \bar{\mathbf{a}}_4^4\; \bar{\mathbf{a}}_5^5$

$$g(0) = 0 \,.\, \boldsymbol{a_0^0}\ a_1^0\ a_2^0\ a_3^0\ a_4^0\ a_5^0\ \cdots$$

$$g(1) = 0 \,.\, a_0^1\ \boldsymbol{a_1^1}\ a_2^1\ a_3^1\ a_4^1\ a_5^1\ \cdots$$

$$g(2) = 0 \,.\, a_0^2\ a_1^2\ \boldsymbol{a_2^2}\ a_3^2\ a_4^2\ a_5^2\ \cdots$$

$$g(3) = 0 \,.\, a_0^3\ a_1^3\ a_2^3\ \boldsymbol{a_3^3}\ a_4^3\ a_5^3\ \cdots$$

$$g(4) = 0 \,.\, a_0^4\ a_1^4\ a_2^4\ a_3^4\ \boldsymbol{a_4^4}\ a_5^4\ \cdots$$

$$g(5) = 0 \,.\, a_0^5\ a_1^5\ a_2^5\ a_3^5\ a_4^5\ \boldsymbol{a_5^5}\ \cdots$$

$$\vdots$$

Diagonal $\boldsymbol{a_0^0\ a_1^1\ a_2^2\ a_3^3\ a_4^4\ a_5^5\ \cdots}$

Set: $z = 0 \,.\, \boldsymbol{\bar{a}_0^0\ \bar{a}_1^1\ \bar{a}_2^2\ \bar{a}_3^3\ \bar{a}_4^4\ \bar{a}_5^5\ \cdots}$

$$
\begin{array}{lcllllllll}
g(0) & = & 0 & . & \mathbf{a_0^0} & a_1^0 & a_2^0 & a_3^0 & a_4^0 & a_5^0 & \cdots \\
g(1) & = & 0 & . & a_0^1 & \mathbf{a_1^1} & a_2^1 & a_3^1 & a_4^1 & a_5^1 & \cdots \\
g(2) & = & 0 & . & a_0^2 & a_1^2 & \mathbf{a_2^2} & a_3^2 & a_4^2 & a_5^2 & \cdots \\
g(3) & = & 0 & . & a_0^3 & a_1^3 & a_2^3 & \mathbf{a_3^3} & a_4^3 & a_5^3 & \cdots \\
g(4) & = & 0 & . & a_0^4 & a_1^4 & a_2^4 & a_3^4 & \mathbf{a_4^4} & a_5^4 & \cdots \\
g(5) & = & 0 & . & a_0^5 & a_1^5 & a_2^5 & a_3^5 & a_4^5 & \mathbf{a_5^5} & \cdots \\
 & \vdots & & & & & & & & & \\
\text{Diagonal} & & & & \mathbf{a_0^0} & \mathbf{a_1^1} & \mathbf{a_2^2} & \mathbf{a_3^3} & \mathbf{a_4^4} & \mathbf{a_5^5} & \cdots
\end{array}
$$

Set: $z = 0 \,.\, \bar{a}_0^0 \; \bar{a}_1^1 \; \bar{a}_2^2 \; \bar{a}_3^3 \; \bar{a}_4^4 \; \bar{a}_5^5 \cdots$

Note: z and the diagonal differ on each digit.

$$g(0) = 0\,.\,a_0^0\; a_1^0\; a_2^0\; a_3^0\; a_4^0\; a_5^0 \cdots$$

$$g(1) = 0\,.\,a_0^1\; a_1^1\; a_2^1\; a_3^1\; a_4^1\; a_5^1 \cdots$$

$$g(2) = 0\,.\,a_0^2\; a_1^2\; a_2^2\; a_3^2\; a_4^2\; a_5^2 \cdots$$

$$g(3) = 0\,.\,a_0^3\; a_1^3\; a_2^3\; a_3^3\; a_4^3\; a_5^3 \cdots$$

$$g(4) = 0\,.\,a_0^4\; a_1^4\; a_2^4\; a_3^4\; a_4^4\; a_5^4 \cdots$$

$$g(5) = 0\,.\,a_0^5\; a_1^5\; a_2^5\; a_3^5\; a_4^5\; a_5^5 \cdots$$

$$\vdots$$

Diagonal $\boldsymbol{a_0^0\; a_1^1\; a_2^2\; a_3^3\; a_4^4\; a_5^5 \cdots}$

Set: $z = 0\,.\,\boldsymbol{\bar{a}_0^0\; \bar{a}_1^1\; \bar{a}_2^2\; \bar{a}_3^3\; \bar{a}_4^4\; \bar{a}_5^5 \cdots}$

