1. Preliminaries:
   (a) The games.
   (b) Extenders, iteration trees.
   (c) Auxiliary game representations.
   (d) Example: $\Sigma^1_2$ determinacy.

2. Games of length $\omega \cdot \omega$ with $\Sigma^1_2$ payoff.

3. Continuously coded games with $\Sigma^1_2$ payoff.
Let $C \subset \mathbb{R}^{<\omega_1}$ be given. Let $f : \mathbb{R} \to \mathbb{N}$, a partial function, be given. $G_{\text{cont}}(C)$ is played as follows:

\[
\begin{array}{c|cccc}
I & \ldots & y_\alpha(0) & y_\alpha(2) \\
II & & y_\alpha(1) & y_\alpha(3) & \ldots
\end{array}
\]

In round $\alpha$, I and II alternate playing natural numbers $y_\alpha(i)$, $i < \omega$, producing a real $y_\alpha$.

If $f(y_\alpha)$ is not defined, the game ends. I wins iff $\langle y_0, y_1, \ldots, y_\alpha \rangle \in C$.

Otherwise we set $n_\alpha = f(y_\alpha)$. If there exists $\xi < \alpha$ so that $n_\alpha = n_\xi$, the game ends. Again I wins iff $\langle y_0, y_1, \ldots, y_\alpha \rangle \in C$.

Otherwise the game continues.

The game ends at a countable $\alpha$; the map $\xi \mapsto n_\xi$ embeds $\alpha$ into $\mathbb{N}$. This map is produced continuously in $\xi$. The game is said to have \textit{continuously coded length}.

*Following standard abuse of notation, we use $\mathbb{R}$ to denote $\mathbb{N}^\omega$.\]
Let \( C \subset \mathbb{R}^\omega = \mathbb{N}^{\omega \cdot \omega} \) be given. In \( G_{\omega \cdot \omega}(C) \) the players play \( \omega \) rounds as follows, producing \( y_k \in \mathbb{R} \) for \( k < \omega \).

\[
\begin{array}{ccc}
I & y_0(0) & \cdots & y_1(0) & \cdots \\
II & y_0(1) & & y_1(1) & \cdots \\
\end{array}
\]

I wins iff \( \langle y_k \mid k < \omega \rangle \) belongs to \( C \).

Let \( C \subset \mathbb{R} = \mathbb{N}^\omega \) be given. In \( G_\omega(C) \) the players play one round as follows, producing \( y \in \mathbb{R} \).

\[
\begin{array}{ccc}
I & y(0) & y(2) & \cdots \\
II & y(1) & y(3) & \cdots \\
\end{array}
\]

I wins iff \( y \in C \).
We intend to prove that $G_{\text{cont-}f}(C)$ are determined, for all continuous $f$ and all $\Sigma^1_2$ payoff sets $C$.

As an illustrative case we will first prove that $G_{\omega \cdot \omega}(C)$ are determined, for all $\Sigma^1_2$ payoff sets $C$.

Before that, we will prove that $G_\omega(C)$ are determined for all $\Sigma^1_2$ sets $C \subset \mathbb{R}$.

Determinacy for games of length $\omega$ was proved by Martin and Steel.

Determinacy for games of fixed length $\omega \cdot \alpha$, $\alpha$ limit, was proved by Woodin.

Determinacy for games of continuously coded length was proved by Neeman.
An **extender** on $\kappa$ is a directed system of measures on $\kappa$. If $E$ is an extender on $\kappa$, we use $\text{dom}(E)$ to denote $\kappa$.

An extender $E$ allows us to form an **ultrapower** of $\mathcal{V}$, denoted $\text{Ult}(\mathcal{V}, E)$, and an elementary **ultrapower embedding** $\pi : \mathcal{V} \rightarrow \text{Ult}(\mathcal{V}, E)$.

We use $P, Q, M, N$ to denote models of ZFC.

We say that $Q$ and $Q^*$ **agree** to $\kappa$ if $\mathcal{P}(\kappa) \cap Q^* = \mathcal{P}(\kappa) \cap Q$.

Suppose $Q \models "E \text{ is an extender on } \kappa"$. Suppose $Q^*$ and $Q$ agree to $\kappa$. Then $E$ can be applied also to $Q^*$: We can form the **ultrapower** $\text{Ult}(Q^*, E)$, and an elementary **ultrapower embedding** $\sigma : Q^* \rightarrow \text{Ult}(Q^*, E)$.

