Determinacy Proofs for Long games

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1.(d) Example: $\Sigma^1_2$ determinacy.

2. Games of length $\omega \cdot \omega$ with $\Sigma^1_2$ payoff.

3. Continuously coded games with $\Sigma^1_2$ payoff.
Recall: $A$ is the set of all reals which satisfy a given $\Sigma^1_2$ statement $\phi$. $\dot{A} \in M$ names the set of reals of $M^{\text{col}(\omega,\delta)}$ which satisfy $\phi$ in $M^{\text{col}(\omega,\delta)}$.

$G$ is the game in which I and II play $x = \langle x_0, x_1, \ldots \rangle \in \mathbb{R}$ and in addition play moves in the auxiliary game $A[x]$.

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<thead>
<tr>
<th></th>
<th>$x_0$</th>
<th>$a_{0-\text{I}}$</th>
<th>$a_{1-\text{I}}$</th>
<th>$x_2$</th>
<th>$\ldots$</th>
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<tbody>
<tr>
<td>I</td>
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<tr>
<td>II</td>
<td></td>
<td>$a_{0-\text{II}}$</td>
<td>$x_1$</td>
<td>$a_{1-\text{II}}$</td>
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The game is played in $M$. Infinite runs of $G$ are won by II.

Using $\sigma_{\text{piv}}$ to ascribe auxiliary moves for II we showed that

**Case 1.** If I wins $G$ in $M$, then (in $V$) I has a winning strategy in $G_\omega(A)$.

Let $\dot{B}$ in $M$ name the set of reals which do not satisfy $\phi$ in $M^{\text{col}(\omega,\delta)}$.

Define $x \mapsto B[x]$ and $x \mapsto B^*[x]$ as before, but changing $\dot{A}$ to $\dot{B}$ and interchanging I and II.
We have $\tau_{\text{gen}}[x, g]$ and $\tau_{\text{piv}}[x, g]$ as before, but with the roles of I and II switched.

Let $H$ be the following game, defined and played inside $M$:

<table>
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<th>$x_0$</th>
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<td>$x_1$</td>
<td>$b_{1-II}$</td>
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</tbody>
</table>

I and II alternate playing natural numbers, producing $x = \langle x_0, x_1, \ldots \rangle \in \mathbb{R}$. In addition they play moves $b_{0-I}, b_{0-II}, \ldots$ in $B[x]$.

This time I is the closed player; she wins if she can last all $\omega$ moves. Otherwise II wins.

**Case 2:** II wins $H$. Then an argument similar to that of Case 1 shows that (in $V$) II has a strategy to get into $B = \mathbb{R} - A$. In other words, II wins $G_\omega(A)$ in $V$. $\square$ (Case 2.)
We showed:

- If I wins $G$ in $M$, then (in $V$) I wins $G_{\omega}(A)$.
- If II wins $H$ in $M$, then (in $V$) II wins $G_{\omega}(A)$.

It is now enough to check that one of these cases must occur.

Suppose not. I.e., assume that, in $M$, II wins $G$ and I wins $H$. Fix strategies $\Sigma^{\text{II}} \in M$ and $\Sigma^{\text{I}} \in M$ witnessing this. We wish to derive a contradiction.
Recall the progress of the games $G$ and $H$:

\[
\begin{array}{c|ccccc}
G: & I & x_0 & a_{0-I} & a_{1-I} & \ldots \\
   & II & a_{0-II} & x_1 & a_{1-II} & \\
\end{array}
\]

\[
\begin{array}{c|ccccc}
H: & I & x_0 & b_{0-I} & b_{1-I} & \ldots \\
   & II & b_{0-II} & x_1 & b_{1-II} & \\
\end{array}
\]

Working in $M[g]$, construct $x = \langle x_0, x_1, \ldots \rangle$, $\vec{a} = \langle a_{0-I}, a_{0-II}, \ldots \rangle$, and $\vec{b} = \langle b_{0-I}, b_{0-II}, \ldots \rangle$ as follows:

- $\Sigma^{II}$ (playing for II in $G$) produces $x_n$ for odd $n$, and $a_{n-II}$ for all $n$.
- $\sigma_{\text{gen}}[x, g]$ produces $a_{n-I}$ for all $n$.
- $\Sigma^{I}$ (playing for I in $H$) produces $x_n$ for even $n$ and $b_{n-I}$ for all $n$.
- $\tau_{\text{gen}}[x, g]$ produces $b_{n-II}$ for all $n$.

