

Determinacy Proofs for Long games

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3. Continuously coded games with Σ_2^1 payoff:
- (a) Preliminaries revisited.
 - (b) Names.
 - (c) Playing for I.

Recall that $G_{\text{cont-}f}(C)$ is played as follows:

I	$y_\alpha(0)$	$y_\alpha(2)$	
II		$y_\alpha(1)$	$y_\alpha(3)$...

I and II alternate playing natural numbers $y_\alpha(i)$, $i < \omega$, producing a real y_α .

If $f(y_\alpha)$ is not defined, the game ends. I wins iff $\langle y_0, y_1, \dots, y_\alpha \rangle \in C$.

Otherwise we set $n_\alpha = f(y_\alpha)$. If there exists $\xi < \alpha$ so that $n_\alpha = n_\xi$, the game ends. Again I wins iff $\langle y_0, y_1, \dots, y_\alpha \rangle \in C$.

Otherwise the game continues.

At any position $\langle y_\xi \mid \xi < \alpha \rangle$, the map $\xi \mapsto n_\xi$ embeds α into \mathbb{N} . This allows coding the position by a real, which we denote x_α or $\ulcorner y_\xi \mid \xi < \alpha \urcorner$.

The payoff set, C , is Σ_2^1 **in the codes** if there is a Σ_2^1 set $A \subset \mathbb{R} \times \mathbb{R}$ so that

$$\langle y_0, \dots, y_\alpha \rangle \in C \iff \langle \ulcorner y_\xi \mid \xi < \alpha \urcorner, y_\alpha \rangle \in A.$$

Our goal is to prove that $G_{\text{cont-}f}(C)$ is determined whenever f is continuous and C is Σ_2^1 in the codes.

Any reasonable use of $\xi \mapsto n_\xi$ to code $\langle y_\xi \mid \xi < \alpha \rangle$ satisfies the following:

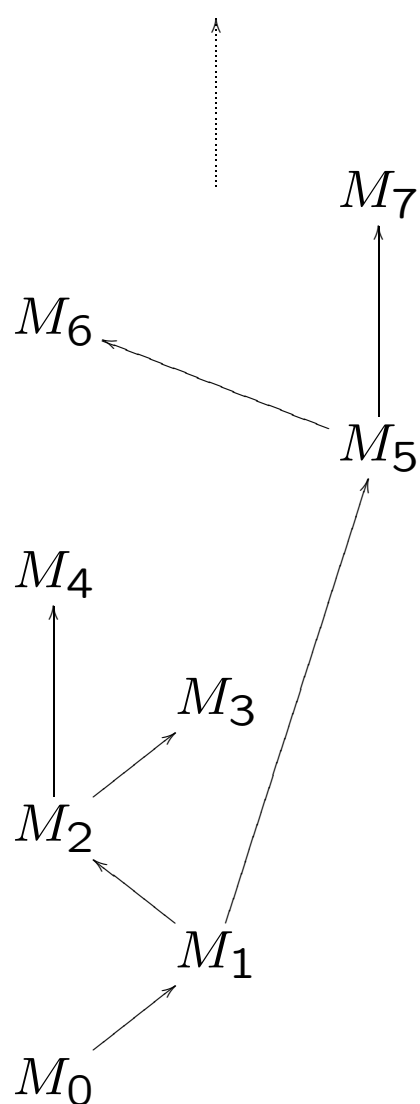
Note 1. The real codes $\ulcorner y_\xi \mid \xi < \alpha \urcorner$ and $\ulcorner y_\xi \mid \xi < \alpha + 1 \urcorner$ agree up to $n_\alpha = f(y_\alpha)$.

Note 2. For any limit λ , $n_\alpha \longrightarrow \infty$ as $\alpha \rightarrow \lambda$.

From this it follows that $x_\alpha = \ulcorner y_\xi \mid \xi < \alpha \urcorner$ converge to x_λ as $\alpha \rightarrow \lambda$.

We will use this later on. (We will have pivots $\mathcal{T}_\alpha, \vec{a}_\alpha$ for x_α . We will make sure they converge to a pivot for the limit of x_α , and use the fact that this is x_λ .)

We will have to work with trees which have more than one even branch.



We say that a branch is **odd** if from some point onwards it contains only odd models.

We say that a branch is **even** if it contains arbitrarily large even models.

Note: we allow “padding” in our trees, for example $M_6 = M_5$ and $j_{5,6} = id$.

In the past we used illfoundedness of the even model to force the iteration strategy to pick an odd branch.

An iteration tree \mathcal{T} is **continuously illfounded on the even models** if it comes equipped with ordinals $\eta_i \in M_i$, $i < \omega$ even, so that

For $k \mathcal{T} l$ both even, $j_{k,l}(\eta_k) > \eta_l$ **strictly**.