$\text{Ult}(Q^*, E)$ needn’t always be wellfounded. If it is wellfounded, we assume it’s transitive.
An **iteration tree** $T$ of length $\omega$ consists of

- a tree order $T$ on $\omega$,
- a sequence of models $\langle M_k \mid k < \omega \rangle$, and
- embeddings $j_{k,l} : M_k \rightarrow M_l$ for $k T l$.

Each model $M_{l+1}$ for $l + 1 > 0$ is an ultrapower of a preceeding model. More precisely: $M_{l+1} = \text{Ult}(M_k, E_l)$, where $E_l$ an extender picked from $M_l$, and $k \leq l$ is the $T$ predecessor of $l + 1$. $j_{k,l+1}$ is the ultrapower embedding.

\[
\begin{array}{c}
M_{l+1} \\
j_{k,l+1} \\
M_k
\end{array}
\]

$E_l \in M_l$

$(M_l$ and $M_k$ must agree to $\text{dom}(E_l)$.)

An iteration tree on $M$ is a tree with $M_0 = M$. 

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Our trees will generally have an **even branch**, \( M_0, M_2, M_4, \ldots \), giving rise to the direct limit \( M_{\text{even}} \).

The tree structure on the odd models will usually be some permutation of \( \omega^{<\omega} \). With each **odd branch** \( b \) we associate the direct limit \( M_b \).

(In this example, \( 0 T 1, 0 T 2, 1 T 3, 0 T 3 \), etc.)
In the iteration game* on $M$, players “good” and “bad” collaborate to produce a sequence of iteration trees as follows:

```
M ←_\mathcal{T}_0 \rightarrow b_0 M_1 ←_\mathcal{T}_1 \rightarrow b_1 M_2 ←_\mathcal{T}_2 \rightarrow b_2 M_3 ---->

----> M_\omega ←_\mathcal{T}_\omega \rightarrow b_\omega M_{\omega+1} ---->
```

“Bad” plays an iteration tree $\mathcal{T}_\xi$ on $M_\xi$. “Good” plays a branch $b_\xi$ through $\mathcal{T}_\xi$. We let $M_{\xi+1}$ be the direct limit model determined by $b_\xi$ and proceed to the next round. For limit $\lambda$ we let $M_\lambda$ be the direct limit of $M_\xi$ for preceding $\xi$. We start with $M_0 = M$.

If ever a model $(M_\xi, \xi < \omega_1)$ is reached which is illfounded, “bad” wins. Otherwise “good” wins.

*The definition given here is specialized to our context. The concept of iteration games is due to Martin–Steel.
We also consider iteration games were round $\xi$ has the following form:

```
M_\xi \xleftarrow{b_\xi} P_\xi \xrightarrow{E_\xi} M_{\xi+1}
```

“Bad” plays an iteration tree $T_\xi$ on $M_\xi$. “Good” plays a branch $b_\xi$, giving rise to the direct limit, $P_\xi$.

Then “good” plays an extender $E_\xi$ in $P_\xi$, with $\text{dom}(E_\xi)$ within the level of agreement between $M_\xi$ and $P_\xi$. We set $M_{\xi+1} = \text{Ult}(M_\xi, E_\xi)$ and continue to the next round.

If ever a model ($P_\xi$ or $M_\xi$, $\xi < \omega_1$) is reached which is illfounded, “bad” wins. Otherwise “good” wins.

We refer to this game too as an **iteration game**.
$M$ is **iterable** if the good player has a winning strategy for each of the iteration games described above. We refer to such winning strategies as **iteration strategies**.

Countable elementary substructures of $V$ are iterable in this sense (Martin–Steel).
Suppose $M \models \text{“}\delta \text{ is a Woodin cardinal”}\text{, and in } V\text{ there are } M\text{-generics for } \text{col}(\omega, \delta)$. Let $\dot{A}$ name a set of reals in $M^{\text{col}(\omega, \delta)}$.

Work with some $x \in \mathbb{R}$. We work to define an auxiliary game, $A[x]$, of $\omega$ moves, taken from $M$. In this game I tries to witness that $x \in \dot{A}[h]$ for some generic $h$. II tries to witness the opposite.

