Determinacy Proofs for Long games

Itay Neeman

Department of Mathematics
University of California Los Angeles
Los Angeles, CA 90095-1555
ineeman@math.ucla.edu

1. Preliminaries:
   (a) The games.
   (b) Extenders, iteration trees.
   (c) Auxiliary game representations.
   (d) Example: Σ^1_2 determinacy.

2. Games of length $\omega \cdot \omega$ with Σ^1_2 payoff.

3. Continuously coded games with Σ^1_2 payoff.

Let $C \subset \mathbb{R}^{<\omega_1}$ be given.* Let $f\colon \mathbb{R} \to \mathbb{N}$, a partial function, be given. $G_{\text{cont}-f}(C)$ is played as follows:

$$\begin{array}{c|ccccccc} \mathrm{I} & \ldots\ldots\ldots & y_\alpha(0) & & y_\alpha(2) & & & \\ \hline \mathrm{II} & & & y_\alpha(1) & & y_\alpha(3) & \cdots \end{array}$$

In round α, I and II alternate playing natural numbers $y_\alpha(i)$, $i < \omega$, producing a real y_α.

If $f(y_\alpha)$ is not defined, the game ends. I wins iff $\langle y_0, y_1, \ldots\ldots, y_\alpha \rangle \in C$.

Otherwise we set $n_\alpha = f(y_\alpha)$. If there exists $\xi < \alpha$ so that $n_\alpha = n_\xi$, the game ends. Again I wins iff $\langle y_0, y_1, \ldots\ldots, y_\alpha \rangle \in C$.

Otherwise the game continues.

The game ends at a countable α; the map $\xi \mapsto n_\xi$ embeds α into $\mathbb{N}$. This map is produced continuously in ξ. The game is said to have **continuously coded length**.

*Following standard abuse of notation, we use $\mathbb{R}$ to denote $\mathbb{N}^\omega$.

Let $C \subset \mathbb{R}^\omega = \mathbb{N}^{\omega \cdot \omega}$ be given. In $G_{\omega \cdot \omega}(C)$ the players plays ω rounds as follows, producing $y_k \in \mathbb{R}$ for $k < \omega$.

$$\begin{array}{c|cccccc}
\mathrm{I} & y_0(0) & & \ldots\ldots & y_1(0) & & \ldots \\
\hline
\mathrm{II} & & y_0(1) & & & y_1(1) & \ldots
\end{array}$$

I wins iff $\langle y_k \mid k < \omega \rangle$ belongs to C.

Let $C \subset \mathbb{R} = \mathbb{N}^\omega$ be given. In $G_\omega(C)$ the players plays one round as follows, producing $y \in \mathbb{R}$.

$$\begin{array}{c|cccccc}
\mathrm{I} & y(0) & & y(2) & & \ldots & \\
\hline
\mathrm{II} & & y(1) & & y(3) & & \ldots
\end{array}$$

I wins iff $y \in C$.

We intend to prove that $G_{\text{cont}-f}(C)$ are determined, for all continuous f and all $\mathbf{\Sigma}^1_2$ payoff sets C.

As an illustrative case we will first prove that $G_{\omega\cdot\omega}(C)$ are determined, for all $\mathbf{\Sigma}^1_2$ payoff sets C.

Before that, we will prove that $G_\omega(C)$ are determined for all $\mathbf{\Sigma}^1_2$ sets $C \subset \mathbb{R}$.

Determinacy for games of length ω was proved by Martin and Steel.

Determinacy for games of fixed length $\omega \cdot \alpha$, α limit, was proved by Woodin.

Determinacy for games of continuously coded length was proved by Neeman.

An **extender** on κ is a directed system of measures on κ. If E is an extender on κ, we use dom(E) to denote κ.

An extender E allows us to form an **ultrapower** of V, denoted Ult(V, E), and an elementary **ultrapower embedding** $\pi\colon \mathrm{V} \to$ Ult(V, E).

We use P, Q, M, N to denote models of ZFC.

We say that Q and Q^* **agree** to κ if $\mathcal{P}(\kappa) \cap Q^* = \mathcal{P}(\kappa) \cap Q$.

Suppose $Q \models$ "E is an extender on κ". Suppose Q^* and Q agree to κ. Then E can be applied also to Q^*: We can form the **ultrapower** Ult(Q^*, E), and an elementary **ultrapower embedding** $\sigma\colon Q^* \to$ Ult(Q^*, E).

Ult(Q^*, E) needn't always be wellfounded. If it is wellfounded, we assume it's transitive.