If \mathcal{T} is cont. illfounded on the even models then all even branches of \mathcal{T} produce illfounded direct limit. An iteration strategy must therefore pick an odd branch.

Note: Being cont. illfounded is a “closed” property: Suppose \mathcal{T}_n are cont. illfounded on the even models, and this is witnessed by $\vec{\eta}^n = \{\eta_i^n\}$. Suppose $\mathcal{T}_n \longrightarrow \mathcal{T}_\infty$ and $\vec{\eta}^n \longrightarrow \vec{\eta}^\infty$. Then \mathcal{T}_∞ is cont. illfounded on the even models, and this is witnessed by $\vec{\eta}^\infty$.

Suppose $M \models \text{“}\delta \text{ is a Woodin cardinal”}$, and in V there are M -generics for $\text{col}(\omega, \delta)$. Let \dot{A} name a subset of $\omega^\omega \times (M \parallel \delta)^\omega$ in $M^{\text{col}(\omega, \delta)}$.

Work with some $x \in \mathbb{R}$. We define an auxiliary game, $\mathcal{A}[x]$, similar to the game we had before. But now, instead of “ $x \in \dot{A}[h]$ ”, I tries to witness that $\langle x, \vec{a} \rangle \in \dot{A}[h]$ where $\vec{a} = \langle a_n \mid n < \omega \rangle$ are the moves played in $\mathcal{A}[x]$.

I		...	l_n, \mathcal{X}_n, p_n	...
II			$\mathcal{F}_n, \mathcal{D}_n$...

In round n I plays

- $l = l_n$, a number $< n$, or $l_n = \text{“new”}$.
- \mathcal{X}_n , a set of **pairs** of $M^{\text{col}(\omega, \delta)}$ -names.
- p_n , a condition in $\text{col}(\omega, \delta)$.

II plays

- \mathcal{F}_n a function from \mathcal{X}_n into the ordinals.
- \mathcal{D}_n , a function from \mathcal{X}_n into $\{\text{dense sets in } \text{col}(\omega, \delta)\}$.

Let a_{n-I} and a_{n-II} denote the moves in round n , played by I and II resp. Let $a_n = \langle a_{n-I}, a_{n-II} \rangle$ and let $\vec{a} = \langle a_n \mid n < \omega \rangle$.

$$\mathcal{A}[x] : \quad \begin{array}{c|ccc} \text{I} & \dots & l_n, \mathcal{X}_n, p_n & \dots \\ \hline \text{II} & & \mathcal{F}_n, \mathcal{D}_n & \dots \end{array}$$

As before I and II play $\mathcal{X}_n, \mathcal{F}_n, \mathcal{D}_n$ **indirectly** by playing types. These types are elements of $M \parallel \delta$. Thus a_n is an element of $M \parallel \delta$ and $\vec{a} \in (M \parallel \delta)^\omega$.

We require (when $l = l_n$ is not “new”) that for every pair $\langle \dot{x}, \dot{a} \rangle \in \mathcal{X}_n$:

1. p_n forces “ $\langle \dot{x}, \dot{a} \rangle \in \dot{A}$ ”.
2. p_n forces “ $\dot{x}(0) = \check{x}_0$ ”, ..., “ $\dot{x}(l) = \check{x}_l$ ”.
3. p_n forces “ $\dot{a}(0) = \check{a}_0$ ”, ..., “ $\dot{a}(l) = \check{a}_l$ ”.
4. p_n belongs to $\mathcal{D}_l(\dot{x}, \dot{a})$.

We make the following requirement on II:

5. $\mathcal{F}_n(\dot{x}, \dot{a}) < \mathcal{F}_l(\dot{x}, \dot{a})$ for every pair $\langle \dot{x}, \dot{a} \rangle \in \mathcal{X}_n$.

Note the addition of condition 3, requiring that \dot{a} must name the actual run of $\mathcal{A}[x]$, \vec{a} .

In this revised $\mathcal{A}[x]$, I tries to witness that there exists some h so that $\langle x, \vec{a} \rangle \in \dot{A}[h]$, where \vec{a} is the sequence of auxiliary moves being played. II tries to witness the opposite.

As before, I can “go over all possible names” by playing in each round the first legal move.

We let $\sigma_{\text{gen}}[x, g]$ be the strategy for I which plays in each round the first legal move. (First with respect to the enumeration given by g .)

We have

Lemma 1. Suppose that \vec{a} is an infinite run of $\mathcal{A}[x]$, played according to $\sigma_{\text{gen}}[x, g]$. Then $\langle x, \vec{a} \rangle \notin \dot{A}[g]$. (This is only useful if $x, \vec{a} \in M[g]$.)

As before, ascribing auxiliary moves for II requires passing to models along an iteration tree.