We get $x \notin \hat{A}[g]$ by Lemma 1. Similarly we get $x \notin \hat{B}[g]$ through our use of $\tau_{\text{gen}}$.

But $\hat{A}$ and $\hat{B}$ name complementary sets. Since $x \in M[g]$ this is a contradiction. □
To sum: Defined in $M$ the game

$$
G : \begin{array}{l|llll}
I & x_0 & a_{0-I} & a_{1-I} & \ldots \\
II & a_{0-II} & x_1 & a_{1-II} & \\
\end{array}
$$

where I, II collaborate to produce $x \in \mathbb{R}$, and in addition play auxiliary moves: I trying to witness $x \in \dot{A}[h]$ for some $h$, II trying to witness the opposite. $G$ is a closed game.

If in $M$ I wins $G$, showed (using $\sigma_{piv}$) that in $V$ I wins to get into some $j_b(\dot{A})[h]$, and hence by absoluteness into $A$.

Defined in $M$ the game

$$
H : \begin{array}{l|llll}
I & x_0 & b_{0-I} & b_{1-I} & \ldots \\
II & b_{0-II} & x_1 & b_{1-II} & \\
\end{array}
$$

This time II is trying to witness $x \in \dot{B}[h]$ for some $h$, and I is trying to witness the opposite.

If in $M$ II wins $H$, showed (using $\tau_{piv}$) that in $V$ II wins to get into some $j_b(\dot{B})[h]$, and hence by absoluteness into $B = \mathbb{R} - A$. 

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Finally, if both cases fail, we worked in $M[g]$ (using $\sigma_{\text{gen}}$ and $\tau_{\text{gen}}$) to construct $x \in M[g]$ which belongs to neither $\dot{A}[g]$ nor $\dot{B}[g]$, a contradiction.
Fix $C \subset \mathbb{R}^\omega$ a $\Sigma^1_2$ set, say the set of all sequences $\langle y_0, y_1, \ldots \rangle \in \mathbb{R}^\omega$ which satisfy the $\Sigma^1_2$ statement $\phi$.

We wish to prove that $G_{\omega^\omega}(C')$ is determined.

Fix $M$ and an increasing sequence $\delta_1, \delta_2, \ldots, \delta_\omega$ so that

- $M$ is a class model.
- Each $\delta_\xi$ is a Woodin cardinal in $M$.
- In $V$ there is $g$ which is $\text{col}(\omega, \delta_\omega)$–gen$/M$.
- $M$ is iterable.

The existence of such $M$ is our large cardinal assumption (needed to prove determinacy). We use $\delta_\infty$ and $g_\infty$ to refer to $\delta_\omega$ and $g$. 
Let $\dot{A}_\infty \in M$ name the set of elements of $\mathbb{R}^\omega \cap M[g_\infty]$ which satisfy $\phi$ in $M[g_\infty]$.

For $\langle y_n \mid n < \omega \rangle \in \mathbb{R}^\omega$ we have the associated game $\mathcal{A}_\infty[y_n \mid n < \omega]$. (Formally we should think of $\langle y_n \mid n < \omega \rangle$ as coded by some real $x$.)

The association is continuous, and we may talk about $\mathcal{A}_\infty[y_0, \ldots, y_{k-1}]$, a game of $k+1$ rounds.

We use $a_0^\infty_{\text{-I}}$, $a_0^\infty_{\text{-II}}$, $a_1^\infty_{\text{-I}}$, etc. to refer to moves in $\mathcal{A}_\infty$.

We use $a_n^\infty$ to denote $\langle a_n^\infty_{\text{-I}}, a_n^\infty_{\text{-II}} \rangle$ and refer to runs of $\mathcal{A}_\infty$ as $\vec{a}^\infty$.

