Determinacy Proofs for Long games

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3. Continuously coded games with Σ^1_2 payoff:
   (a) Preliminaries revisited.
   (b) Names.
   (c) Playing for I.

Recall that $G_{\text{cont}-f}(C)$ is played as follows:

I	....	$y_\alpha(0)$		$y_\alpha(2)$		
II			$y_\alpha(1)$		$y_\alpha(3)$	$\cdots$

I and II alternate playing natural numbers $y_\alpha(i)$, $i < \omega$, producing a real y_α.

If $f(y_\alpha)$ is not defined, the game ends. I wins iff $\langle y_0, y_1, \ldots\ldots, y_\alpha\rangle \in C$.

Otherwise we set $n_\alpha = f(y_\alpha)$. If there exists $\xi < \alpha$ so that $n_\alpha = n_\xi$, the game ends. Again I wins iff $\langle y_0, y_1, \ldots\ldots, y_\alpha\rangle \in C$.

Otherwise the game continues.

At any position $\langle y_\xi \mid \xi < \alpha\rangle$, the map $\xi \mapsto n_\xi$ embeds α into $\mathbb{N}$. This allows coding the position by a real, which we denote x_α or $\ulcorner y_\xi \mid \xi < \alpha \urcorner$.

The payoff set, C, is Σ^1_2 **in the codes** if there is a Σ^1_2 set $A \subset \mathbb{R} \times \mathbb{R}$ so that

$$\langle y_0, \ldots, y_\alpha \rangle \in C \iff \langle \ulcorner y_\xi \mid \xi < \alpha \urcorner, y_\alpha \rangle \in A.$$

Our goal is to prove that $G_{\text{cont}-f}(C)$ is determined whenever f is continuous and C is Σ^1_2 in the codes.

Any reasonable use of $\xi \mapsto n_\xi$ to code $\langle y_\xi \mid \xi < \alpha \rangle$ satisfies the following:

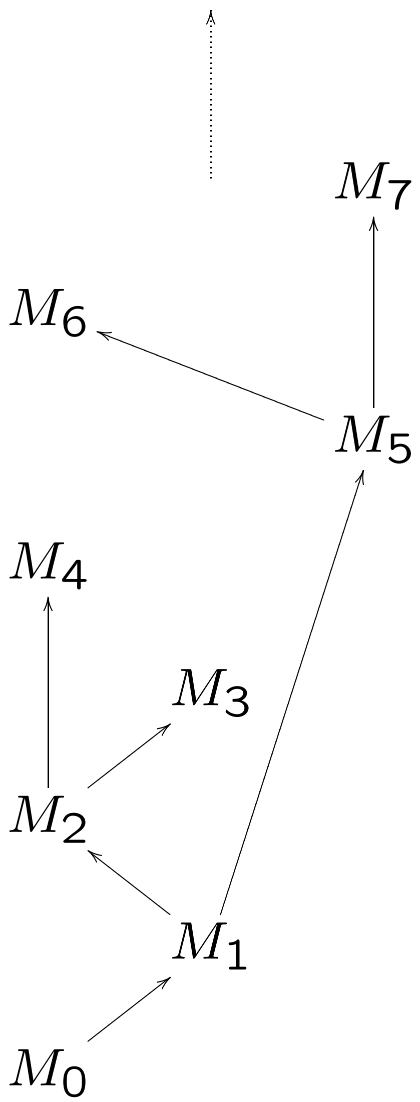
Note 1. The real codes $\ulcorner y_\xi \mid \xi < \alpha \urcorner$ and $\ulcorner y_\xi \mid \xi < \alpha + 1 \urcorner$ agree up to $n_\alpha = f(y_\alpha)$.

Note 2. For any limit λ, $n_\alpha \longrightarrow \infty$ as $\alpha \to \lambda$.

From this it follows that $x_\alpha = \ulcorner y_\xi \mid \xi < \alpha \urcorner$ converge to x_λ as $\alpha \to \lambda$.

We will use this later on. (We will have pivots $\mathcal{T}_\alpha, \vec{a}_\alpha$ for x_α. We will make sure they converge to a pivot for the limit of x_α, and use the fact that this is x_λ.)

We will have to work with trees which have more than one even branch.



We say that a branch is **odd** if from some point onwards it contains only odd models.

We say that a branch is **even** if it contains arbitrarily large even models.

Note: we allow "padding" in our trees, for example $M_6 = M_5$ and $j_{5,6} = id$.

In the past we used illfoundedness of the even model to force the iteration strategy to pick an odd branch.

An iteration tree $\mathcal{T}$ is **continuously illfounded on the even models** if it comes equipped with ordinals $\eta_i \in M_i$, $i < \omega$ even, so that

For $k\ T\ l$ both even, $j_{k,l}(\eta_k) > \eta_l$ **strictly**.