$$g(0) = 0\ .\ \boldsymbol{a_0^0}\ a_1^0\ a_2^0\ a_3^0\ a_4^0\ a_5^0\ \cdots$$

$$g(1) = 0\ .\ a_0^1\ \boldsymbol{a_1^1}\ a_2^1\ a_3^1\ a_4^1\ a_5^1\ \cdots$$

$$g(2) = 0\ .\ a_0^2\ a_1^2\ \boldsymbol{a_2^2}\ a_3^2\ a_4^2\ a_5^2\ \cdots$$

$$g(3) = 0\ .\ a_0^3\ a_1^3\ a_2^3\ \boldsymbol{a_3^3}\ a_4^3\ a_5^3\ \cdots$$

$$g(4) = 0\ .\ a_0^4\ a_1^4\ a_2^4\ a_3^4\ \boldsymbol{a_4^4}\ a_5^4\ \cdots$$

$$g(5) = 0\ .\ a_0^5\ a_1^5\ a_2^5\ a_3^5\ a_4^5\ \boldsymbol{a_5^5}\ \cdots$$

$$\vdots$$

Diagonal $\boldsymbol{a_0^0\ a_1^1\ a_2^2\ a_3^3\ a_4^4\ a_5^5\ \cdots}$

Set: $z = 0\ .\ \boldsymbol{\bar{a}_0^0\ \bar{a}_1^1\ \bar{a}_2^2\ \bar{a}_3^3\ \bar{a}_4^4\ \bar{a}_5^5\ \cdots}$

Hence z and $g(n)$ differ on digit number n.

z and $g(n)$ *differ* on digit number n.

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It follows that $z \neq g(n)$.

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Hence $g \colon \mathbb{N} \to [0, 1]$ is *not* onto.

This completes the proof of Cantor's theorem.

Theorem (Cantor, 1873). $\mathbb{R}$ and $\mathbb{N}$ are *not* of the same size.

Proof. Suppose for contradiction that $\mathbb{R} \approx \mathbb{N}$.

Then there is a function $f : \mathbb{N} \to \mathbb{R}$, so that f is one-to-one and *onto*.

Let $[0,1]$ denote the interval $\{x \mid 0 \leq x \leq 1\}$.

Define $g(x) = f(x) - \lfloor f(x) \rfloor$.

Note that $g : \mathbb{N} \to [0,1]$ is *onto*.

Consider the following table:

z and $g(n)$ *differ* on digit number n.

It follows that $z \neq g(n)$.

This is true for each $n \in \mathbb{N}$.

So z, which belongs to the interval $[0,1]$, is *not* in the range of g.

Hence $g \colon \mathbb{N} \to [0,1]$ is *not* onto.

This completes the proof of Cantor's theorem.

$$
\begin{array}{llllllllllll}
g(0) & = & 0 & . & \mathbf{a_0^0} & a_1^0 & a_2^0 & a_3^0 & a_4^0 & a_5^0 & \cdots \\
g(1) & = & 0 & . & a_0^1 & \mathbf{a_1^1} & a_2^1 & a_3^1 & a_4^1 & a_5^1 & \cdots \\
g(2) & = & 0 & . & a_0^2 & a_1^2 & \mathbf{a_2^2} & a_3^2 & a_4^2 & a_5^2 & \cdots \\
g(3) & = & 0 & . & a_0^3 & a_1^3 & a_2^3 & \mathbf{a_3^3} & a_4^3 & a_5^3 & \cdots \\
g(4) & = & 0 & . & a_0^4 & a_1^4 & a_2^4 & a_3^4 & \mathbf{a_4^4} & a_5^4 & \cdots \\
g(5) & = & 0 & . & a_0^5 & a_1^5 & a_2^5 & a_3^5 & a_4^5 & \mathbf{a_5^5} & \cdots \\
 & \vdots & & & & & & & & & \\
\text{Diagonal} & & & & \mathbf{a_0^0} & \mathbf{a_1^1} & \mathbf{a_2^2} & \mathbf{a_3^3} & \mathbf{a_4^4} & \mathbf{a_5^5} & \cdots \\
 & & & & & & & & & & \\
\text{Set:} \quad z & = & 0 & . & \bar{\mathbf{a}}_0^0 & \bar{\mathbf{a}}_1^1 & \bar{\mathbf{a}}_2^2 & \bar{\mathbf{a}}_3^3 & \bar{\mathbf{a}}_4^4 & \bar{\mathbf{a}}_5^5 & \cdots
\end{array}
$$

Cantor named the smallest infinite size "$\aleph_0$", the next infinite size "$\aleph_1$", etc.