Definition. A **Pivot** for x is a pair \mathcal{T}, \vec{a} so that

1. \mathcal{T} is an iteration tree on M with an even branch.
2. \vec{a} is an infinite play of $j_{\text{even}}(\mathcal{A})[x]$.
3. For every odd branch b of \mathcal{T} , there exists some h so that
 - (a) h is $\text{col}(\omega, j_b(\delta))$ –generic/ M_b ; and
 - (b) $\langle x, \vec{a} \rangle \in j_b(\dot{A})[h]$.

(Note the change in 3(b) from “ $x \in \dots$ ” to “ $\langle x, \vec{a} \rangle \in \dots$ ”.)

As before there is a strategy $\sigma_{\text{piv}}[x, g]$, playing for II in \mathcal{A}^* , so that all runs according to $\sigma_{\text{piv}}[x, g]$ are pivots.

But now this is not enough. We need a stronger method for ascribing moves for II. The method must be able to handle “changes of play” (also called “mixing”) imposed by I.

Suppose we have an assignment $\gamma \mapsto \dot{A}[\gamma]$ in M . We define a game $\mathcal{A}_{\text{mix}}^*[x]$, played as follows:

At the start of round n we have an even number $k(n)$; an iteration tree $\mathcal{T} \upharpoonright k(n)+1$ with final model $M_{k(n)}$; an ordinal γ_n in $M_{k(n)}$; and a position P_n of n rounds in $\mathcal{A}^s[\gamma_n, x]$, the auxiliary game associated to $\dot{A}^s[\gamma_n]$ and x , inside $M_{k(n)}$.

(We start with $k(0) = 0$ and a given γ_0 .)

I plays l_n, \mathcal{X}_n, p_n in $M_{k(n)}$, a legal move in $\mathcal{A}^s[\gamma_n, x]$ following P_n .

II plays extenders $E_{k(n)}, E_{k(n)+1}$ extending the iteration tree to create the models $M_{k(n)+1}, M_{k(n)+2}$, and the embedding $j = j_{k(n), k(n)+2}$ from $M_{k(n)}$ into $M_{k(n)+2}$.

(The T -predecessor of $k(n) + 1$ is $k(l_n) + 1$ if $l_n \neq \text{"new"}$, and $k(n)$ otherwise.)

We set $Q_n = j(P_n \text{---}, l_n, \mathcal{X}_n, p_n)$.

II plays $\mathcal{F}_n, \mathcal{D}_n$ in $M_{k(n)+2}$, a legal move in $\mathcal{A}^s[j(\gamma_n), x]$ following Q_n .

We set $P_{n+1} = Q_{n-}, \mathcal{F}_n, \mathcal{D}_n$.

So far we essentially followed the rules of \mathcal{A}^* .

I has two options now.

I can set $k(n+1) = k(n) + 2$, and $\gamma_{n+1} = j(\gamma_n)$.
We then pass to the next round.