If $\mathcal{T}$ is cont. illfounded on the even models then all even branches of $\mathcal{T}$ produce illfounded direct limit. An iteration strategy must therefore pick an odd branch.

Note: Being cont. illfounded is a "closed" property: Suppose $\mathcal{T}_n$ are cont. illfounded on the even models, and this is witnessed by $\vec{\eta}^n = \{\eta_i^n\}$. Suppose $\mathcal{T}_n \longrightarrow \mathcal{T}_\infty$ and $\vec{\eta}^n \longrightarrow \vec{\eta}^\infty$. Then $\mathcal{T}_\infty$ is cont. illfounded on the even models, and this is witnessed by $\vec{\eta}^\infty$.

Suppose $M \models$ "δ is a Woodin cardinal", and in V there are M–generics for col(ω, δ). Let $\dot{A}$ name a subset of $\omega^\omega \times (M\|\,\delta)^\omega$ in $M^{\text{col}(\omega,\delta)}$.

Work with some $x \in \mathbb{R}$. We define an auxiliary game, $\mathcal{A}[x]$, similar to the game we had before. But now, instead of "$x \in \dot{A}[h]$", I tries to witness that $\langle x, \vec{a} \rangle \in \dot{A}[h]$ where $\vec{a} = \langle a_n \mid n < \omega \rangle$ are the moves played in $\mathcal{A}[x]$.

I	$\dots$	$l_n, \mathcal{X}_n, p_n$		$\dots$	
II			$\mathcal{F}_n, \mathcal{D}_n$		$\dots$

In round n I plays

- $l = l_n$, a number $< n$, or $l_n =$ "new".
- $\mathcal{X}_n$, a set of **pairs** of $M^{\text{col}(\omega,\delta)}$–names.
- p_n, a condition in col(ω, δ).

II plays

- $\mathcal{F}_n$ a function from $\mathcal{X}_n$ into the ordinals.
- $\mathcal{D}_n$, a function from $\mathcal{X}_n$ into {dense sets in col(ω, δ)}.

Let $a_{n-\mathrm{I}}$ and $a_{n-\mathrm{II}}$ denote the moves in round n, played by I and II resp. Let $a_n = \langle a_{n-\mathrm{I}}, a_{n-\mathrm{II}} \rangle$ and let $\vec{a} = \langle a_n \mid n < \omega \rangle$.

$$\mathcal{A}[x]: \quad \begin{array}{c|cccc} \mathrm{I} & \ldots & l_n, \mathcal{X}_n, p_n & & \ldots \\ \hline \mathrm{II} & & & \mathcal{F}_n, \mathcal{D}_n & \ldots \end{array}$$

As before I and II play $\mathcal{X}_n$, $\mathcal{F}_n$, $\mathcal{D}_n$ **indirectly** by playing types. These types are elements of $M \| \delta$. Thus a_n is an element of $M \| \delta$ and $\vec{a} \in (M \| \delta)^\omega$.

We require (when $l = l_n$ is not "new") that for every pair $\langle \dot{x}, \dot{a} \rangle \in \mathcal{X}_n$:

1. p_n forces "$\langle \dot{x}, \dot{a} \rangle \in \dot{A}$".
2. p_n forces "$\dot{x}(0) = \check{x_0}$", ..., "$\dot{x}(l) = \check{x_l}$".
3. p_n forces "$\dot{a}(0) = \check{a_0}$", ..., "$\dot{a}(l) = \check{a_l}$".
4. p_n belongs to $\mathcal{D}_l(\dot{x}, \dot{a})$.

We make the following requirement on II:

5. $\mathcal{F}_n(\dot{x}, \dot{a}) < \mathcal{F}_l(\dot{x}, \dot{a})$ for every pair $\langle \dot{x}, \dot{a} \rangle \in \mathcal{X}_n$.

Note the addition of condition 3, requiring that $\dot{a}$ must name the actual run of $\mathcal{A}[x]$, $\vec{a}$.

In this revised $\mathcal{A}[x]$, I tries to witness that there exists some h so that $\langle x, \vec{a} \rangle \in \dot{A}[h]$, where $\vec{a}$ is the sequence of auxiliary moves being played. II tries to witness the opposite.

As before, I can "go over all possible names" by playing in each round the first legal move.

We let $\sigma_{\text{gen}}[x, g]$ be the strategy for I which plays in each round the first legal move. (First with respect to the enumeration given by g.)

We have

Lemma 1. Suppose that $\vec{a}$ is an infinite run of $\mathcal{A}[x]$, played according to $\sigma_{\text{gen}}[x, g]$. Then $\langle x, \vec{a} \rangle \notin \dot{A}[g]$. (This is only useful if $x, \vec{a} \in M[g]$.)

