Determinacy Proofs for Long games

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1.(d) Example: Σ^1_2 determinacy.

2. Games of length $\omega \cdot \omega$ with Σ^1_2 payoff.

3. Continuously coded games with Σ^1_2 payoff.

Recall: A is the set of all reals which satisfy a given Σ^1_2 statement ϕ. $\dot{A} \in M$ names the set of reals of $M^{\mathrm{col}(\omega,\delta)}$ which satisfy ϕ in $M^{\mathrm{col}(\omega,\delta)}$.

G is the game in which I and II play $x = \langle x_0, x_1, \ldots \rangle \in \mathbb{R}$ and in addition play moves in the auxiliary game $\mathcal{A}[x]$.

$$\begin{array}{c|cccccccc} \mathrm{I} & x_0 & a_{0-\mathrm{I}} & & & a_{1-\mathrm{I}} & & x_2 & \ldots \\ \hline \mathrm{II} & & & a_{0-\mathrm{II}} & x_1 & & a_{1-\mathrm{II}} & & \end{array}$$

The game is played in M. Infinite runs of G are won by II.

Using σ_{piv} to ascribe auxiliary moves for II we showed that

Case 1. If I wins G in M, then (in V) I has a winning strategy in $G_\omega(A)$.

Let $\dot{B}$ in M name the set of reals which do **not** satisfy ϕ in $M^{\mathrm{col}(\omega,\delta)}$.

Define $x \mapsto \mathcal{B}[x]$ and $x \mapsto \mathcal{B}^*[x]$ as before, but changing $\dot{A}$ to $\dot{B}$ and interchanging I and II.

We have $\tau_{\mathsf{gen}}[x,g]$ and $\tau_{\mathsf{piv}}[x,g]$ as before, but with the roles of I and II switched.

Let H be the following game, defined and played inside M:

$$\begin{array}{c|cccccccc} \mathrm{I} & x_0 & & b_{0-\mathrm{I}} & & & b_{1-\mathrm{I}} & x_2 & \ldots \\ \hline \mathrm{II} & & b_{0-\mathrm{II}} & & x_1 & b_{1-\mathrm{II}} & & & \end{array}$$

I and II alternate playing natural numbers, producing $x = \langle x_0, x_1, \ldots \rangle \in \mathbb{R}$. In addition they play moves $b_{0-\mathrm{I}}, b_{0-\mathrm{II}}, \ldots$ in $\mathcal{B}[x]$.

This time I is the closed player; she wins if she can last all ω moves. Otherwise II wins.

Case 2: II wins H. Then an argument similar to that of Case 1 shows that (in V) II has a strategy to get into $B = \mathbb{R} - A$. In other words, II wins $G_\omega(A)$ in V. $\square$(Case 2.)

We showed:

- If I wins G in M, then (in V) I wins $G_\omega(A)$.
- If II wins H in M, then (in V) II wins $G_\omega(A)$.

It is now enough to check that one of these cases must occur.

Suppose not. I.e., assume that, in M, II wins G and I wins H. Fix strategies $\Sigma^{\mathrm{II}} \in M$ and $\Sigma^{\mathrm{I}} \in M$ witnessing this. We wish to derive a contradiction.

Recall the progress of the games G and H:

$$G: \begin{array}{c|cccccccc} \mathrm{I} & x_0 & a_{0-\mathrm{I}} & & & a_{1-\mathrm{I}} & & \ldots \\ \hline \mathrm{II} & & & a_{0-\mathrm{II}} & x_1 & & a_{1-\mathrm{II}} & \end{array}$$

$$H: \begin{array}{c|cccccccc} \mathrm{I} & x_0 & & b_{0-\mathrm{I}} & & & b_{1-\mathrm{I}} & \ldots \\ \hline \mathrm{II} & & b_{0-\mathrm{II}} & & x_1 & b_{1-\mathrm{II}} & & \end{array}$$