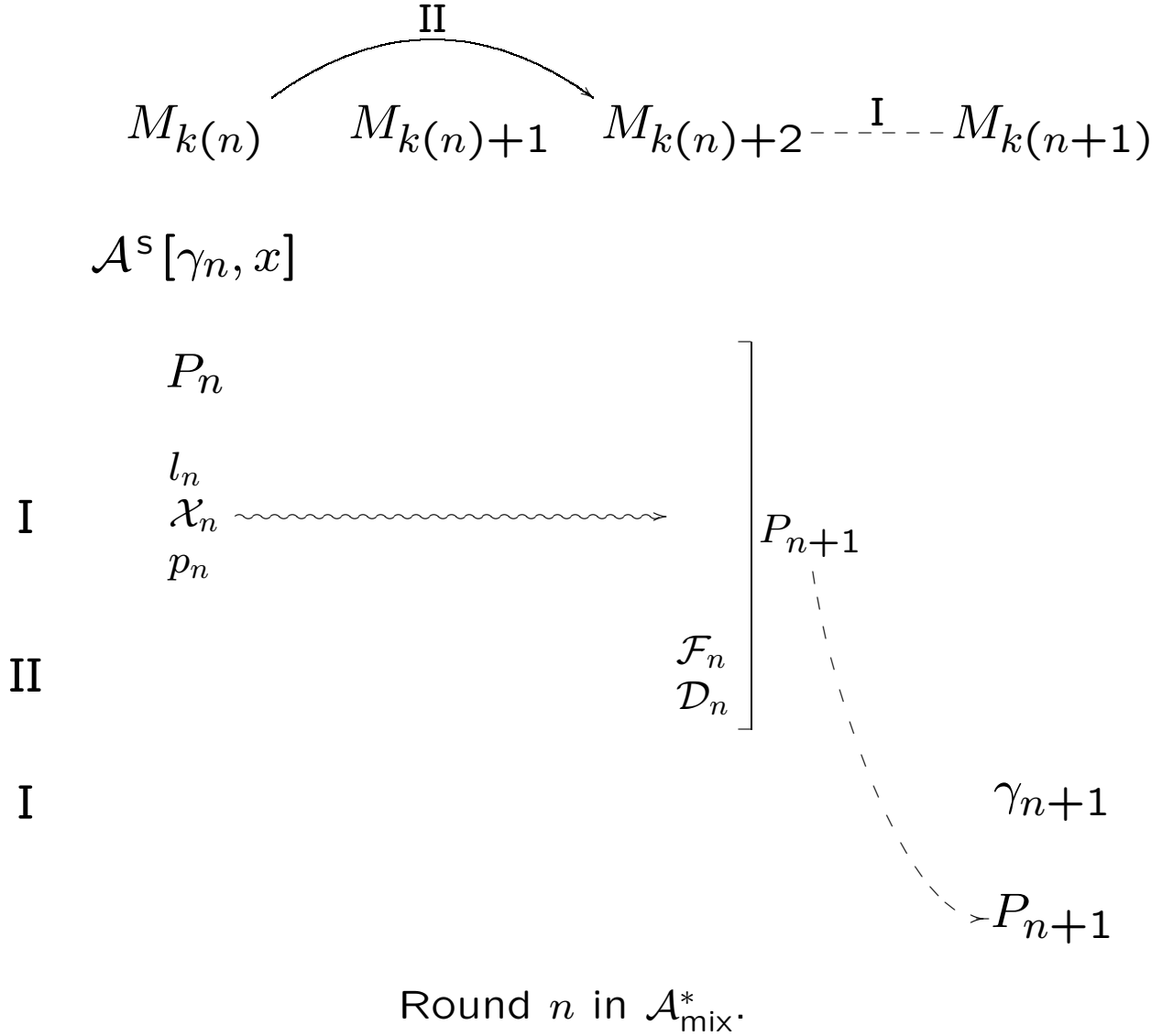
(This amounts to following the old \mathcal{A}^* .)

Alternatively, I can play $k(n+1) > k(n) + 2$,
extend the existing iteration tree to form
 $M_{k(n+1)}$, and play $\gamma_{n+1} \in M_{k(n+1)}$ subject to
the following rule:

- P_{n+1} is a legal position in $\mathcal{A}^s[\gamma_{n+1}, x]$. (\mathcal{A}
here is shifted to $M_{k(n+1)}$.)

This is the “change of game”.

Restriction: When extending $\mathcal{T} \upharpoonright k(n) + 3$, I is
not allowed to apply extenders to models in
 $\bigcup_{\bar{n} < n} [k(\bar{n}) + 2, k(\bar{n} + 1))$.



(I may set $k(n+1) = k(n) + 2$ and $\gamma_{n+1} = j_{k(n), k(n)+2}(\gamma_n)$. **But** I may also set $k(n+1) > k(n) + 2$ and start a fresh $\mathcal{A}^s[\gamma_{n+1}]$.)

Suppose $\mathcal{T}, \vec{a}, \{k(n), \gamma_n\}_{n < \omega}$ is a run of $\mathcal{A}_{\text{mix}}^*[x]$.

For an odd branch b of \mathcal{T} , note that the largest even model in b has the form $k(n)$ for some n . We use $n(b)$ to denote this n , and $k(b)$ to denote $k(n)$. We have $j_{k(b),b}: M_{k(b)} \rightarrow M_b$.

Definition. $\mathcal{T}, \vec{a}, \{k(n), \gamma_n\}$ is a **mixed pivot** for x if for every odd branch b of \mathcal{T} there exists some h so that

- h is $\text{col}(\omega, j_b(\delta))$ –generic/ M_b ; and
- $\langle x, \vec{a} \rangle \in \dot{A}^s[j_{k(b),b}(\gamma_{n(b)})][h]$.

Lemma 2. There exists $\sigma_{\text{mix}}[x, g]$, a strategy for II in $\mathcal{A}_{\text{mix}}^*$, so that every run according to $\sigma_{\text{mix}}[x, g]$ is a mixed pivot for x .

The association $x, g \mapsto \sigma_{\text{mix}}[x, g]$ is continuous.

As before, the proof of Lemma 2 draws heavily on techniques of Martin–Steel’s “A proof of projective determinacy”. The assumption that δ is a Woodin cardinal is crucial.

Fix a continuous function $f: \mathbb{R} \rightarrow \mathbb{N}$.

For $s \in \omega^{<\omega}$ put $\bar{f}(s) = n$ iff $f(x) = n$ for all x extending s . Wlog \bar{f} is recursive.