As before, ascribing auxiliary moves for II requires passing to models along an iteration tree.

Definition. A **Pivot** for x is a pair $\mathcal{T}, \vec{a}$ so that

1. $\mathcal{T}$ is an iteration tree on M with an even branch.
2. $\vec{a}$ is an infinite play of $j_{\text{even}}(\mathcal{A})[x]$.
3. For every odd branch b of $\mathcal{T}$, there exists some h so that
   (a) h is $\text{col}(\omega, j_b(\delta))$–generic/$M_b$; and
   (b) $\langle x, \vec{a} \rangle \in j_b(\dot{A})[h]$.

(Note the change in 3(b) from "$x \in \cdots$" to "$\langle x, \vec{a} \rangle \in \cdots$".)

As before there is a strategy $\sigma_{\text{piv}}[x, g]$, playing for II in $\mathcal{A}^*$, so that all runs according to $\sigma_{\text{piv}}[x, g]$ are pivots.

But now this is not enough. We need a stronger method for ascribing moves for II. The method must be able to handle "changes of play" (also called "mixing") imposed by I.

Suppose we have an assignment $\gamma \mapsto \dot{A}[\gamma]$ in M. We define a game $\mathcal{A}^*_{\text{mix}}[x]$, played as follows:

At the start of round n we have an even number $k(n)$; an iteration tree $\mathcal{T} \restriction k(n)+1$ with final model $M_{k(n)}$; an ordinal γ_n in $M_{k(n)}$; and a position P_n of n rounds in $\mathcal{A}^{\mathsf{s}}[\gamma_n, x]$, the auxiliary game associated to $\dot{A}^{\mathsf{s}}[\gamma_n]$ and x, inside $M_{k(n)}$.

(We start with $k(0) = 0$ and a given γ_0.)

I plays $l_n, \mathcal{X}_n, p_n$ in $M_{k(n)}$, a legal move in $\mathcal{A}^{\mathsf{s}}[\gamma_n, x]$ following P_n.

II plays extenders $E_{k(n)}, E_{k(n)+1}$ extending the iteration tree to create the models $M_{k(n)+1}$, $M_{k(n)+2}$, and the embedding $j = j_{k(n),k(n)+2}$ from $M_{k(n)}$ into $M_{k(n)+2}$.

(The T–predecessor of $k(n) + 1$ is $k(l_n) + 1$ if $l_n \neq$ "new", and $k(n)$ otherwise.)

We set $Q_n = j(P_n—, l_n, \mathcal{X}_n, p_n)$.

II plays $\mathcal{F}_n, \mathcal{D}_n$ in $M_{k(n)+2}$, a legal move in $\mathcal{A}^{\mathsf{s}}[j(\gamma_n), x]$ following Q_n.

We set $P_{n+1} = Q_n —, \mathcal{F}_n, \mathcal{D}_n$.

So far we essentially followed the rules of $\mathcal{A}^*$.

I has two options now.

I can set $k(n+1) = k(n)+2$, and $\gamma_{n+1} = j(\gamma_n)$. We then pass to the next round.