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0

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0 1

Cantor named the smallest infinite size "$\aleph_0$", the next infinite size "$\aleph_1$", etc.

0 1 2

Cantor named the smallest infinite size "$\aleph_0$", the next infinite size "$\aleph_1$", etc.

$0 \quad 1 \quad 2 \quad \cdots$

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$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0$$

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$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2$$

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$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots$$

Cantor named the smallest infinite size "$\aleph_0$", the next infinite size "$\aleph_1$", etc.

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega$$

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$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1}$$

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$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

$\mathbb{N}$ has size $\aleph_0$.

Cantor named the smallest infinite size "$\aleph_0$", the next infinite size "$\aleph_1$", etc.

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

$\mathbb{N}$ has size $\aleph_0$.

By Cantor's theorem, $\mathbb{R}$ has size *at least* $\aleph_1$.

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It is impossible to prove $\mathbb{R} \approx \aleph_1$ (Cohen, 1963), and it is also impossible to prove $\mathbb{R} \not\approx \aleph_1$ (Gödel, 1938).

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Impossible here really means impossible (and provably so).

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

Let V denote the entire universe of sets.

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

Let V denote the entire universe of sets.

An **elementary embedding of the universe** is a function $\pi\colon V \to M$

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

Let V denote the entire universe of sets.

An **elementary embedding of the universe** is a function $\pi\colon V \to M$ (where $M \subseteq V$)

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

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Precisely this means that for all sets $x_1, \ldots, x_k$, any statement true of $x_1, \ldots, x_k$ in V is also true of $\pi(x_1), \ldots, \pi(x_k)$ in M.

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

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$\pi(0) = 0$.

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

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$\pi(0) = 0$.

Consider the statement "x is the smallest size".

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

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Consider the statement "x is the smallest size". It is true of 0.

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

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$\pi(0) = 0$.

Consider the statement "x is the smallest size". It is true of 0. By preservation of truth it is true also of $\pi(0)$.

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

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$\pi(0) = 0$.

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

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$\pi(0) = 0$. $\pi(1) = 1$.

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

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$\pi(0) = 0$. $\pi(1) = 1$.

1 is the next size above 0 in V.

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

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$\pi(0) = 0$. $\pi(1) = 1$.

1 is the next size above 0 in V.
Hence $\pi(1)$ is the next size above $\pi(0)$ in M.

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

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$\pi(0) = 0$. $\pi(1) = 1$.

1 is the next size above 0 in V.
Hence $\pi(1)$ is the next size above $\pi(0) = 0$.

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

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$\pi(0) = 0$. $\pi(1) = 1$.

1 is the next size above 0 in V.
Hence $\pi(1)$ is the next size above $\pi(0) = 0$.

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

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$\pi(0) = 0$. $\pi(1) = 1$.

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

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$\pi(0) = 0$. $\pi(1) = 1$. $\pi(2) = 2$

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

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$\pi(0) = 0$. $\pi(1) = 1$. $\pi(2) = 2 \ldots\ldots\ldots\ldots\ldots$

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

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$\pi(0) = 0$. $\pi(1) = 1$. $\pi(2) = 2 \ldots\ldots\ldots\ldots\ldots$
$\pi(\aleph_0) = \aleph_0$.

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

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$\pi(0) = 0$. $\pi(1) = 1$. $\pi(2) = 2$ . . . .
$\pi(\aleph_0) = \aleph_0$.

$\aleph_0$ is the first infinite size.

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

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$\pi(0) = 0$. $\pi(1) = 1$. $\pi(2) = 2$ . . . .
$\pi(\aleph_0) = \aleph_0$.

$\aleph_0$ is the first infinite size.
By preservation of truth so is $\pi(\aleph_0)$.

$$0 \quad 1 \quad 2 \quad \cdots \quad \aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad \aleph_\omega \quad \aleph_{\omega+1} \quad \cdots \quad \aleph_{\omega+\omega} \quad \cdots\cdots$$

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$\pi(0) = 0$. $\pi(1) = 1$. $\pi(2) = 2 \ldots\ldots\ldots\ldots\ldots$
$\pi(\aleph_0) = \aleph_0$.

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The first size which is actually moved by π cannot be described from below. It must be extremely large.

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The smallest size moved by an elementary embedding of the universe is referred to as a **large cardinal**.

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Statements asserting the existence of elementary embeddings of the universe are called **large cardinal axioms**.