Fix a Σ_2^1 set $A \subset \mathbb{R} \times \mathbb{R}$, say the set of all pairs satisfying the Σ_2^1 statement ϕ .

Let C be the set of sequences $\langle y_\xi \mid \xi \leq \alpha \rangle$ so that $(\ulcorner y_\xi \mid \xi < \alpha \urcorner, y_\alpha) \in A$.

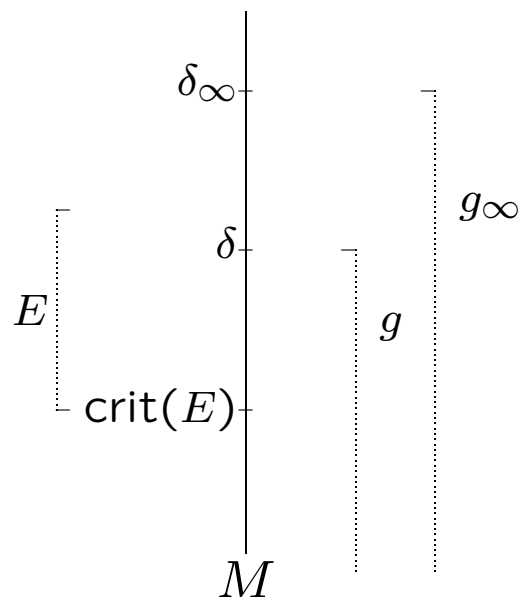
We wish to show that $G_{\text{cont}-f}(C)$ is determined.

Fix M , $\delta < \delta_\infty$, and E so that

1. M is an iterable class model.
2. δ and δ_∞ are Woodin cardinals of M .
3. In V there is g_∞ , $\text{col}(\omega, \delta_\infty)$ –generic/ M .
4. E is an extender of M , $\text{crit}(E) < \delta$, the ult embedding sends $\text{crit}(E)$ above δ , and $\text{Ult}(M, E)$ contains all subsets of δ in M .

The existence of such a model is our large cardinal assumption.

For expository simplicity fix g which is $\text{col}(\omega, \delta)$ –generic/ M , and g_∞ which is $\text{col}(\omega, \delta_\infty)$ –generic/ M , with $g \in M[g_\infty]$.



Note: If $x \in \mathbb{R}$ belongs to $M[g]$, then by 4 x belongs also to $\text{Ult}(M, E)[g]$.

Let $\dot{A}_\infty \in M$ name the set of pairs of reals in $M[g_\infty]$ which satisfy ϕ in $M[g_\infty]$.

We have the associated auxiliary games, $\mathcal{A}_\infty[x, y]$, where I tries to witness $\langle x, y \rangle \in A$ and II tries to witness the opposite.

We work to define a class $A \subset \text{ON} \times \mathbb{R} \times (M \parallel \delta)^\omega$ in $M[g]$. For $\gamma \in \text{ON}$ we let $A[\gamma]$ denote the set

$$\{\langle x, \vec{a} \rangle \in \mathbb{R} \mid \langle \gamma, x, \vec{a} \rangle \in A\}.$$

This is a subset of $\omega^\omega \times (M \parallel \delta)^\omega$ in $M[g]$.

Really we are defining names, so we will have names $\dot{A}[\gamma]$ for $A[\gamma]$. The association $\gamma \mapsto \dot{A}[\gamma]$ will belong to M .

We let $\mathcal{A}[\gamma, x]$ be the corresponding auxiliary games: I tries to witness that $\langle \gamma, x, \vec{a} \rangle$ belongs to $\dot{A}[h]$ for some h , where \vec{a} are the auxiliary moves, and II tries to witness the opposite.

To define A : work with $x = \ulcorner y_\xi \mid \xi < \alpha \urcorner$, γ , and \vec{a} , all in $M[g]$. Put

$\langle \gamma, x, \vec{a} \rangle \in A$ iff I has a winning strategy in $G(\gamma, x, \vec{a})$

where $G(\gamma, x, \vec{a})$ is played as follows:

I and II collaborate as usual playing $y_\alpha = \langle y_\alpha(i) \mid i < \omega \rangle \in \mathbb{R}$. In addition they play moves in the auxiliary game $\mathcal{A}_\infty[x, y_\alpha]$.

They continue until (if ever) $i < \omega$ is reached so that $\bar{f}(y_\alpha \upharpoonright i)$ is defined.

Set $n_\alpha = \bar{f}(y_\alpha \upharpoonright i)$. If there exists $\xi < \alpha$ so that $n_\alpha = f(y_\xi)$, the players simply continue playing y_α and the auxiliary moves of $\mathcal{A}_\infty[x, y_\alpha]$.

(Intuitively: as long as it seems that α is the last round, the players play the auxiliary moves of \mathcal{A}_∞ , I trying to witness $\langle x, y_\alpha \rangle \in A$ and II trying to witness the opposite.)