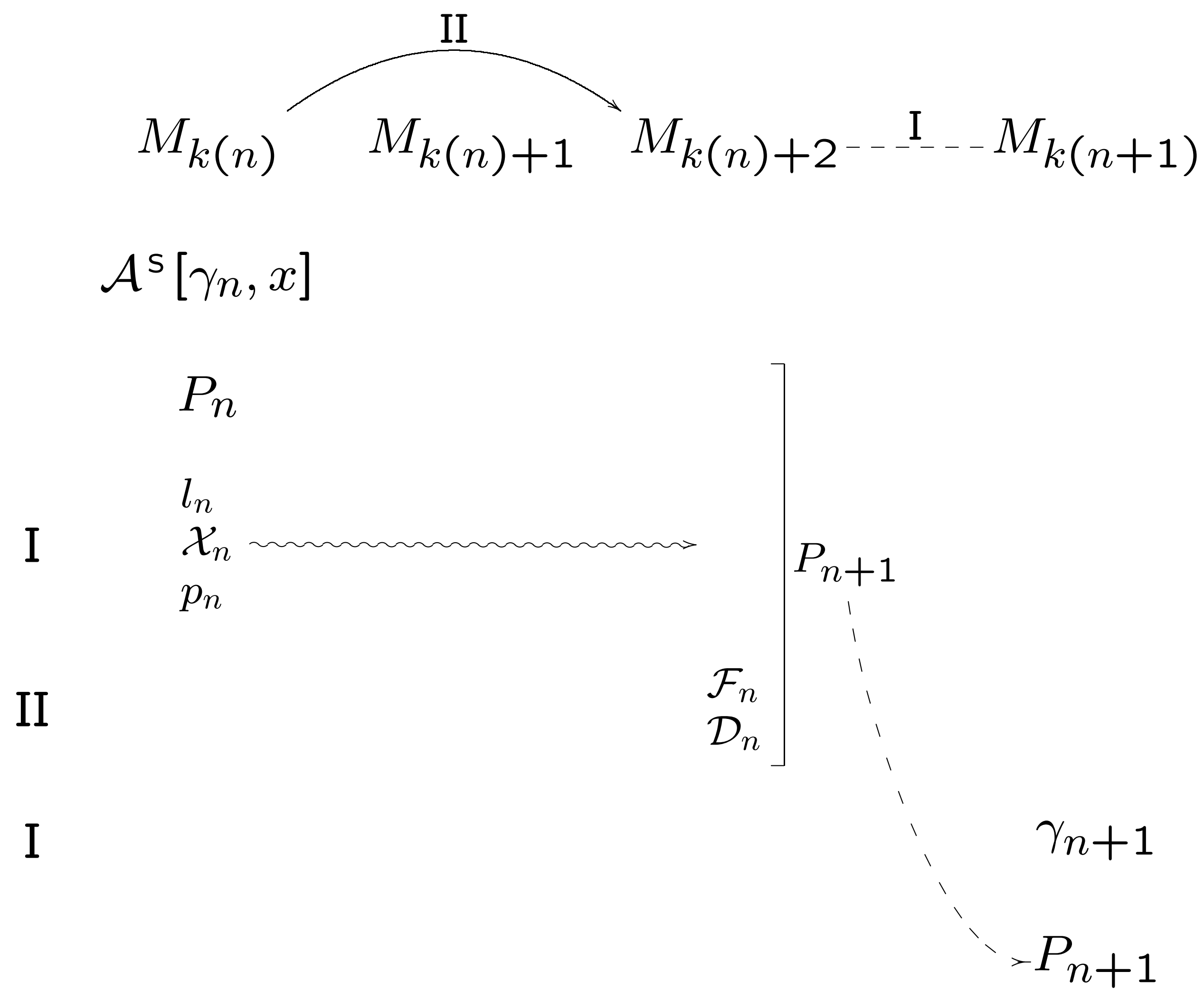
(This amounts to following the old $\mathcal{A}^*$.)

Alternatively, I can play $k(n+1) > k(n)+2$, extend the existing iteration tree to form $M_{k(n+1)}$, and play $\gamma_{n+1} \in M_{k(n+1)}$ subject to the following rule:

- P_{n+1} is a legal position in $\mathcal{A}^{\mathsf{s}}[\gamma_{n+1}, x]$. ($\mathcal{A}$ here is shifted to $M_{k(n+1)}$.)

This is the "change of game".

Restriction: When extending $\mathcal{T} \restriction k(n)+3$, I is not allowed to apply extenders to models in $\bigcup_{\bar{n}<n}[\, k(\bar{n})+2\, ,\ k(\bar{n}+1)\,)$.



Round n in $\mathcal{A}^*_{\mathsf{mix}}$.

(I may set $k(n+1) = k(n)+2$ and $\gamma_{n+1} = j_{k(n),k(n)+2}(\gamma_n)$. **But** I may also set $k(n+1) > k(n)+2$ and start a fresh $\mathcal{A}^{\mathsf{s}}[\gamma_{n+1}]$.)

Suppose $\mathcal{T}, \vec{a}, \{k(n), \gamma_n\}_{n<\omega}$ is a run of $\mathcal{A}^*_{\text{mix}}[x]$.

For an odd branch b of $\mathcal{T}$, note that the largest even model in b has the form $k(n)$ for some n. We use $n(b)$ to denote this n, and $k(b)$ to denote $k(n)$. We have $j_{k(b),b}\colon M_{k(b)} \to M_b$.

Definition. $\mathcal{T}, \vec{a}, \{k(n), \gamma_n\}$ is a **mixed pivot** for x if for every odd branch b of $\mathcal{T}$ there exists some h so that

- h is col$(\omega, j_b(\delta))$–generic/M_b; and
- $\langle x, \vec{a}\rangle \in \dot{A}^{\text{s}}[j_{k(b),b}(\gamma_{n(b)})][h]$.

Lemma 2. There exists $\sigma_{\text{mix}}[x, g]$, a strategy for II in $\mathcal{A}^*_{\text{mix}}$, so that every run according to $\sigma_{\text{mix}}[x, g]$ is a mixed pivot for x.
The association $x, g \mapsto \sigma_{\text{mix}}[x, g]$ is continuous.

As before, the proof of Lemma 2 draws heavily on techniques of Martin–Steel's "A proof of projective determinacy". The assumption that δ is a Woodin cardinal is crucial.