An **iteration tree** $\mathcal{T}$ of length ω consists of

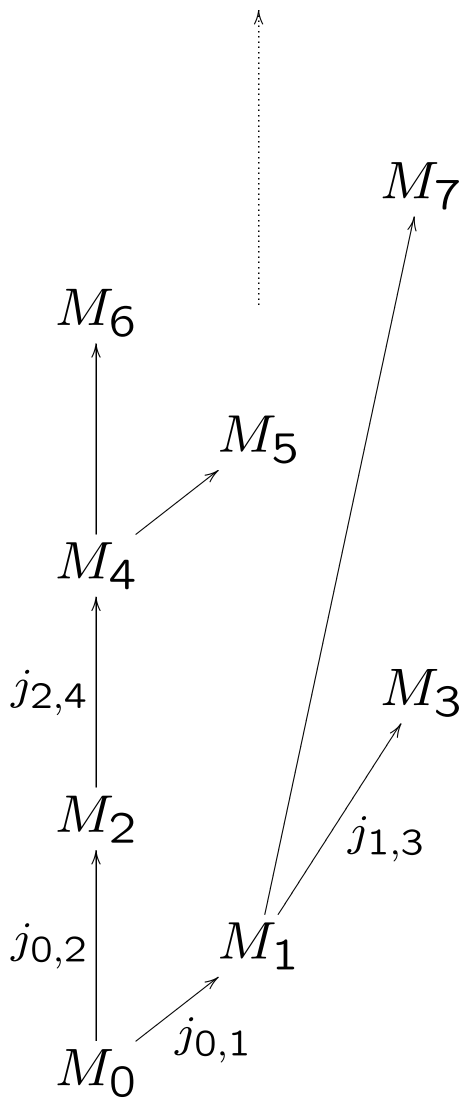
- a tree order T on ω,
- a sequence of models $\langle M_k \mid k < \omega \rangle$, and
- embeddings $j_{k,l}\colon M_k \to M_l$ for $k\,T\,l$.

Each model M_{l+1} for $l+1 > 0$ is an ultrapower of a preceeding model. More precisely: $M_{l+1} = \mathrm{Ult}(M_k, E_l)$, where E_l an extender picked from M_l, and $k \le l$ is the T predecessor of $l+1$. $j_{k,l+1}$ is the ultrapower embedding.

$$\begin{tikzcd} M_{l+1} & \\ & E_l \in M_l \\ M_k \arrow[uu, "j_{k,l+1}"] & \end{tikzcd}$$

(M_l and M_k must agree to $\mathrm{dom}(E_l)$.)

An iteration tree on M is a tree with $M_0 = M$.



Our trees will generally have an **even branch**, $M_0, M_2, M_4, \ldots,$ giving rise to the direct limit M_{even}.

The tree structure on the odd models will usually be some permutation of $\omega^{<\omega}$. With each **odd branch** b we associate the direct limit M_b.

(In this example, 0 T 1, 0 T 2, 1 T 3, 0 T 3, etc.)

In the **iteration game*** on M, players "good" and "bad" collaborate to produce a sequence of iteration trees as follows:

$$M \xrightarrow[\mathcal{T}_0]{b_0} M_1 \xrightarrow[\mathcal{T}_1]{b_1} M_2 \xrightarrow[\mathcal{T}_2]{b_2} M_3 \dashrightarrow$$

$$\dashrightarrow M_\omega \xrightarrow[\mathcal{T}_\omega]{b_\omega} M_{\omega+1} \dashrightarrow$$

"Bad" plays an iteration tree $\mathcal{T}_\xi$ on M_ξ. "Good" plays a branch b_ξ through $\mathcal{T}_\xi$. We let $M_{\xi+1}$ be the direct limit model determined by b_ξ and proceed to the next round. For limit λ we let M_λ be the direct limit of M_ξ for preceding ξ. We start with $M_0 = M$.

If ever a model (M_ξ, $\xi < \omega_1$) is reached which is illfounded, "bad" wins. Otherwise "good" wins.

*The definition given here is specialized to our context. The concept of iteration games is due to Martin–Steel.

We also consider iteration games were round ξ has the following form:

$$\begin{tikzcd} {} \arrow[r, dashed] & M_\xi \arrow[r, "b_\xi", "\mathcal{T}_\xi"'] \arrow[rr, bend left, "E_\xi"] & P_\xi & M_{\xi+1} \arrow[r, dashed] & {} \end{tikzcd}$$

"Bad" plays an iteration tree $\mathcal{T}_\xi$ on M_ξ. "Good" plays a branch b_ξ, giving rise to the direct limit, P_ξ.

Then "good" plays an extender E_ξ in P_ξ, with dom(E_ξ) within the level of agreement between M_ξ and P_ξ. We set $M_{\xi+1} = \text{Ult}(M_\xi, E_\xi)$ and continue to the next round.

If ever a model (P_ξ or M_ξ, $\xi < \omega_1$) is reached which is illfounded, "bad" wins. Otherwise "good" wins.

We refer to this game too as an **iteration game**.