If $n_\alpha = \bar{f}(y_\alpha \upharpoonright i)$ does not equal any previous n_ξ :

Let $N = \text{Ult}(M, E)$, let $\pi: M \rightarrow N$ be the ultrapower embedding, let $\gamma' = \pi(\gamma')$, $\mathcal{A}' = \pi(\mathcal{A})$, and $\delta' = \pi(\delta)$. Let $a' = \pi(\vec{a} \upharpoonright n_\alpha)$.

We set $x' = \ulcorner y_\xi \mid \xi < \alpha + 1 \urcorner$. (We obtain x' continuously as y_α is played. Note x' and x agree to n_α .)

I plays $\gamma^* < \gamma'$, so that

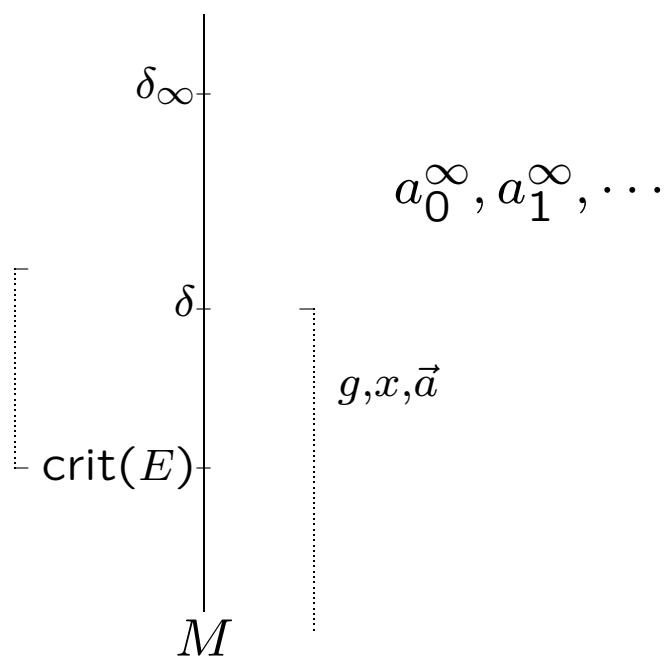
- a' is a legal position in $\mathcal{A}'[\gamma^*, x']$.

(Note: no knowledge of y_α is needed for the first n_α rounds of $\mathcal{A}'[\gamma^*, x']$.)

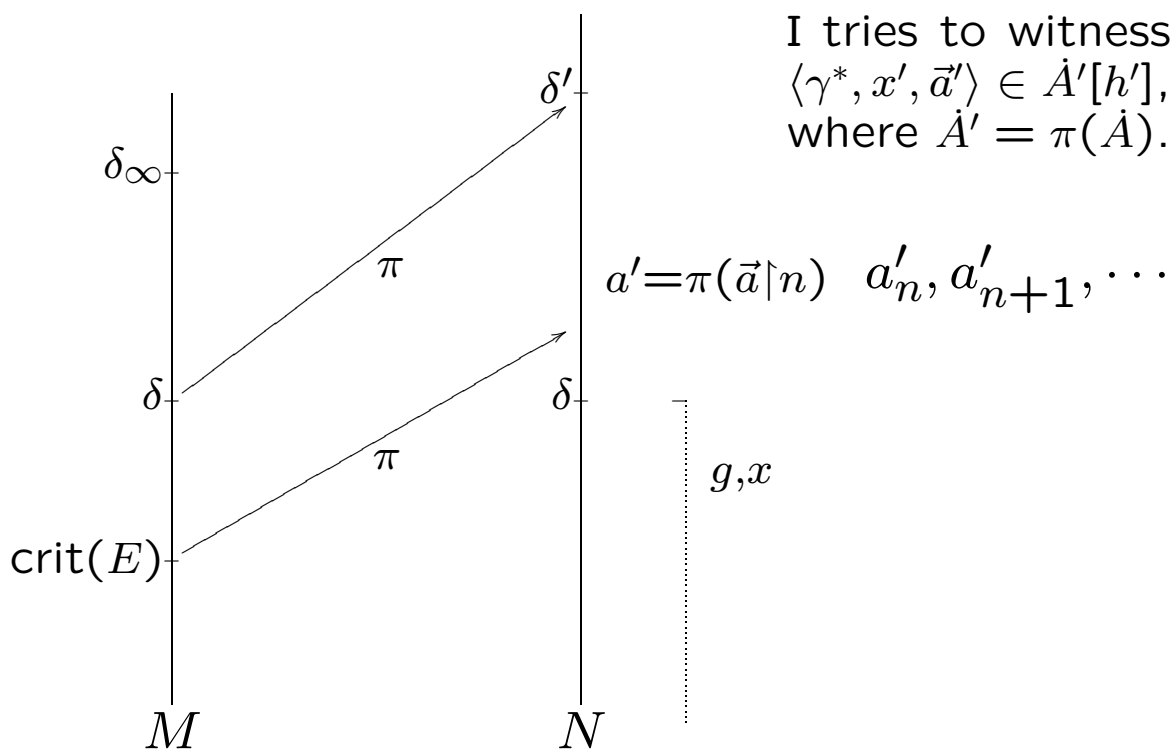
The players now continue playing y_α (extending the previously played $y_\alpha \upharpoonright i$).