Fix a continuous function $f\colon \mathbb{R} \to \mathbb{N}$.

For $s \in \omega^{<\omega}$ put $\bar{f}(s) = n$ iff $f(x) = n$ for all x extending s. Wlog $\bar{f}$ is recursive.

Fix a Σ^1_2 set $A \subset \mathbb{R} \times \mathbb{R}$, say the set of all pairs satisfying the Σ^1_2 statement ϕ.

Let C be the set of sequences $\langle y_\xi \mid \xi \leq \alpha \rangle$ so that $(\ulcorner y_\xi \mid \xi < \alpha \urcorner, y_\alpha) \in A$.

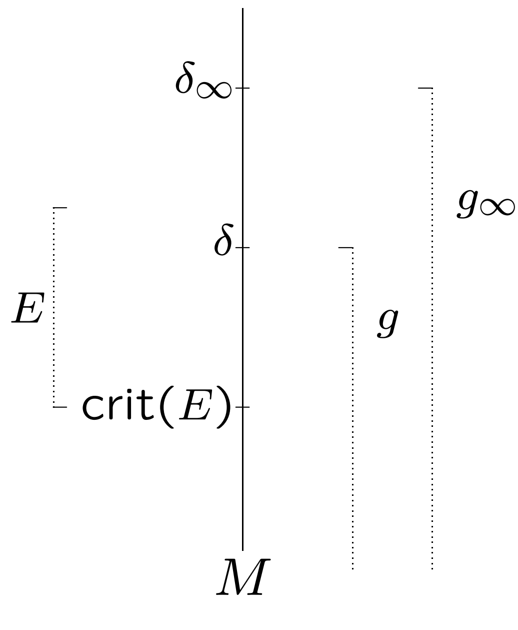
We wish to show that $G_{\text{cont}-f}(C)$ is determined.

Fix M, $\delta < \delta_\infty$, and E so that

1. M is an iterable class model.
2. δ and δ_∞ are Woodin cardinals of M.
3. In V there is g_∞, col(ω, δ_∞)–generic/M.
4. E is an extender of M, crit$(E) < \delta$, the ult embedding sends crit(E) above δ, and Ult(M, E) contains all subsets of δ in M.

The existence of such a model is our large cardinal assumption.

For expository simplicity fix g which is col(ω, δ)–generic/M, and g_∞ which is col(ω, δ_∞)–generic/M, with $g \in M[g_\infty]$.



Note: If $x \in \mathbb{R}$ belongs to $M[g]$, then by 4 x belongs also to Ult$(M, E)[g]$.

Let $\dot{A}_\infty \in M$ name the set of pairs of reals in $M[g_\infty]$ which satisfy ϕ in $M[g_\infty]$.

We have the associated auxiliary games, $\mathcal{A}_\infty[x, y]$, where I tries to witness $\langle x, y\rangle \in A$ and II tries to witness the opposite.

We work to define a class $A \subset \mathrm{ON} \times \mathbb{R} \times (M \| \delta)^\omega$ in $M[g]$. For $\gamma \in \mathrm{ON}$ we let $A[\gamma]$ denote the set

$$\{\langle x, \vec{a}\rangle \in \mathbb{R} \mid \langle \gamma, x, \vec{a}\rangle \in A\}.$$

This is a subset of $\omega^\omega \times (M \| \delta)^\omega$ in $M[g]$.

Really we are defining names, so we will have names $\dot{A}[\gamma]$ for $A[\gamma]$. The association $\gamma \mapsto \dot{A}[\gamma]$ will belong to M.

We let $\mathcal{A}[\gamma, x]$ be the corresponding auxiliary games: I tries to witness that $\langle \gamma, x, \vec{a}\rangle$ belongs to $\dot{A}[h]$ for some h, where $\vec{a}$ are the auxiliary moves, and II tries to witness the opposite.

To define A: work with $x = \ulcorner y_\xi \mid \xi < \alpha \urcorner$, γ, and $\vec{a}$, all in $M[g]$. Put

$\langle \gamma, x, \vec{a} \rangle \in A$ iff I has a winning strategy in $G(\gamma, x, \vec{a})$

where $G(\gamma, x, \vec{a})$ is played as follows:

I and II collaborate as usual playing $y_\alpha = \langle y_\alpha(i) \mid i < \omega \rangle \in \mathbb{R}$. In addition they play moves in the auxiliary game $\mathcal{A}_\infty[x, y_\alpha]$.

They continue until (if ever) $i < \omega$ is reached so that $\bar{f}(y_\alpha \restriction i)$ is defined.