M is **iterable** if the good player has a winning strategy for each of the iteration games described above. We refer to such winning strategies as **iteration strategies**.

Countable elementary substructures of V are iterable in this sense (Martin–Steel).

Suppose $M \models$ "δ is a Woodin cardinal", and in V there are M–generics for col(ω, δ). Let $\dot{A}$ name a set of reals in $M^{\text{col}(\omega,\delta)}$.

Work with some $x \in \mathbb{R}$. We work to define an auxiliary game, $\mathcal{A}[x]$, of ω moves, taken from M. In this game I tries to witness that $x \in \dot{A}[h]$ for some generic h. II tries to witness the opposite.

The auxiliary game is played as follows:

$$\begin{array}{c|ccccc} \text{I} & \ldots & l_n, \mathcal{X}_n, p_n & & \ldots & \\ \hline \text{II} & & & \mathcal{F}_n, \mathcal{D}_n & & \ldots \end{array}$$

In round n I plays

- $l = l_n$, a number $< n$, or $l_n =$ "new".
- $\mathcal{X}_n$, a set of names for reals of $M^{\text{col}(\omega,\delta)}$.
- p_n, a condition in col(ω, δ).

II plays

- $\mathcal{F}_n$ a function from $\mathcal{X}_n$ into the ordinals.
- $\mathcal{D}_n$, a function from $\mathcal{X}_n$ into {dense sets in col(ω, δ)}.

$$\mathcal{A}[x]: \quad \begin{array}{c|ccccc} \mathrm{I} & \ldots & l_n, \mathcal{X}_n, p_n & & \ldots & \\ \hline \mathrm{II} & & & \mathcal{F}_n, \mathcal{D}_n & & \ldots \end{array}$$

If $l_n =$ "new" we make no requirements on I. Otherwise, we require $p_n < p_l$ and $\mathcal{X}_n \subset \mathcal{X}_l$. We further require that for every name $\dot{x} \in \mathcal{X}_n$:

1. p_n forces "$\dot{x} \in \dot{A}$".
2. p_n forces "$\dot{x}(0) = \check{x_0}$", ..., "$\dot{x}(l) = \check{x_l}$".
3. p_n belongs to $\mathcal{D}_l(\dot{x})$.

We make the following requirement on II:

4. For every name $\dot{x} \in \mathcal{X}_n$, $\mathcal{F}_n(\dot{x}) < \mathcal{F}_l(\dot{x})$.

If there is h so that $x \in \dot{A}[h]$, I can pick a name for x, play $\mathcal{X}_n$ containing this name, and play $p_n \in h$. Condition 4 ensures defeat for II.

On the other hand, if there is an infinite run of $\mathcal{A}[x]$ where I covered all possible names and chains of conditions, condition 4 ensures that $x \notin \dot{A}[h]$ for all generic h.

Note 1. Rather than play the sets $\mathcal{X}_n$ directly, I plays their *type*. I plays $\kappa_n < \delta$, and a set u_n of formulae with parameters in $M\|\,\kappa_n \cup \{\kappa_n, \delta, \dot{A}\}$.*
We take $\mathcal{X}_n$ to be the set of names which satisfy all these formulae.

The fact that this still allows I enough control over her choice of $\mathcal{X}_n$ has to do with our assumption that δ is a Woodin cardinal.

$\mathcal{F}_n$ and $\mathcal{D}_n$ are played similarly.

Observe that moves in $\mathcal{A}[x]$ are therefore elements of $M\|\,\delta$.

Note 2. The association $x \mapsto \mathcal{A}[x]$ is continuous: The rules governing the first $n+1$ rounds of $\mathcal{A}[x]$ depend only on $x\restriction n$.

We in fact defined an association $s \mapsto \mathcal{A}[s]$ ($s \in \omega^{<\omega}$, $\mathcal{A}[s]$ a game of $\mathrm{lh}(s)+1$ many rounds). This association belongs to M.

*By $M\|\,\kappa_n$ we mean $\mathrm{V}^M_{\kappa_n}$.

Recall that g is col(ω, δ)–generic/M. We alternate between thinking of g as a generic enumeration of δ, and as a generic enumeration of $M \| \delta$.

Let $\sigma_{\text{gen}}[x, g]$, a strategy for I in $\mathcal{A}[x]$ be defined as follows:

$\sigma_{\text{gen}}[x, g]$ plays in each round the *first* (with respect to the enumeration g) legal move.

Note. The association $x, g \mapsto \sigma_{\text{gen}}[x, g]$ is continuous.

Lemma 1. Suppose that there exists an infinite run of $\mathcal{A}[x]$, played according to $\sigma_{\text{gen}}[x, g]$. Then $x \notin \dot{A}[g]$. (This is only useful if $x \in M[g]$.)