Working in $M[g]$, construct $x = \langle x_0, x_1, \ldots \rangle$, $\vec{a} = \langle a_{0-\mathrm{I}}, a_{0-\mathrm{II}}, \ldots \rangle$, and $\vec{b} = \langle b_{0-\mathrm{I}}, b_{0-\mathrm{II}}, \ldots \rangle$ as follows:

- Σ^{II} (playing for II in G) produces x_n for odd n, and $a_{n-\mathrm{II}}$ for all n.
- $\sigma_{\text{gen}}[x, g]$ produces $a_{n-\mathrm{I}}$ for all n.
- Σ^{I} (playing for I in H) produces x_n for even n and $b_{n-\mathrm{I}}$ for all n.
- $\tau_{\text{gen}}[x, g]$ produces $b_{n-\mathrm{II}}$ for all n.

We get $x \notin \dot{A}[g]$ by Lemma 1. Similarly we get $x \notin \dot{B}[g]$ through our use of τ_{gen}.

But $\dot{A}$ and $\dot{B}$ name complementary sets. Since $x \in M[g]$ this is a contradiction. $\square$

To sum: Defined in M the game

$$G: \begin{array}{c|cccccccc} \mathrm{I} & x_0 & a_{0-\mathrm{I}} & & & a_{1-\mathrm{I}} & & \ldots \\ \hline \mathrm{II} & & & a_{0-\mathrm{II}} & x_1 & & a_{1-\mathrm{II}} & \end{array}$$

where I, II collaborate to produce $x \in \mathbb{R}$, and in addition play auxiliary moves: I trying to witness $x \in \dot{A}[h]$ for some h, II trying to witness the opposite. G is a closed game.

If in M I wins G, showed (using σ_{piv}) that in V I wins to get into some $j_b(\dot{A})[h]$, and hence by absoluteness into A.

Defined in M the game

$$H: \begin{array}{c|cccccccc} \mathrm{I} & x_0 & & b_{0-\mathrm{I}} & & & b_{1-\mathrm{I}} & \ldots \\ \hline \mathrm{II} & & b_{0-\mathrm{II}} & & x_1 & b_{1-\mathrm{II}} & & \end{array}$$

This time II is trying to witness $x \in \dot{B}[h]$ for some h, and I is trying to witness the opposite.

If in M II wins H, showed (using τ_{piv}) that in V II wins to get into some $j_b(\dot{B})[h]$, and hence by absoluteness into $B = \mathbb{R} - A$.

Finally, if both cases fail, we worked in $M[g]$ (using σ_{gen} and τ_{gen}) to construct $x \in M[g]$ which belongs to neither $\dot{A}[g]$ nor $\dot{B}[g]$, a contradiction.

Fix $C \subset \mathbb{R}^\omega$ a Σ^1_2 set, say the set of all sequences $\langle y_0, y_1, \ldots \rangle \in \mathbb{R}^\omega$ which satisfy the Σ^1_2 statement ϕ.

We wish to prove that $G_{\omega \cdot \omega}(C)$ is determined.

Fix M and an increasing sequence $\delta_1, \delta_2, \ldots, \delta_\omega$ so that

- M is a class model.
- Each δ_ξ is a Woodin cardinal in M.
- In V there is g which is col(ω, δ_ω)–gen/M.
- M is iterable.

The existence of such M is our large cardinal assumption (needed to prove determinacy). We use δ_∞ and g_∞ to refer to δ_ω and g.

Let $\dot{A}_\infty \in M$ name the set of elements of $\mathbb{R}^\omega \cap M[g_\infty]$ which satisfy ϕ in $M[g_\infty]$.

For $\langle y_n \mid n < \omega \rangle \in \mathbb{R}^\omega$ we have the associated game $\mathcal{A}_\infty[y_n \mid n < \omega]$. (Formally we should think of $\langle y_n \mid n < \omega \rangle$ as coded by some real x.)

The association is continuous, and we may talk about $\mathcal{A}_\infty[y_0, \ldots, y_{k-1}]$, a game of $k+1$ rounds.