Set $n_\alpha = \bar{f}(y_\alpha \restriction i)$. If there exists $\xi < \alpha$ so that $n_\alpha = f(y_\xi)$, the players simply continue playing y_α and the auxiliary moves of $\mathcal{A}_\infty[x, y_\alpha]$.

(Intuitively: as long as it seems that α is the last round, the players play the auxiliary moves of $\mathcal{A}_\infty$, I trying to witness $\langle x, y_\alpha \rangle \in A$ and II trying to witness the opposite.)

If $n_\alpha = \bar{f}(y_\alpha \restriction i)$ does not equal any previous n_ξ:

Let $N = \mathrm{Ult}(M, E)$, let $\pi\colon M \to N$ be the ultrapower embedding, let $\gamma' = \pi(\gamma')$, $\mathcal{A}' = \pi(\mathcal{A})$, and $\delta' = \pi(\delta)$. Let $a' = \pi(\vec{a} \restriction n_\alpha)$.

We set $x' = \ulcorner y_\xi \mid \xi < \alpha + 1 \urcorner$. (We obtain x' continuously as y_α is played. Note x' and x agree to n_α.)

I plays $\gamma^* < \gamma'$, so that

- a' is a legal position in $\mathcal{A}'[\gamma^*, x']$.

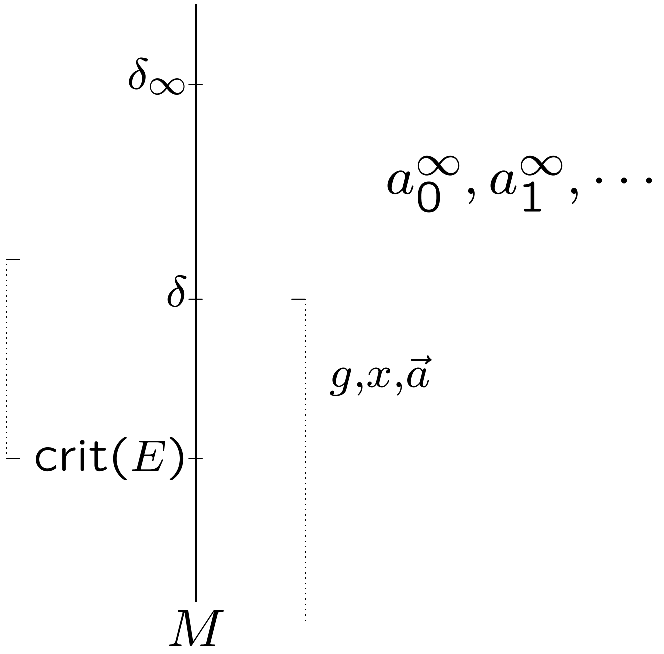
(Note: no knowledge of y_α is needed for the first n_α rounds of $\mathcal{A}'[\gamma^*, x']$.)

The players now continue playing y_α (extending the previously played $y_\alpha \restriction i$).

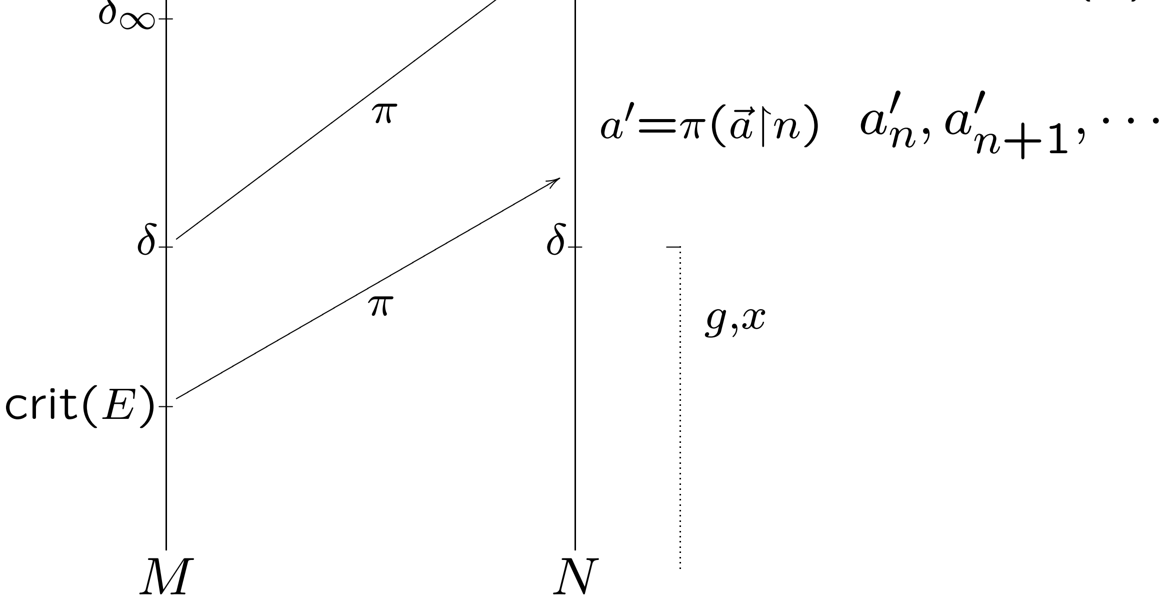
In addition they play auxiliary moves in the game $\mathcal{A}'[\gamma^*, x']$, continuing from a'.