(Recall that moves in $\mathcal{A}_\infty[y_n \mid n < \omega]$ are arranged so that I tries to witness $\langle y_n \mid n < \omega \rangle \in \dot{A}_\infty[h]$ for some $h$, and II tries to witness the opposite.)
A \textbf{k-sequences} is a sequence

\[ \langle y_0, \ldots, y_{k-1}, a_0^\infty, \ldots, a_{k-1}^\infty, \gamma \rangle \]

so that

\begin{itemize}
  \item Each \( y_i \) is a real;
  \item \( a_0^\infty, \ldots, a_{k-1}^\infty \) is a position in the auxiliary game \( A^\infty[y_0, \ldots, y_{k-1}] \); and
  \item \( \gamma \) is an ordinal.
\end{itemize}

We use \( S \) to denote \( k \)-sequences.
A valid extension for a $k$-sequence is a triplet $y_k, a_k^\infty, \gamma^*$ so that

- $y_k$ is a real;
- $a_k^\infty = \langle a_{k-\mathrm{I}}^\infty, a_{k-\mathrm{II}}^\infty \rangle$ where $a_{k-\mathrm{I}}^\infty$ and $a_{k-\mathrm{II}}^\infty$ are legal moves for I and II respectively in the game $A_\infty[y_0, \ldots, y_{k-1}],^*$ following the position $a_0^\infty, \ldots, a_{k-1}^\infty$; and
- $\gamma^*$ is an ordinal smaller than $\gamma$.

We use $S_{--}, y_k, a_k^\infty, \gamma^*$ to denote the $k + 1$-sequence

$$\langle y_0, \ldots, y_{k-1}, y_k, a_0^\infty, \ldots, a_{k-1}^\infty, a_k^\infty, \gamma^* \rangle.$$

*Observe that knowledge of $y_k$ is not needed to determine the rules for round $k$ of this game.*
For expository simplicity, fix for each $n$ some $g_n$ which is $\text{col}(\omega, \delta_n)$–generic$/M$. Do this so that the sequence $\langle g_n \mid n < \omega \rangle$ belongs to $M[g_\infty]$ and each $g_n$ belongs to $M[g_{n+1}]$.

Below we define sets in $M[g_n]$ where strictly speaking we should be defining names in $M^{\text{col}(\omega,\delta_n)}$.

We work to define sets $A_k$ in $M[g_k]$ ($k \geq 1$). $A_k$ will be a set of $k$-sequences.

Given $a_0^\infty, \ldots, a_{k-1}^\infty, \gamma$ we let $A_k[a_0^\infty, \ldots, a_{k-1}^\infty, \gamma]$ be the set of tuples $\langle y_0, \ldots, y_{k-1} \rangle$ so that

$$\langle y_0, \ldots, y_{k-1}, a_0^\infty, \ldots, a_{k-1}^\infty, \gamma \rangle \in A_k.$$ 

$A_k[a_0^\infty, \ldots, a_{k-1}^\infty, \gamma]$ then is a subset of $\mathbb{R}^k$ in $M[g_k]$. Really we are defining names, not sets. So we have a name $\dot{A}_k[a_0^\infty, \ldots, a_{k-1}^\infty, \gamma]$. 

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Let \( A_k[y_0, \ldots, y_{k-1}, a_0^\infty, \ldots, a_{k-1}^\infty, \gamma] \) be auxiliary game associated to \( \langle y_0, \ldots, y_{k-1} \rangle \) and the name \( \dot{A}_k[a_0^\infty, \ldots, a_{k-1}^\infty, \gamma] \).

We use \( A_k[S] \) to denote this game, and use \( a_{0-I}^k, a_{0-II}^k \) etc. to denote moves in the game.

(Recall that these moves are such that I tries to witness that \( S \) belongs to \( \dot{A}_k[h] \) for some \( h \). II tries to witness the opposite.)

Given \( S = \langle y_0, \ldots, y_{k-1}, a_0^\infty, \ldots, a_{k-1}^\infty, \gamma \rangle \), a \( k \)-sequence, define a game \( G_k(S) \) in which:

I and II play a valid extension \( \gamma^*, a_k^\infty, y_k \). In addition I tries to witness that the extended sequence, \( S—, y_k, a_k^\infty, \gamma^* \), belongs to \( A_{k+1} \). II tries to witness the opposite.
\[ G_k(S): \]

<table>
<thead>
<tr>
<th></th>
<th>( \gamma^* ), ( a_k^\infty )</th>
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<th>( a_{\infty-1} )</th>
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<td>( a_{\infty-0} )</td>
</tr>
<tr>
<td>II</td>
<td>( a_k^\infty )</td>
<td>( y_k(1) )</td>
<td>( a_{\infty-1} )</td>
<td>( a_{\infty-0} )</td>
</tr>
</tbody>
</table>

I and II play

- \( \gamma^* \),
- \( a_k^\infty = \langle a_k^\infty, a_k^\infty \rangle \), and
- \( y_k = \langle y_k(0), y_k(1), \ldots \rangle \)

which form a valid extension of \( S \). (In particular \( \gamma^* \) is **smaller** than \( \gamma \).)