We use $a^\infty_{0-\mathrm{I}}$, $a^\infty_{0-\mathrm{II}}$, $a^\infty_{1-\mathrm{I}}$, etc. to refer to moves in $\mathcal{A}_\infty$.

We use a^∞_n to denote $\langle a^\infty_{n-\mathrm{I}}, a^\infty_{n-\mathrm{II}} \rangle$ and refer to runs of $\mathcal{A}_\infty$ as $\vec{a}^\infty$.

(Recall that moves in $\mathcal{A}_\infty[y_n \mid n < \omega]$ are arranged so that I tries to witness $\langle y_n \mid n < \omega \rangle \in \dot{A}_\infty[h]$ for some h, and II tries to witness the opposite.)

A k-**sequences** is a sequence

$$\langle y_0, \dots, y_{k-1}, a_0^\infty, \dots, a_{k-1}^\infty, \gamma \rangle$$

so that

- Each y_i is a real;
- $a_0^\infty, \dots, a_{k-1}^\infty$ is a position in the auxiliary game $\mathcal{A}_\infty[y_0, \dots, y_{k-1}]$; and
- γ is an ordinal.

We use S to denote k-sequences.

A **valid extension** for a k-sequence is a triplet $y_k, a_k^\infty, \gamma^*$ so that

- y_k is a real;
- $a_k^\infty = \langle a_{k-\mathrm{I}}^\infty, a_{k-\mathrm{II}}^\infty \rangle$ where $a_{k-\mathrm{I}}^\infty$ and $a_{k-\mathrm{II}}^\infty$ are legal moves for I and II respectively in the game $\mathcal{A}_\infty[y_0, \ldots, y_{k-1}]$,* following the position $a_0^\infty, \ldots, a_{k-1}^\infty$; and
- γ^* is an ordinal **smaller** than γ.

We use S—$, y_k, a_k^\infty, \gamma^*$ to denote the $k+1$-sequence

$$\langle y_0, \ldots, y_{k-1}, y_k, a_0^\infty, \ldots, a_{k-1}^\infty, a_k^\infty, \gamma^* \rangle.$$

*Observe that knowledge of y_k is not needed to determine the rules for round k of this game.

For expository simplicity, fix for each n some g_n which is col(ω, δ_n)–generic/M. Do this so that the sequence $\langle g_n \mid n < \omega \rangle$ belongs to $M[g_\infty]$ and each g_n belongs to $M[g_{n+1}]$.

Below we define sets in $M[g_n]$ where strictly speaking we should be defining names in $M^{\mathrm{col}(\omega,\delta_n)}$.

We work to define sets A_k in $M[g_k]$ $(k \geq 1)$. A_k will be a set of k-sequences.

Given $a_0^\infty, \ldots, a_{k-1}^\infty$, γ we let $A_k[a_0^\infty, \ldots, a_{k-1}^\infty, \gamma]$ be the set of tuples $\langle y_0, \ldots, y_{k-1} \rangle$ so that

$$\langle y_0, \ldots, y_{k-1}, a_0^\infty, \ldots, a_{k-1}^\infty, \gamma \rangle \in A_k.$$

$A_k[a_0^\infty, \ldots, a_{k-1}^\infty, \gamma]$ then is a subset of $\mathbb{R}^k$ in $M[g_k]$. Really we are defining names, not sets. So we have a name $\dot{A}_k[a_0^\infty, \ldots, a_{k-1}^\infty, \gamma]$.

Let $\mathcal{A}_k[y_0, \ldots, y_{k-1}, a_0^\infty, \ldots, a_{k-1}^\infty, \gamma]$ be auxiliary game associated to $\langle y_0, \ldots, y_{k-1} \rangle$ and the name $\dot{A}_k[a_0^\infty, \ldots, a_{k-1}^\infty, \gamma]$.

We use $\mathcal{A}_k[S]$ to denote this game, and use $a^k_{0-\mathrm{I}}$, $a^k_{0-\mathrm{II}}$ etc. to denote moves in the game.

(Recall that these moves are such that I tries to witness that S belongs to $\dot{A}_k[h]$ for some h. II tries to witness the opposite.)