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Statements asserting the existence of elementary embeddings of the universe are called **large cardinal axioms**.

They cannot be proved from the basic axioms of set theory.

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Players I and II alternate playing digits a_n,

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(A **strategy** is a complete recipe that instructs the player precisely how to play in each conceivable situation.)

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Determinacy is now accepted as a natural hypothesis in the study of definable sets of reals.

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Other operations which increase complexity include complementation (passing from A to $[0, 1] - A$), and projection.

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From k onwards I is guaranteed to win no matter how she plays.

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If there is no k so that $\langle a_0, \ldots, a_k \rangle \in Q$, then $0.a_0\, a_1\, a_2\, a_3 \cdots$ is won by player II.

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If $z = 0.a_0\, a_1\, a_2\, a_3 \cdots$ is won by player I, then there exists k so that $\langle a_0, \ldots, a_k \rangle \in Q$. This uses the assumption that A is finitely supported.

If there is no k so that $\langle a_0, \ldots, a_k \rangle \in Q$, then $0.a_0\, a_1\, a_2\, a_3 \cdots$ is won by player II.

If II can avoid positions in Q for the entire game, then she wins.

I	a_0		a_2		a_4		a_6		a_8		$\cdots$	
II		a_1		a_3		a_5		a_7		a_9		$\cdots$

Q = the set of positions from which player I has a winning strategy.

$$\begin{array}{c|cccccccccccc} I & a_0 & & a_2 & & a_4 & & a_6 & & a_8 & & \cdots & \\ \hline II & & a_1 & & a_3 & & a_5 & & a_7 & & a_9 & & \cdots \end{array}$$

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Claim. Let $p = \langle a_0, \ldots, a_{2k} \rangle$ be a position not in Q. Then there is a move a_{2k+1} for II so that $p^\frown a_{2k+1}$ is also not in Q.

I	a_0		a_2		a_4		a_6		a_8		$\cdots$	
II		a_1		a_3		a_5		a_7		a_9		$\cdots$

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Proof.

I	a_0		a_2		a_4		a_6		a_8		$\cdots$	
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Proof. Otherwise, for all a_{2k+1},

I	a_0		a_2		a_4		a_6		a_8		$\cdots$	
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Proof. Otherwise, for all a_{2k+1}, $p^\frown a_{2k+1} \in Q$

I	a_0		a_2		a_4		a_6		a_8		$\cdots$	
II		a_1		a_3		a_5		a_7		a_9		$\cdots$

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I	a_0		a_2		a_4		a_6		a_8		$\cdots$	
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Proof. Otherwise, for all a_{2k+1}, I has a winning strategy from $p^\frown a_{2k+1}$.

I	a_0		a_2		a_4		a_6		a_8		$\cdots$	
II		a_1		a_3		a_5		a_7		a_9		$\cdots$

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Proof. Otherwise, for all a_{2k+1}, I has a winning strategy from $p^\frown a_{2k+1}$. These strategies combine to a winning strategy for I from p.

I	a_0		a_2		a_4		a_6		a_8		$\cdots$	
II		a_1		a_3		a_5		a_7		a_9		$\cdots$

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I	a_0		a_2
II		a_1	

I	a_0		a_2		a_4		a_6		a_8		$\cdots$
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I	a_0		a_2	
II		a_1		a_3

I	a_0		a_2		a_4		a_6		a_8		$\cdots$	
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Proof. Otherwise, for all a_{2k+1}, I has a winning strategy from $p^\frown a_{2k+1}$. These strategies combine to a winning strategy for I from p.

I	a_0		a_2		I has a winning strategy from $a_0, \ldots, a_3$.
II		a_1		a_3	

I	a_0		a_2		a_4		a_6		a_8		$\cdots$	
II		a_1		a_3		a_5		a_7		a_9		$\cdots$

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Proof. Otherwise, for all a_{2k+1}, I has a winning strategy from $p^\frown a_{2k+1}$. These strategies combine to a winning strategy for I from p.