The auxiliary game is played as follows:

<table>
<thead>
<tr>
<th>I</th>
<th>$\ldots$ $l_n, X_n, p_n$ $\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>$\mathcal{F}_n, \mathcal{D}_n$ $\ldots$</td>
</tr>
</tbody>
</table>

In round $n$ I plays

- $l = l_n$, a number $< n$, or $l_n =$ “new”.
- $X_n$, a set of names for reals of $M^{\text{col}(\omega, \delta)}$.
- $p_n$, a condition in $\text{col}(\omega, \delta)$.

II plays

- $\mathcal{F}_n$ a function from $X_n$ into the ordinals.
- $\mathcal{D}_n$, a function from $X_n$ into $\{\text{dense sets in } \text{col}(\omega, \delta)\}$.
\[ \mathcal{A}[x] : \begin{array}{c|c c c} & I & \ldots & l_n, \mathcal{X}_n, p_n & \ldots \\ \hline II & & & \mathcal{F}_n, \mathcal{D}_n & \ldots \end{array} \]

If \( l_n = \text{“new”} \) we make no requirements on I. Otherwise, we require \( p_n < p_l \) and \( \mathcal{X}_n \subset \mathcal{X}_l \). We further require that for every name \( \dot{x} \in \mathcal{X}_n \):

1. \( p_n \) forces \( \dot{x} \in \dot{\mathcal{A}} \).
2. \( p_n \) forces \( \dot{x}(0) = \tilde{x}_0 \), \ldots, \( \dot{x}(l) = \tilde{x}_l \).
3. \( p_n \) belongs to \( \mathcal{D}_l(\dot{x}) \).

We make the following requirement on II:

4. For every name \( \dot{x} \in \mathcal{X}_n \), \( \mathcal{F}_n(\dot{x}) < \mathcal{F}_l(\dot{x}) \).

If there is \( h \) so that \( x \in \dot{\mathcal{A}}[h] \), I can pick a name for \( x \), play \( \mathcal{X}_n \) containing this name, and play \( p_n \in h \). Condition 4 ensures defeat for II.

On the other hand, if there is an infinite run of \( \mathcal{A}[x] \) where I covered all possible names and chains of conditions, condition 4 ensures that \( x \not\in \dot{\mathcal{A}}[h] \) for all generic \( h \).
**Note 1.** Rather than play the sets \( \mathcal{X}_n \) directly, I plays their *type*. I plays \( \kappa_n < \delta \), and a set \( u_n \) of formulae with parameters in \( M \| \kappa_n \cup \{ \kappa_n, \delta, \dot{A} \} \).* We take \( \mathcal{X}_n \) to be the set of names which satisfy all these formulae.

The fact that this still allows I enough control over her choice of \( \mathcal{X}_n \) has to do with our assumption that \( \delta \) is a Woodin cardinal.

\( \mathcal{F}_n \) and \( \mathcal{D}_n \) are played similarly.

Observe that moves in \( A[x] \) are therefore elements of \( M \| \delta \).

**Note 2.** The association \( x \mapsto A[x] \) is continuous: The rules governing the first \( n + 1 \) rounds of \( A[x] \) depend only on \( x \upharpoonright n \).

We in fact defined an association \( s \mapsto A[s] \) (\( s \in \omega^{<\omega} \), \( A[s] \) a game of \( \text{lh}(s) + 1 \) many rounds). This association belongs to \( M \).

*By \( M \| \kappa_n \) we mean \( V^{M}_{\kappa_n} \).
Recall that \( g \) is \( \text{col}(\omega, \delta) \)-generic/\( M \). We alternate between thinking of \( g \) as a generic enumeration of \( \delta \), and as a generic enumeration of \( M\|\delta \).

Let \( \sigma_{\text{gen}}[x, g] \), a strategy for I in \( A[x] \) be defined as follows:

\( \sigma_{\text{gen}}[x, g] \) plays in each round the first (with respect to the enumeration \( g \)) legal move.

**Note.** The association \( x, g \mapsto \sigma_{\text{gen}}[x, g] \) is continuous.

**Lemma 1.** Suppose that there exists an infinite run of \( A[x] \), played according to \( \sigma_{\text{gen}}[x, g] \). Then \( x \notin \dot{A}[g] \). (This is only useful if \( x \in M[g] \).)