In addition they play auxiliary moves in the game \( A_{k+1}[S \longrightarrow, y_k, a_k^\infty, \gamma^*] \).

II is the closed player; she wins if she can last \( \omega \) moves. Otherwise I wins.
Define the sets $A_k$ by:

$$S \in A_k \text{ iff } I \text{ has a winning strategy in } G_k(S)$$

(for a $k$-sequence $S \in M[g_k]$).

If $S$ belongs to $A_k$, we expect to be able to extend to $S^* = S\langle *, \ldots, *, \gamma \rangle$ which belongs to a “shift” of $A_{k+1}$.

Our definition of $A_k$ depends on some knowledge of $A_{k+1}$. (We need knowledge of $G_k$, which involves the auxiliary game $A_{k+1}$.)

The definition is by induction, not on $k$, but on $\gamma$.

Figuring out the rules of $G_k(S)$, where $S = \langle *, \ldots, *, \gamma \rangle$, requires knowledge of the sets $A_{k+1}[a_0^\infty, \ldots, a_k^\infty, \gamma^*]$, but only for $\gamma^* < \gamma$.

Determining whether $S$ belongs to $A_k$ thus requires knowledge of $A_{k+1}$, but only for $k + 1$-sequences ending with $\gamma^* < \gamma$. 

A 0-sequence is simply an ordinal $\gamma$. We have for each $\gamma$ the game $G_0(\gamma)$. This game belongs to $M$.

**Case 1**: There exists some $\gamma$ so that (in $M$) $I$ has a winning strategy in $G_0(\gamma)$.

We will show that (in $V$) $I$ has a winning strategy in $G_{\omega \cdot \omega}(C)$.

Fix $\Sigma_0 \in M$, a winning strategy for $I$ (the open player) in $G_0(\gamma)$.

Fix an imaginary opponent, playing for $II$ in $G_{\omega \cdot \omega}(C)$.

We will use $\Sigma_0$, the strategies $\sigma_{piv-1}, \sigma_{piv-2}, \ldots$, the strategy $\sigma_{piv-\infty}$, and an iteration strategy for $M$, to play against the imaginary opponent.
\( G_0(\gamma):\)

<table>
<thead>
<tr>
<th>I</th>
<th>(\gamma_0^*, a_{0-I}^\infty)</th>
<th>(y_0(0))</th>
<th>(a_{0-I}^1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>(a_{0-II}^\infty)</td>
<td>(a_{0-II}^1)</td>
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\[
y_0(1) \quad a_{1-II}^1 \quad y_0(2) \quad \ldots
\]

Our opponent, \(\Sigma_0, \sigma_{\text{piv} - \infty}\) (for the first round), and \(\sigma_{\text{piv} - 1}\) (for the remaining rounds) cover all moves in the game.

We obtain an iteration tree \(\mathcal{U}^0\) of length 3, played by \(\sigma_{\text{piv} - \infty}\), with final model \(P_2^0\), embedding \(\pi_{0,2}^0: M \to P_2^0\), and moves \(\tilde{\gamma}_0^*, \tilde{a}_0^\infty\) in \(P_2^0\).

We obtain \(y_0 \in \mathbb{R}\), and an iteration tree \(T_0\) (played by \(\sigma_{\text{piv} - 1}\)) with illfounded even model.

The iteration strategy picks an odd branch, \(b_0\) say. Let \(M_1\) be the direct limit along \(b_0\) and let \(j_{0,1}\) be the direct limit embedding.
Let $\mathcal{U}^1 = j_{0,1}(\mathcal{U}^0)$, and similarly with $P_2^1$, $\pi_0^{1,2}$. Let $\gamma_*^0 = j_{0,1}(\tilde{\gamma}_0^*)$ and similarly $a_\infty^0$.

Our use of $\sigma_{\text{piv}^{-1}}$ guarantees that there exists some $h_1$ so that

1. $h_1$ is $\text{col}(\omega, \delta_1^s)$–generic/$M_1$, and
2. $\langle y_0, a_\infty^0, \gamma_*^0 \rangle \in \dot{A}_1^s[h_1]$.

(*$^s$ denotes $j_{0,1}(\pi_0^{0,2}(*)$).)