(I tries to witness that $\langle \gamma^*, x', \vec{a}' \rangle \in \dot{A}'[h']$ for some h', generic for the collapse of δ'.)

I tries to witness
$\langle x, y_\alpha \rangle \in A$



I tries to witness
$\langle \gamma^*, x', \vec{a}' \rangle \in \dot{A}'[h']$,
where $\dot{A}' = \pi(\dot{A})$.



As always player II is the closed player. She wins if she can last ω moves. As usual the definition is by induction on γ.

The part of $G(\gamma, x, \vec{a})$ involving $\mathcal{A}_\infty[x, y_\alpha]$ we call the "first half". The part involving $\mathcal{A}'[\gamma^*, x']$ we call the "second half".

Note: the second half of G is a game which belongs to $N[g]$. (To decide the rules of the second half we need knowledge of $x = \ulcorner y_\xi \mid \xi < \alpha \urcorner$, so that we can figure x' as we are given y_α. x belongs to $N[g]$ because of our initial assumption on the strength of E.)

This note is important. $N[g]$ is a "small" generic extension of N; small with respect to the Woodin cardinal $\delta' = \pi(\delta)$. If II wins the second half, we can find a winning strategy in $N[g]$, and this strategy can shifted along the even models of an iteration given by $\pi(\sigma_{\mathsf{mix}})$.

Case 1: There exists some γ so that (in M) I wins $G(\gamma, x_0, \emptyset)$. (Where $x_0 = \ulcorner \emptyset \urcorner$.)

We will show that (in V) I wins $G_{\text{cont}-f}(C)$.

Fix an imaginary opponent playing for II in $G_{\text{cont}-f}(C)$.

Working against the imaginary opponent we construct:

- $y_\xi \in \mathbb{R}$. We set $x_\alpha = \ulcorner y_\xi \mid \xi < \alpha \urcorner$.
- Iterates M_α of M, with $j_{0,\alpha} \colon M \to M_\alpha$.
- Mixed pivots $\mathcal{T}_\alpha, \vec{a}_\alpha, \{k^\alpha(n), \gamma_n^\alpha\}$ for x_α over the model M_α, played according to $j_{0,\alpha}(\sigma_{\text{mix}})$.

$\mathcal{T}_\alpha$ will be continuously illfounded on the even models. (This will follow from our requirement in the second half of $G(\gamma, x, \vec{a})$, that $\gamma^* < \gamma'$.)

The construction is fairly similar to the kinds of constructions handled before. The key point is the following:

Key point: The pivot at $\alpha+1$ agrees with the $j_{\alpha,\alpha+1}$ image of the pivot at α, up to n_α.

(Similarly for the witness of continuous illfoundedness of the even branches.)

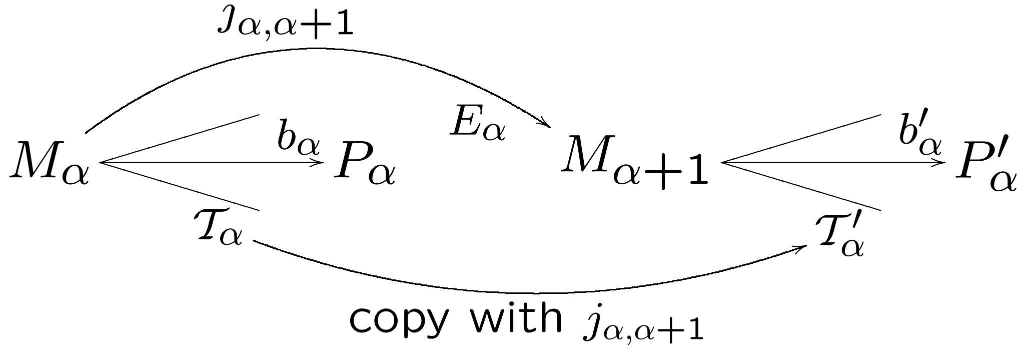
When reaching a **limit** ordinal λ, we can therefore let the pivot at λ be the limit of (the appropriate images of) the pivots at α, as $\alpha \to \lambda$.

This makes sense because of our key point, because $n_\alpha \longrightarrow \infty$ as $\alpha \to \lambda$, and because $x_\alpha \longrightarrow x_\lambda$ as $\alpha \to \lambda$.