Given $S = \langle y_0, \ldots, y_{k-1}, a_0^\infty, \ldots, a_{k-1}^\infty, \gamma \rangle$, a k-sequence, define a game $G_k(S)$ in which:

I and II play a valid extension $\gamma^*, a_k^\infty, y_k$. In addition I tries to witness that the extended sequence, $S —, y_k, a_k^\infty, \gamma^*$, belongs to A_{k+1}. II tries to witness the opposite.

$G_k(S)$:

$$
\begin{array}{c|cccccccc}
\mathrm{I} & \gamma^*, a^{\infty}_{k-\mathrm{I}} & & y_k(0) & a^{k+1}_{0-\mathrm{I}} & & & & \\
\hline
\mathrm{II} & & a^{\infty}_{k-\mathrm{II}} & & & a^{k+1}_{0-\mathrm{II}} & & & \\
\end{array}
$$

$$
\begin{array}{ccccc}
 & a^{k+1}_{1-\mathrm{I}} & & y_k(2) & \ldots \\
\hline
y_k(1) & & a^{k+1}_{1-\mathrm{II}} & & \\
\end{array}
$$

I and II play

- γ^*,
- $a^{\infty}_k = \langle a^{\infty}_{k-\mathrm{I}}, a^{\infty}_{k-\mathrm{II}} \rangle$, and
- $y_k = \langle y_k(0), y_k(1), \ldots \rangle$

which form a valid extension of S. (In particular γ^* is **smaller** than γ.)

In addition they play auxiliary moves in the game $\mathcal{A}_{k+1}[S—, y_k, a^{\infty}_k, \gamma^*]$.

II is the closed player; she wins if she can last ω moves. Otherwise I wins.

Define the sets A_k by:

$S \in A_k$ iff I has a winning strategy in $G_k(S)$

(for a k-sequence $S \in M[g_k]$).

If S belongs to A_k, we expect to be able to extend to $S^* = S—, y_k, a_k^\infty, \gamma^*$ which belongs to a "shift" of A_{k+1}.

Our definition of A_k depends on some knowledge of A_{k+1}. (We need knowledge of G_k, which involves the auxiliary game $\mathcal{A}_{k+1}$.)

The definition is by induction, not on k, but on γ.

Figuring out the rules of $G_k(S)$, where $S = \langle *, \ldots, *, \gamma \rangle$, requires knowledge of the sets $A_{k+1}[a_0^\infty, \ldots, a_k^\infty, \gamma^*]$, but only for $\gamma^* < \gamma$.

Determining whether S belongs to A_k thus requires knowledge of A_{k+1}, but only for $k+1$-sequences ending with $\gamma^* < \gamma$.

A 0-sequence is simply an ordinal γ. We have for each γ the game $G_0(\gamma)$. This game belongs to M.

Case 1: There exists some γ so that (in M) I has a winning strategy in $G_0(\gamma)$.

We will show that (in V) I has a winning strategy in $G_{\omega \cdot \omega}(C)$.

Fix $\Sigma_0 \in M$, a winning strategy for I (the open player) in $G_0(\gamma)$.

Fix an imaginary opponent, playing for II in $G_{\omega \cdot \omega}(C)$.

We will use Σ_0, the strategies $\sigma_{\text{piv}-1}, \sigma_{\text{piv}-2}, \ldots$, the strategy $\sigma_{\text{piv}-\infty}$, and an iteration strategy for M, to play against the imaginary opponent.

$G_0(\gamma)$:

$$
\begin{array}{c|ccccc}
\mathrm{I} & \gamma_0^*, a^{\infty}_{0-\mathrm{I}} & & y_0(0) & a^1_{0-\mathrm{I}} & \\
\hline
\mathrm{II} & & a^{\infty}_{0-\mathrm{II}} & & & a^1_{0-\mathrm{II}}
\end{array}
$$

$$
\begin{array}{ccccc}
 & a^1_{1-\mathrm{I}} & & y_0(2) & \ldots \\
\hline
y_0(1) & & a^1_{1-\mathrm{II}} & &
\end{array}
$$

Our opponent, Σ_0, $\sigma_{\mathrm{piv}-\infty}$ (for the first round), and $\sigma_{\mathrm{piv}-1}$ (for the remaining rounds) cover all moves in the game.