Note that by 2, player I (the open player) has a winning strategy in $G_1^s(y_0, a_\infty^0, \gamma_*^0)$. Fix $\Sigma_1 \in M_1[h_1]$, a strategy for I witnessing this.
\[
\begin{array}{c|ccc}
I & \gamma_1^*, a_{1-I}^\infty & y_1(0) & a_{0-I}^2 \\
II & a_{1-II}^\infty & a_{0-II}^2 & \\
\hline & a_{1-I}^2 & y_1(2) & \ldots \\
& y_1(1) & a_{1-II}^2 & \\
\end{array}
\]

Note, \(\Sigma_1 \) belongs to \(M_1[h_1]\), a \textbf{small} generic extension of \(M\). (Small with respect to \(\delta_2\) and \(\delta_\infty\).) This allows us to shift \(\Sigma_1\) along the even branch of trees given by \(\sigma_{\text{piv}-\infty}\) and \(\sigma_{\text{piv}-2}\).

Using \(\Sigma_1\) and \(j_{0,1}(\sigma_{\text{piv}-\infty})\) we get

\[
M \begin{array}{c} b_0 \end{array} M_1 = P_0^1 \begin{array}{c} \pi_{0,2}^0 \end{array} P_1^1 \begin{array}{c} \pi_{2,4}^0 \end{array} P_2^1 \begin{array}{c} \pi_{2,4}^0 \end{array} P_3^1 \begin{array}{c} \pi_{0,4}^2 \end{array} P_4^1
\]

Then using \(j_{0,1}(\sigma_{\text{piv}-2})\) and \textbf{shifts} of \(\Sigma_1\) get

\[
M \begin{array}{c} b_0 \end{array} M_1 \begin{array}{c} b_1 \end{array} M_2 = P_0^2 \begin{array}{c} \pi_{0,4}^2 \end{array} P_4^2
\]

(where \(P_0^2 = j_{1,2}(P_0^1)\), etc.).
We get $\gamma_1^*, a_1^\infty$, and $y_1$. Our use of $\sigma_{\text{piv} - 2}$ guarantees that there exists $h_2$ so that

1. $h_2$ is col$(\omega, \delta_2^s)$–generic/$M_2$, and

2. $\langle y_0, y_1, a_0^{\infty-s}, a_1^\infty, \gamma_1^* \rangle \in \dot{A}_2^s[h_2]$. (A second $^s$ stands for application of $j_{1,2} \circ \pi_{2,4}^1$.)

By 2, player I (the open player) wins $G_2^s(y_0, y_1, a_0^{\infty-s}, a_1^\infty, \gamma_1^*)$.

This game belongs to $M[h_2]$. Fix $\Sigma_2 \in M[h_2]$, a strategy witnessing that I wins.

Continue as before.
In general we have:

\[
M_k = P_0^k \
\]

\[
\pi_{0,2k}^k 
\]

\[
P_{2k}^k 
\]

\[
\pi_{2k,2k+2}^k 
\]

\[
P_{2k+1}^k 
\]

\[
P_{2k+2}^k 
\]

\[
T_k 
\]

\[
j_{k,k+1} 
\]

\[
M_{k+1} = P_0^{k+1} 
\]

\[
\pi_{0,2k+2}^{k+1} 
\]

\[
j_{k,k+1} 
\]

In \( P_{2k}^k \) we have the \( k \)-sequence

\[
S_k = \langle y_0, \ldots, y_{k-1}, a_0^\infty-s \ldots s, \ldots, a_{k-1}^\infty-\ldots-\gamma_{k-1}^* \rangle.
\]

\( S_{k+1} \) (in \( P_{2k+2}^{k+1} \)) is obtained as a valid extension of \( j_{k,k+1}(\pi_{2k,2k+2}^k(S_k)) \). In particular:

\( \uparrow \) \( \gamma_{k}^* \) is smaller than \( j_{k,k+1}(\pi_{2k,2k+2}^k(\gamma_{k-1}^*)) \).
We end with a sequence of reals $\langle y_n \mid n < \omega \rangle$, a sequence of iteration trees

$$
\begin{align*}
M_0 & \xrightarrow{b_0} M_1 \xleftarrow{b_1} M_2 \quad \cdots \quad \longrightarrow M_\infty \\
\text{T}_0 & \quad \text{T}_1
\end{align*}
$$

and an iteration tree $\mathcal{U}_\infty$ on $M_\infty$ as follows:

$$
M_\infty = P_0^\infty \xrightarrow{\pi^\infty_{0,2}} P_1^\infty \xrightarrow{\pi^\infty_{2,4}} P_2^\infty \xrightarrow{\pi^\infty_3} P_3^\infty \xrightarrow{\pi^\infty_4} \cdots
$$

By (†) the even branch of $\mathcal{U}_\infty$ is illfounded.