Proof: In playing for I, \( \sigma_{\text{gen}}[g, x] \) goes over all possible names and all possible generics. (This uses the genericity of the enumeration \( g \).) So in fact \( x \notin \dot{A}[h] \) for all generic \( h \). \( \square \)
We wish to phrase a similar lemma with a strategy for II, which puts $x$ in $A$. To do this we have to give II additional control. We let II “shift” the play board along an even branch of an iteration tree.

\[
M = M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow M_4 \rightarrow M_5 \rightarrow M_6 \rightarrow \ldots
\]
The game $\mathcal{A}^*[x]$ is played as follows:

<table>
<thead>
<tr>
<th>I</th>
<th>$\ldots$</th>
<th>$l_n, x_n, p_n$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>$E_{2n}, E_{2n+1}, F_n, D_n$</td>
<td>$\ldots$</td>
<td></td>
</tr>
</tbody>
</table>

At the start of round $n$ we have a model $M_{2n}$, an embedding $j_{0,2n}: M \rightarrow M_{2n}$, and a position $P_n$ of $n$ rounds in $j_{0,2n}(\mathcal{A})[x]$.

I plays $l_n, x_n, p_n$, a legal move in $j_{0,2n}(\mathcal{A})[x]$ following $P_n$.

II plays extenders $E_{2n}, E_{2n+1}$ giving rise to models $M_{2n+1}, M_{2n+2}$, and to an embedding $j_{2n,2n+2}: M_{2n} \rightarrow M_{2n+2}$. (The $T$–predecessor of $2n + 1$ is $2l_n + 1$ if $l_n \neq \text{“new”}$ and $2n$ otherwise.)

We let $Q_n = j_{2n,2n+2}(P_n \leftarrow, l_n, x_n, p_n)$. (This is the “shifting” mentioned before.)

II plays $F_n, D_n$, a legal move in $j_{0,2n+2}(\mathcal{A})[x]$ following $Q_n$.

We let $P_{n+1} = Q_n \leftarrow, F_n, D_n$ and proceed to the next round.
**Definition.** A **pivot** for $x$ is a pair $\mathcal{T}, \vec{a}$ so that

1. $\mathcal{T}$ is an iteration tree on $M$, with an even branch.
2. $\vec{a}$ is a run of $j_{\text{even}}(A)[x]$.
3. For every odd branch $b$ of $\mathcal{T}$, there exists some $h$ so that
   
   (a) $h$ is $\text{col}(\omega, j_b(\delta))$–generic$/M_b$; and
   (b) $x \in j_b(A)[h]$.

Any run of $A^*[x]$ produces $\mathcal{T}, \vec{a}$ which satisfy conditions 1 and 2.

**Lemma 2.** There exists $\sigma_{\text{piv}}[x, g]$, a strategy for II in $A^*[x]$, so that every run according to $\sigma_{\text{piv}}[x, g]$ is a pivot.

The association $x, g \mapsto \sigma_{\text{piv}}[x, g]$ is continuous.

The proof of Lemma 2 draws heavily on the techniques of Martin–Steel’s “A proof of projective determinacy”. The assumption that $\delta$ is a Woodin cardinal is crucial.
To sum: Have continuous associations
\[ x \mapsto A[x]; \quad x, g \mapsto \sigma_{\text{gen}}[x, g]; \quad x \mapsto A^*[x]; \quad \text{and} \]
\[ x, g \mapsto \sigma_{\text{piv}}[x, g]. \]

\( \sigma_{\text{gen}}[x, g] \) is a strategy for I in \( A[x] \).

If \( \bar{a} \) is an infinite run of \( A[x] \) according to \( \sigma_{\text{gen}}[x, g] \), then \( x \notin \dot{A}[g] \).

\( \sigma_{\text{piv}}[x, g] \) is a strategy for II in \( A^*[x] \).

If \( \mathcal{T}, \bar{a} \) is an infinite run of \( A^*[x] \) according to \( \sigma_{\text{piv}}[x, g] \), then

for every odd branch \( b \) of \( \mathcal{T} \), there exists some \( h \) so that

- \( h \) is \( \text{col}(\omega, j_b(\delta)) \)-generic/\( M_b \); and
- \( x \in j_b(\dot{A})[h] \).
\( \Sigma^1_2\) determinacy:

Fix \( A \subset \mathbb{R} \), a \( \Sigma^1_2 \) set (say the set of reals which satisfy a given \( \Sigma^1_2 \) statement \( \phi \)).

Suppose there is an iterable class model \( M \) with a Woodin cardinal \( \delta \). Suppose that (in \( V \)) there is \( g \) which is \( \text{col}(\omega, \delta)\)–generic/\( M \).