We obtain an iteration tree $\mathcal{U}^0$ of length 3, played by $\sigma_{\mathrm{piv}-\infty}$, with final model P_2^0, embedding $\pi^0_{0,2}\colon M \to P_2^0$, and moves $\bar{\gamma}_0^*, \bar{a}_0^{\infty}$ in P_2^0.

We obtain $y_0 \in \mathbb{R}$, and an iteration tree $\mathcal{T}_0$ (played by $\sigma_{\mathrm{piv}-1}$) with illfounded even model.

The iteration strategy picks an odd branch, b_0 say. Let M_1 be the direct limit along b_0 and let $j_{0,1}$ be the direct limit embedding.

$$
\begin{tikzcd}
M = P^0_0 \arrow[rr, bend left, "\pi^0_{0,2}"] & P^0_1 & P^0_2
\end{tikzcd}
$$

$$
\begin{tikzcd}
M \arrow[r, "b_0", "\mathcal{T}_0"'] & M_1 = P^1_0 \arrow[rr, bend left, "\pi^1_{0,2}"] & P^1_1 & P^1_2
\end{tikzcd}
$$

Let $\mathcal{U}^1 = j_{0,1}(\mathcal{U}^0)$, and similarly with P^1_2, $\pi^1_{0,2}$. Let $\gamma^*_0 = j_{0,1}(\bar{\gamma}^*_0)$ and similarly a^∞_0.

Our use of $\sigma_{\text{piv}-1}$ guarantees that there exists some h_1 so that

1. h_1 is col$(\omega, \delta^{\mathsf{s}}_1)$–generic/$M_1$, and
2. $\langle y_0, a^\infty_0, \gamma^*_0 \rangle \in \dot{A}^{\mathsf{s}}_1[h_1]$.

($*^{\mathsf{s}}$ denotes $j_{0,1}(\pi^0_{0,2}(*))$.)

Note that by 2, player I (the open player) has a winning strategy in $G^{\mathsf{s}}_1(y_0, a^\infty_0, \gamma^*_0)$. Fix $\Sigma_1 \in M_1[h_1]$, a strategy for I witnessing this.

$$
\begin{array}{c|cccccc}
\mathrm{I} & \gamma_1^*, a^\infty_{1-\mathrm{I}} & & y_1(0) & a^2_{0-\mathrm{I}} & & \\
\hline
\mathrm{II} & & a^\infty_{1-\mathrm{II}} & & & a^2_{0-\mathrm{II}} &
\end{array}
$$

$$
\begin{array}{cccccc}
 & a^2_{1-\mathrm{I}} & & y_1(2) & \ldots \\
\hline
y_1(1) & & a^2_{1-\mathrm{II}} & & &
\end{array}
$$

Note, Σ_1 belongs to $M_1[h_1]$, a **small** generic extension of M. (Small with respect to δ_2 and δ_∞.) This allows us to shift Σ_1 along the even branch of trees given by $\sigma_{\mathrm{piv}-\infty}$ and $\sigma_{\mathrm{piv}-2}$.

Using Σ_1 and $j_{0,1}(\sigma_{\mathrm{piv}-\infty})$ we get

$$
\begin{tikzcd}
M \arrow[r, "b_0", "\mathcal{T}_0"'] & M_1 = P^1_0 \arrow[rr, bend left, "\pi^0_{0,2}"] & P^1_1 & P^1_2 \arrow[rr, bend left, "\pi^0_{2,4}"] & P^1_3 & P^1_4
\end{tikzcd}
$$

Then using $j_{0,1}(\sigma_{\mathrm{piv}-2})$ and **shifts** of Σ_1 get

$$
\begin{tikzcd}
M \arrow[r, "b_0", "\mathcal{T}_0"'] & M_1 \arrow[r, "b_1", "\mathcal{T}_1"'] & M_2 = P^2_0 \arrow[r, dotted, bend left, "\pi^2_{0,4}"] & P^2_4
\end{tikzcd}
$$

(where $P^2_0 = j_{1,2}(P^1_0)$, etc.).