In addition they play auxiliary moves in the game $\mathcal{A}'[\gamma^*, x']$, continuing from a' .

(I tries to witness that $\langle \gamma^*, x', \vec{a}' \rangle \in \dot{A}'[h']$ for some h' , generic for the collapse of δ' .)



I tries to witness
 $\langle x, y_\alpha \rangle \in A$



I tries to witness
 $\langle \gamma^*, x', \vec{a}' \rangle \in \dot{A}'[h']$,
 where $\dot{A}' = \pi(\dot{A})$.

As always player II is the closed player. She wins if she can last ω moves. As usual the definition is by induction on γ .

The part of $G(\gamma, x, \vec{a})$ involving $\mathcal{A}_\infty[x, y_\alpha]$ we call the “first half”. The part involving $\mathcal{A}'[\gamma^*, x']$ we call the “second half”.

Note: the second half of G is a game which belongs to $N[g]$. (To decide the rules of the second half we need knowledge of $x = \ulcorner y_\xi \mid \xi < \alpha \urcorner$, so that we can figure x' as we are given y_α . x belongs to $N[g]$ because of our initial assumption on the strength of E .)

This note is important. $N[g]$ is a “small” generic extension of N ; small with respect to the Woodin cardinal $\delta' = \pi(\delta)$. If II wins the second half, we can find a winning strategy in $N[g]$, and this strategy can be shifted along the even models of an iteration given by $\pi(\sigma_{\text{mix}})$.

Case 1: There exists some γ so that (in M) I wins $G(\gamma, x_0, \emptyset)$. (Where $x_0 = \ulcorner \emptyset \urcorner$.)

We will show that (in V) I wins $G_{\text{cont}-f}(C)$.

Fix an imaginary opponent playing for II in $G_{\text{cont}-f}(C)$.

Working against the imaginary opponent we construct:

- $y_\xi \in \mathbb{R}$. We set $x_\alpha = \ulcorner y_\xi \mid \xi < \alpha \urcorner$.
- Iterates M_α of M , with $j_{0,\alpha}: M \rightarrow M_\alpha$.
- Mixed pivots $\mathcal{T}_\alpha, \vec{a}_\alpha, \{k^\alpha(n), \gamma_n^\alpha\}$ for x_α over the model M_α , played according to $j_{0,\alpha}(\sigma_{\text{mix}})$.

\mathcal{T}_α will be continuously illfounded on the even models. (This will follow from our requirement in the second half of $G(\gamma, x, \vec{a})$, that $\gamma^* < \gamma'$.)

The construction is fairly similar to the kinds of constructions handled before. The key point is the following:

Key point: The pivot at $\alpha + 1$ agrees with the $j_{\alpha, \alpha+1}$ image of the pivot at α , up to n_α .

(Similarly for the witness of continuous illfoundedness of the even branches.)

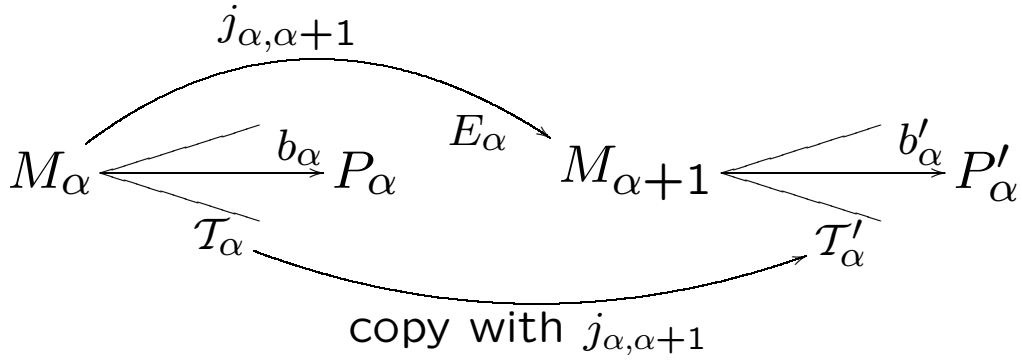
When reaching a **limit** ordinal λ , we can therefore let the pivot at λ be the limit of (the appropriate images of) the pivots at α , as $\alpha \rightarrow \lambda$.