We intend to prove that (in \( V \)) \( G_\omega(A) \) is determined.

Let \( \dot{A} \in M \) name \( A \). More precisely, \( \dot{A} \) names the set of reals of \( M^{\text{col}(\omega, \delta)} \) which satisfy \( \phi \) in \( M^{\text{col}(\omega, \delta)} \).

We have \( x \mapsto A[x], x, g \mapsto \sigma_{\text{gen}}[x, g] \), etc. as before.
Let $G$ be the following game, defined and played inside $M$:

<table>
<thead>
<tr>
<th>I</th>
<th>$x_0$</th>
<th>$a_{0-I}$</th>
<th>$a_{1-I}$</th>
<th>$x_2$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td></td>
<td>$a_{0-II}$</td>
<td>$x_1$</td>
<td>$a_{1-II}$</td>
<td></td>
</tr>
</tbody>
</table>

I and II alternate playing natural numbers, producing together $x = \langle x_0, x_1, \ldots \rangle \in \mathbb{R}$. In addition they play moves $a_{0-I}, a_{0-II}, \ldots$ in $\mathcal{A}[x]$.

II is the closed player; she wins if she can last all $\omega$ moves. Otherwise I wins.

$G$ is a closed game, hence determined. A winning strategy exists in $M$.

**Case 1**: I wins $G$. Fix $\Sigma \in M$ a winning strategy for I (the open player).

We wish to show that I wins $G_\omega(A)$ in V. Let us play $G_\omega(A)$ against an imaginary opponent. We describe how to play, and win.
We construct a run \( x \in \mathbb{R} \) of \( G_\omega(A) \). At the same time we construct \( T, \bar{a} \), a run of \( A^*[x] \).

The participants in our construction are:

- The imaginary opponent: playing \( x_n \) for odd \( n \).
- The strategy \( \sigma_{\text{piv}}[g,x] \): playing for II in \( A^*[x] \).
- The strategy \( \Sigma \) and its shifts along the even branch of \( T \): playing \( x_n \) for even \( n \) and playing for I in \( A^*[x] \) (i.e. playing for I in shifts of \( A[x] \)).

We obtain \( x \in \mathbb{R} \) and \( T, \bar{a} \) a run of \( A^*[x] \) according to \( \sigma_{\text{piv}}[x,g] \).

We must check that \( x \) belongs to \( A \).
\[ M = M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow M_4 \rightarrow M_5 \rightarrow M_6 \rightarrow \cdots \]

\[ \Sigma \xrightarrow{x_0} \]

\[ \Sigma \xrightarrow{l_0} X_0 \xrightarrow{p_0} \]

\[ \sigma_{\text{piv}} \quad \begin{array}{c} F_0 \\ D_0 \end{array} \]

\[ \text{Oppnt} \xrightarrow{x_1} \]

\[ j_{0,2}(\Sigma) \xrightarrow{l_1} X_1 \xrightarrow{p_1} \]

\[ \sigma_{\text{piv}} \quad \begin{array}{c} F_1 \\ D_1 \end{array} \]

\[ j_{0,4}(\Sigma) \xrightarrow{x_2} \]

\[ j_{0,4}(\Sigma) \xrightarrow{l_2} X_2 \xrightarrow{p_2} \]

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Note that $x, \bar{a}$ is an infinite run of $j_{\text{even}}(G)$ according to $j_{\text{even}}(\Sigma)$.

Now $\Sigma$ is a strategy for the open player in $G$. So there are no infinite runs according to $\Sigma$. But there is an infinite run according to $j_{\text{even}}(\Sigma)$. Thus $M_{\text{even}}$ is illfounded.

$M$ is iterable. So there exists some branch $b$ of $T$ so that $M_b$ is wellfounded. $b$ must be an odd branch.

By Lemma 2, $T$, $\bar{a}$ is a pivot for $x$. Thus there is $h$ so that

- $h$ is $\text{col}(\omega, j_b(\delta))$--generic/$M_b$ and
- $x \in j_b(\dot{A})[h]$.

This means that in $M_b[h]$, $x$ satisfies the $\Sigma^1_2$ statement $\phi$.

By absoluteness, $x$ satisfies $\phi$ in $V$. (This uses the wellfoundedness of $M_b$.)

So $x \in A$ as required. $\square$(Case 1.)