Once the pivot at λ is defined:

The iteration strategy picks an odd branch of $\mathcal{T}_\lambda$. The play so far is generic over the direct limit and belongs to a (shift of) $\dot{A}[\gamma][h]$ for some γ and h. This allows us to proceed as usual.

Why mixed pivots?

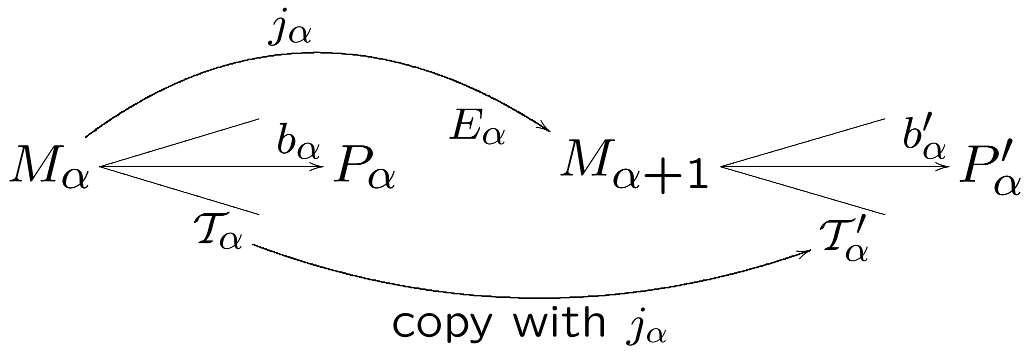


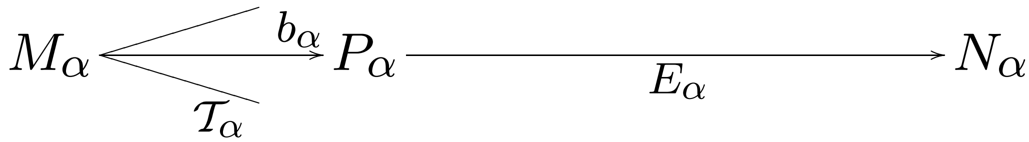
$\mathcal{T}_{\alpha+1}$ starts out like $\mathcal{T}'_\alpha$.

The pivot at $\alpha+1$ will start out like the $j_{\alpha,\alpha+1}$ image of the pivot at α, and will continue this way long enough to construct the $j_{\alpha,\alpha+1}$ image of $\vec{a}_\alpha \restriction n_\alpha$.

Only then will the pivot at $\alpha + 1$ change — passing from the γ of the pivot at α, to some smaller γ^*.

Thus, the change from γ to γ^* occurs in the "middle" of the pivot.





Further, the change from γ to γ^* occurs on an odd model of $\mathcal{T}'_\alpha$. In fact a model along b'_α.

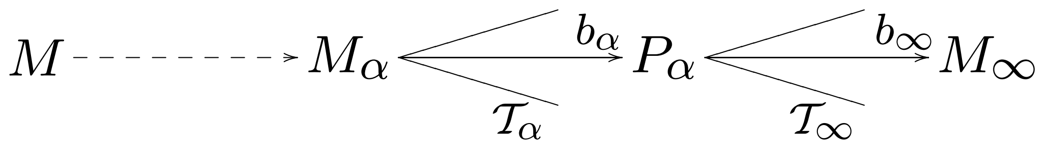
(This has to do with the fact that P'_α, the direct limit of models along b'_α, exactly equals $\text{Ult}(P_\alpha, E_\alpha)$.)

So the change from the pivot at α to the pivot at $\alpha + 1$ involves skipping from the even model of $\mathcal{T}'_\alpha$ where the image of $\vec{a}_\alpha \restriction n_\alpha$ is first constructed, to some later odd model on b'_α. (Then pad to make this model "even".)

We continue the construction until we reach an α so that, when playing the (appropriate shift of) $G(*, x_\alpha, \vec{a}_\alpha)$, we stay within the "first half".

When playing the first half of $G(*, x_\alpha, \vec{a}_\alpha)$, we use (the appropriate shift of) $\sigma_{\mathsf{piv}-\infty}$ to ascribe auxiliary moves for II.

We obtain y_α and $\mathcal{T}_\infty$ which is part of a pivot for $\langle x, y_\alpha \rangle$ and the name $\dot{A}_\infty$ (shifted).



The iteration strategy produces an odd branch b_∞, and we conclude that $\langle x, y_\alpha \rangle \in A$.

(Remember, in the first half of $G(*, x, \vec{a})$ I tries to witness that $\langle x, y \rangle$ belongs to the Σ^1_2 set A.)

Thus I wins $G_{\mathsf{cont}-f}$ and we are done.