We get γ_1^*, a_1^∞, and y_1. Our use of $\sigma_{\mathrm{piv}-2}$ guarantees that there exists h_2 so that

1. h_2 is col$(\omega, \delta_2^{\mathrm{s\,s}})$–generic/$M_2$, and
2. $\langle y_0, y_1, a_0^{\infty-\mathrm{s}}, a_1^\infty, \gamma_1^* \rangle \in \dot{A}_2^{\mathrm{s\,s}}[h_2]$.

(A second $^{\mathrm{s}}$ stands for application of $j_{1,2} \circ \pi^1_{2,4}$.)

By 2, player I (the open player) wins

$$G_2^{\mathrm{s\,s}}(y_0, y_1, a_0^{\infty-\mathrm{s}}, a_1^\infty, \gamma_1^*).$$

This game belongs to $M[h_2]$. Fix $\Sigma_2 \in M[h_2]$, a strategy witnessing that I wins.

Continue as before.

In general we have:

$$\begin{tikzcd}
M_k = P_0^k \arrow[rr, dotted, bend left, "\pi^k_{0,2k}"] \arrow[d, "j_{k,k+1}"'] & & P^k_{2k} \arrow[rr, bend left, "\pi^k_{2k,2k+2}"] & P^k_{2k+1} & P^k_{2k+2} \arrow[d, "j_{k,k+1}"] \\
M_{k+1} = P_0^{k+1} \arrow[rrrr, dotted, bend left, "\pi^{k+1}_{0,2k+2}"] & & & & P^{k+1}_{2k+2}
\end{tikzcd}$$

$\mathcal{T}_k$

In P^k_{2k} we have the k-sequence

$S_k = \langle y_0, \ldots, y_{k-1}, a_0^{\infty^{-\mathsf{s}\cdots\mathsf{s}}}, \ldots, a_{k-1}^{\infty^{--\cdots-}}, \gamma^*_{k-1}\rangle.$

S_{k+1} (in P^{k+1}_{2k+2}) is obtained as a valid extension of $j_{k,k+1}(\pi^k_{2k,2k+2}(S_k))$. In particular:

(†) γ^*_k is **smaller** than $j_{k,k+1}(\pi^k_{2k,2k+2}(\gamma^*_{k-1}))$.

We end with a sequence of reals $\langle y_n \mid n < \omega \rangle$, a sequence of iteration trees

$$\begin{tikzcd} M_0 \arrow[r, "b_0", "\mathcal{T}_0"'] & M_1 \arrow[r, "b_1", "\mathcal{T}_1"'] & M_2 & \cdots & {} \arrow[r, dashed] & M_\infty \end{tikzcd}$$

and an iteration tree $\mathcal{U}_\infty$ on M_∞ as follows:

$$\begin{tikzcd} M_\infty = P_0^\infty \arrow[rr, bend left, "\pi^\infty_{0,2}"] & P_1^\infty & P_2^\infty \arrow[rr, bend left, "\pi^\infty_{2,4}"] & P_3^\infty & P_4^\infty \arrow[rr, bend left, dashed] & \cdots & {} \end{tikzcd}$$

By (†) the even branch of $\mathcal{U}_\infty$ is illfounded.

The iteration strategy for M produces an odd branch c of $\mathcal{U}_\infty$. Let M_c be the direct limit, and let $\pi_c \colon M_\infty \to M_c$ be the direct limit embedding. Note M_c, played by an iteration strategy, is **wellfounded**.

Now $\mathcal{U}_\infty$ is part of a play according to $j_{0,\infty}(\sigma_{\mathrm{piv}-\infty})[y_n \mid n < \omega]$.