This makes sense because of our key point, because $n_\alpha \rightarrow \infty$ as $\alpha \rightarrow \lambda$, and because $x_\alpha \rightarrow x_\lambda$ as $\alpha \rightarrow \lambda$.

Once the pivot at λ is defined:

The iteration strategy picks an odd branch of \mathcal{T}_λ . The play so far is generic over the direct limit and belongs to a (shift of) $\dot{A}[\gamma][h]$ for some γ and h . This allows us to proceed as usual.

Why mixed pivots?

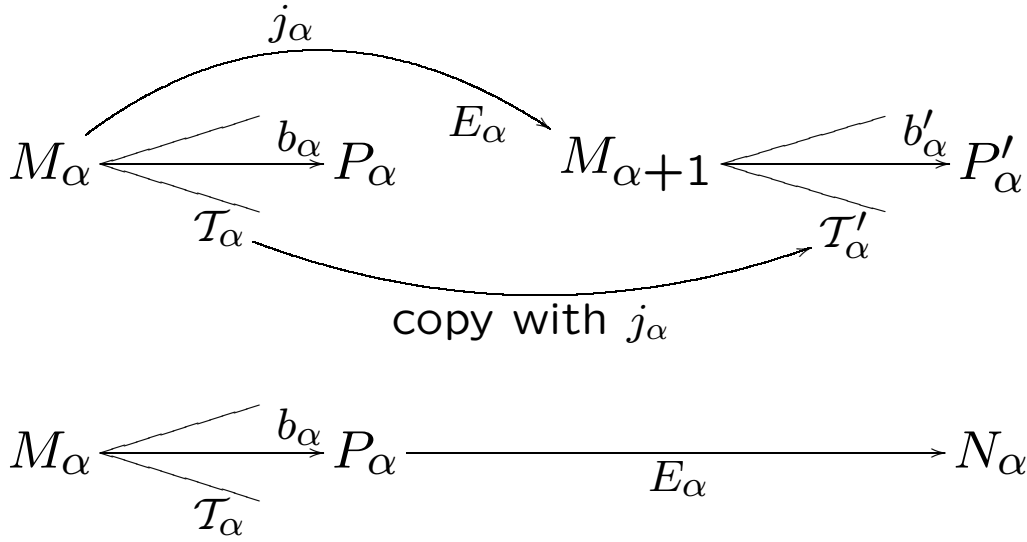


$\mathcal{T}_{\alpha+1}$ starts out like \mathcal{T}'_{α} .

The pivot at $\alpha + 1$ will start out like the $j_{\alpha, \alpha+1}$ image of the pivot at α , and will continue this way long enough to construct the $j_{\alpha, \alpha+1}$ image of $\vec{a}_{\alpha} \upharpoonright n_{\alpha}$.

Only then will the pivot at $\alpha + 1$ change — passing from the γ of the pivot at α , to some smaller γ^* .

Thus, the change from γ to γ^* occurs in the “middle” of the pivot.



Further, the change from γ to γ^* occurs on an odd model of \mathcal{T}'_α . In fact a model along b'_α .

(This has to do with the fact that P'_α , the direct limit of models along b'_α , exactly equals $\text{Ult}(P_\alpha, E_\alpha)$.)

So the change from the pivot at α to the pivot at $\alpha + 1$ involves skipping from the even model of \mathcal{T}'_α where the image of $\vec{a}_\alpha \restriction n_\alpha$ is first constructed, to some later odd model on b'_α . (Then pad to make this model “even”.)

We continue the construction until we reach an α so that, when playing the (appropriate shift of) $G(*, x_\alpha, \vec{a}_\alpha)$, we stay within the “first half”.

When playing the first half of $G(*, x_\alpha, \vec{a}_\alpha)$, we use (the appropriate shift of) $\sigma_{\text{piv}-\infty}$ to ascribe auxiliary moves for II.

We obtain y_α and \mathcal{T}_∞ which is part of a pivot for $\langle x, y_\alpha \rangle$ and the name \dot{A}_∞ (shifted).

$$M \dashrightarrow M_\alpha \begin{array}{c} \nearrow^{b_\alpha} \\ \searrow_{\mathcal{T}_\alpha} \end{array} P_\alpha \begin{array}{c} \nearrow^{b_\infty} \\ \searrow_{\mathcal{T}_\infty} \end{array} M_\infty$$

The iteration strategy produces an odd branch b_∞ , and we conclude that $\langle x, y_\alpha \rangle \in A$.

(Remember, in the first half of $G(*, x, \vec{a})$ I tries to witness that $\langle x, y \rangle$ belongs to the Σ_2^1 set A .)

Thus I wins $G_{\text{cont}-f}$ and we are done.