The iteration strategy for $M$ produces an odd branch $c$ of $\mathcal{U}_\infty$. Let $M_c$ be the direct limit, and let $\pi_c : M_\infty \to M_c$ be the direct limit embedding. Note $M_c$, played by an iteration strategy, is wellfounded.
Now $\mathcal{U}_\infty$ is part of a play according to $j_{0,\infty}(\sigma_{\text{piv}-\infty})[y_n \mid n < \omega]$.

Our use of $j_{0,\infty}(\sigma_{\text{piv}-\infty})[y_n \mid n < \omega]$ guarantees that there exists some $h_\infty$ so that

1. $h_\infty$ is $\text{col}(\omega, \pi_c(j_{0,\infty}(\delta_\infty)))$–generic$/M_c$, and
2. $\langle y_n \mid n < \omega \rangle \in \pi_c(j_{0,\infty}(\dot{A}_\infty))[h_\infty]$.

From 2 we see that $\langle y_n \mid n < \omega \rangle$ satisfies the $\Sigma^1_2$ statement $\phi$, inside $M_c[h_\infty]$.

By absoluteness $\phi$ is satisfied in $V$.

So $\langle y_n \mid n < \omega \rangle \in C$ and I won, as required.

□(Case 1.)

Assuming there is some $\gamma$ so that (in $M$) I wins $G_0(\gamma)$, we showed that (in $V$) I wins $G_{\omega \cdot \omega}(C)$. 

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Fix $\gamma_L < \gamma_H$ indiscernibles for $M$, above $\delta_\infty$.

Suppose $S = \langle y_0, \ldots, y_{k-1}, a_0^\infty, \ldots, a_{k-1}^\infty, \gamma_L \rangle$ is a $k$-sequence in $M[g_k]$ and does not belong $A_k$.

So II wins $G_k(S)$. By indiscernibility II also wins $G_k(S_H)$ where $S_H = \langle *, \ldots, \gamma_H \rangle$. Fix a winning strategy $\Sigma_{II-k}$.

$$
\begin{array}{c|ccc}
I & \gamma^*, a_{k-1}^\infty & y_k(0) & a_{0-1}^{k+1} \\
II & a_k^\infty & y_k(2) & \ldots \\
 & a_{1-1}^{k+1} & a_{1-2}^{k+1} & y_k(1) \\
\end{array}
$$

Play $\gamma^* = \gamma_L$. Use $\Sigma_{II-k}$, $\sigma_{gen-\infty}$, $\sigma_{gen-(k+1)}$ to obtain $a_k^\infty$, $a_{k+1}$, and (half of) $y_k$ in $M[g_{k+1}]$.

Our use of $\sigma_{gen-(k+1)}$ guarantees that $S' = \langle S_H --, y_k, a_k^\infty, \gamma_L \rangle \notin A_{k+1}[g_{k+1}]$.

If $S'$ belongs to $M[g_{k+1}]$ this means that II wins $G_{k+1}(S')$. 

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Continue this way. Our use of $\sigma_{\text{gen} - \infty}$ guarantees that $\langle y_n \mid n < \omega \rangle$ does not belong to $\dot{A}[g_\infty]$.

If there is $\gamma$ so that I wins the closed game $G_0(\gamma)$, then I has a winning strategy in $G_{\omega \cdot \omega}(C)$.

Mirroring this with sets $B_k$ and games $H_k$ we get:

If there is $\gamma$ so that II wins the closed game $H_0(\gamma)$, then II has a winning strategy in $G_{\omega \cdot \omega}(C)$.

Finally, if II wins $G_0(\gamma_L)$ and I wins $H_0(\gamma_L)$, we can work in $M[g_n]$, $n < \omega$, and produce $\langle y_n \mid n < \omega \rangle \in M[g_\infty]$ * which belongs to neither $\dot{A}[g_\infty]$ nor $\dot{B}[g_\infty]$, a contradiction.

It follows that $G_{\omega \cdot \omega}(C)$ is determined.

*Note $\langle g_n \mid n < \omega \rangle \in M[g_\infty]$. 24