Our use of $j_{0,\infty}(\sigma_{\mathrm{piv}-\infty})[y_n \mid n < \omega]$ guarantees that there exists some h_∞ so that

1. h_∞ is $\mathrm{col}(\omega, \pi_c(j_{0,\infty}(\delta_\infty)))$–generic/$M_c$, and
2. $\langle y_n \mid n < \omega \rangle \in \pi_c(j_{0,\infty}(\dot{A}_\infty))[h_\infty]$.

From 2 we see that $\langle y_n \mid n < \omega \rangle$ satisfies the Σ^1_2 statement ϕ, inside $M_c[h_\infty]$.

By absoluteness ϕ is satisfied in V.

So $\langle y_n \mid n < \omega \rangle \in C$ and I won, as required.

$\square$(Case 1.)

Assuming there is some γ so that (in M) I wins $G_0(\gamma)$, we showed that (in V) I wins $G_{\omega \cdot \omega}(C)$.

Fix $\gamma_{\mathsf{L}} < \gamma_{\mathsf{H}}$ indiscernibles for M, above δ_∞.

Suppose $S = \langle y_0, \dots, y_{k-1}, a_0^\infty, \dots, a_{k-1}^\infty, \gamma_{\mathsf{L}} \rangle$ is a k-sequence in $M[g_k]$ and does not belong A_k.

So II wins $G_k(S)$. By indiscernibility II also wins $G_k(S_{\mathsf{H}})$ where $S_{\mathsf{H}} = \langle *, \dots, \gamma_{\mathsf{H}} \rangle$. Fix a winning strategy $\Sigma_{\mathrm{II}-k}$.

$$
\begin{array}{c|cccccc}
\mathrm{I} & \gamma^*, a^\infty_{k-\mathrm{I}} & & y_k(0) & a^{k+1}_{0-\mathrm{I}} & \\
\hline
\mathrm{II} & & a^\infty_{k-\mathrm{II}} & & & a^{k+1}_{0-\mathrm{II}}
\end{array}
$$

$$
\begin{array}{cccccc}
 & a^{k+1}_{1-\mathrm{I}} & & y_k(2) & \dots \\
\hline
y_k(1) & & a^{k+1}_{1-\mathrm{II}} & & &
\end{array}
$$

Play $\gamma^* = \gamma_{\mathsf{L}}$. Use $\Sigma_{\mathrm{II}-k}$, $\sigma_{\mathsf{gen}-\infty}$, $\sigma_{\mathsf{gen}-(k+1)}$ to obtain a_k^∞, $\vec{a}^{k+1}$, and (half of) y_k in $M[g_{k+1}]$.

Our use of $\sigma_{\mathsf{gen}-(k+1)}$ guarantees that $S' = \langle S_{\mathsf{H}}—, y_k, a_k^\infty, \gamma_{\mathsf{L}} \rangle \notin \dot{A}_{k+1}[g_{k+1}]$.

If S' belongs to $M[g_{k+1}]$ this means that II wins $G_{k+1}(S')$.

Continue this way. Our use of $\sigma_{\text{gen}-\infty}$ guarantees that $\langle y_n \mid n < \omega\rangle$ does not belong to $\dot{A}[g_\infty]$.

If there is γ so that I wins the closed game $G_0(\gamma)$, then I has a winning strategy in $G_{\omega\cdot\omega}(C)$.

Mirroring this with sets B_k and games H_k we get:

If there is γ so that II wins the closed game $H_0(\gamma)$, then II has a winning strategy in $G_{\omega\cdot\omega}(C)$.

Finally, if II wins $G_0(\gamma_{\mathsf{L}})$ and I wins $H_0(\gamma_{\mathsf{L}})$, we can work in $M[g_n]$, $n < \omega$, and produce $\langle y_n \mid n < \omega\rangle \in M[g_\infty]$ * which belongs to neither $\dot{A}[g_\infty]$ nor $\dot{B}[g_\infty]$, a contradiction.

It follows that $G_{\omega\cdot\omega}(C)$ is determined.

*Note $\langle g_n \mid n < \omega\rangle \in M[g_\infty]$.