I	a_0		a_2		Follow this strategy.
II		a_1		a_3	

I	a_0		a_2		a_4		a_6		a_8		$\cdots$	
II		a_1		a_3		a_5		a_7		a_9		$\cdots$

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II		a_1		a_3

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I	a_0		a_2		a_4
II		a_1		a_3	

I	a_0		a_2		a_4		a_6		a_8		$\cdots$	
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I	a_0		a_2		a_4	
II		a_1		a_3		a_5

I	a_0		a_2		a_4		a_6		a_8		$\cdots$	
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II		a_1		a_3		a_5	

I	a_0		a_2		a_4		a_6		a_8		$\cdots$	
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I	a_0		a_2		a_4		a_6	
II		a_1		a_3		a_5		a_7

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I	a_0		a_2		a_4		a_6		a_8
II		a_1		a_3		a_5		a_7	

I	a_0		a_2		a_4		a_6		a_8		$\cdots$	
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I	a_0		a_2		a_4		a_6		a_8		$\cdots$
II		a_1		a_3		a_5		a_7		a_9	

I	a_0		a_2		a_4		a_6		a_8		$\cdots$	
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I	a_0		a_2		a_4		a_6		a_8		$\cdots$	
II		a_1		a_3		a_5		a_7		a_9		$\cdots$

I	a_0		a_2		a_4		a_6		a_8		$\cdots$	
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Proof. Otherwise, for all a_{2k+1}, I has a winning strategy from $p^\frown a_{2k+1}$. These strategies combine to a winning strategy for I from p. But then $p \in Q$,

I	a_0		a_2		a_4		a_6		a_8		$\cdots$	
II		a_1		a_3		a_5		a_7		a_9		$\cdots$

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Proof. Otherwise, for all a_{2k+1}, I has a winning strategy from $p^\frown a_{2k+1}$. These strategies combine to a winning strategy for I from p. But then $p \in Q$, contradiction.

I	a_0		a_2		a_4		a_6		a_8		$\cdots$	
II		a_1		a_3		a_5		a_7		a_9		$\cdots$

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I	a_0		a_2		a_4		a_6		a_8		$\cdots$	
II		a_1		a_3		a_5		a_7		a_9		$\cdots$

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Claim. Let $p = \langle a_0, \ldots, a_{2k-1} \rangle$ be a position not in Q. Then for every move a_{2k} for I, $p^\frown a_{2k}$ is also not in Q.

$Q =$ the set of positions from which player I does not have a winning strategy.

The empty position is not in Q.

If $p = \langle a_0, \ldots, a_{2k} \rangle$ is not in Q then **there is** a move a_{2k+1} for II so that $p^\frown a_{2k+1}$ is also not in Q.

If $p = \langle a_0, \ldots, a_{2k-1} \rangle$ is not in Q then for every move a_{2k} for I, $p^\frown a_{2k}$ is also not in Q.

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It follows that II has a strategy that stays outside Q for the entire game.

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I	
II	

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I	a_0
II	

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I	a_0	
II		$\boldsymbol{a}_1$

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It follows that II has a strategy that stays outside Q for the entire game.

I	a_0		a_2
II		$\boldsymbol{a_1}$	

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The empty position is not in Q.

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I	a_0		a_2	
II		$\boldsymbol{a}_1$		$\boldsymbol{a}_3$

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I	a_0		a_2		a_4		
II		$\boldsymbol{a}_1$		$\boldsymbol{a}_3$			

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I	a_0		a_2		a_4	
II		$\boldsymbol{a}_1$		$\boldsymbol{a}_3$		$\boldsymbol{a}_5$

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I	a_0		a_2		a_4		$\cdots$	
II		$\boldsymbol{a_1}$		$\boldsymbol{a_3}$		$\boldsymbol{a_5}$		$\boldsymbol{\cdots}$

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I	a_0		a_2		a_4		$\cdots$	
II		$\boldsymbol{a}_1$		$\boldsymbol{a}_3$		$\boldsymbol{a}_5$		$\cdots$

This strategy is winning for II in $G(A)$.

Theorem (Gale–Stewart, 1953). Let $A \subseteq [0,1]$ be finitely supported. Then $G(A)$ is determined.

Proof. Suppose that player I does not have a winning strategy in $G(A)$. We prove that player II does.

Let Q be the set of positions from which player I has a winning strategy. (By assumption, the empty position is not in Q.)

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If II can avoid positions in Q for the entire game, then she wins.

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Proof. Suppose that player I does not have a winning strategy in $G(A)$. We prove that player II does.

Let Q be the set of positions from which player I has a winning strategy. (By assumption, the empty position is not in Q.)

If $z = 0.a_0\, a_1\, a_2\, a_3 \cdots$ is won by player I, then there exists k so that $\langle a_0, \ldots, a_k \rangle \in Q$. This uses the assumption that A is finitely supported.

If there is no k so that $\langle a_0, \ldots, a_k \rangle \in Q$, then $0.a_0\, a_1\, a_2\, a_3 \cdots$ is won by player II.

If II can avoid positions in Q for the entire game, then she wins.

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The End

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