Proof: In playing for I, $\sigma_{\text{gen}}[g, x]$ goes over all possible names and all possible generics. (This uses the genericity of the enumeration g.) So in fact $x \notin \dot{A}[h]$ for all generic h. $\square$

We wish to phrase a similar lemma with a strategy for II, which puts x in A. To do this we have to give II additional control. We let II "shift" the play board along an even branch of an iteration tree.

$$\begin{tikzcd} M = M_0 \arrow[rr, bend left] & M_1 & M_2 \arrow[rr, bend left] & M_3 & M_4 \arrow[rr, bend left, dotted] & M_5 & M_6 & \cdots \end{tikzcd}$$

$$\begin{array}{llll}
 & l_0 & & \\
\mathrm{I} & \mathcal{X}_0 \rightsquigarrow & & \\
 & p_0 & & \\
\mathrm{II} & & \mathcal{F}_0 & \\
 & & \mathcal{D}_0 & \\
 & & l_1 & \\
\mathrm{I} & & \mathcal{X}_1 \rightsquigarrow & \\
 & & p_1 & \\
\mathrm{II} & & & \mathcal{F}_1 \\
 & & & \mathcal{D}_1 \\
 & & & l_2 \\
\mathrm{I} & & & \mathcal{X}_2 \rightsquigarrow \\
 & & & p_2
\end{array}$$

The game $\mathcal{A}^*[x]$ is played as follows:

I	$\ldots$	$l_n, \mathcal{X}_n, p_n$		$\ldots$	
II			$E_{2n}, E_{2n+1}, \mathcal{F}_n, \mathcal{D}_n$		$\ldots$

At the start of round n we have a model M_{2n}, an embedding $j_{0,2n}\colon M \to M_{2n}$, and a position P_n of n rounds in $j_{0,2n}(\mathcal{A})[x]$.

I plays $l_n, \mathcal{X}_n, p_n$, a legal move in $j_{0,2n}(\mathcal{A})[x]$ following P_n.

II plays extenders E_{2n}, E_{2n+1} giving rise to models M_{2n+1}, M_{2n+2}, and to an embedding $j_{2n,2n+2}\colon M_{2n} \to M_{2n+2}$. (The T–predecessor of $2n+1$ is $2l_n+1$ if $l_n \neq$ "new" and $2n$ otherwise.)

We let $Q_n = j_{2n,2n+2}(P_n—, l_n, \mathcal{X}_n, p_n)$. (This is the "shifting" mentioned before.)

II plays $\mathcal{F}_n, \mathcal{D}_n$, a legal move in $j_{0,2n+2}(\mathcal{A})[x]$ following Q_n.

We let $P_{n+1} = Q_n—, \mathcal{F}_n, \mathcal{D}_n$ and proceed to the next round.

Definition. A **pivot** for x is a pair $\mathcal{T}$, $\vec{a}$ so that

1. $\mathcal{T}$ is an iteration tree on M, with an even branch.
2. $\vec{a}$ is a run of $j_{\text{even}}(\mathcal{A})[x]$.
3. For every odd branch b of $\mathcal{T}$, there exists some h so that
   (a) h is col$(\omega, j_b(\delta))$–generic/M_b; and
   (b) $x \in j_b(\dot{A})[h]$.

Any run of $\mathcal{A}^*[x]$ produces $\mathcal{T}$, $\vec{a}$ which satisfy conditions 1 and 2.

Lemma 2. There exists $\sigma_{\text{piv}}[x, g]$, a strategy for II in $\mathcal{A}^*[x]$, so that every run according to $\sigma_{\text{piv}}[x, g]$ is a pivot.
The association $x, g \mapsto \sigma_{\text{piv}}[x, g]$ is continuous.

The proof of Lemma 2 draws heavily on the techniques of Martin–Steel's "A proof of projective determinacy". The assumption that δ is a Woodin cardinal is crucial.

To sum: Have continuous associations $x \mapsto \mathcal{A}[x]$; $x, g \mapsto \sigma_{\mathsf{gen}}[x, g]$; $x \mapsto \mathcal{A}^*[x]$; and $x, g \mapsto \sigma_{\mathsf{piv}}[x, g]$.

$\sigma_{\mathsf{gen}}[x, g]$ is a strategy for I in $\mathcal{A}[x]$.

If $\vec{a}$ is an infinite run of $\mathcal{A}[x]$ according to $\sigma_{\mathsf{gen}}[x, g]$, then $x \notin \dot{A}[g]$.

$\sigma_{\mathsf{piv}}[x, g]$ is a strategy for II in $\mathcal{A}^*[x]$.

If $\mathcal{T}$, $\vec{a}$ is an infinite run of $\mathcal{A}^*[x]$ according to $\sigma_{\mathsf{piv}}[x, g]$, then

for every odd branch b of $\mathcal{T}$, there exists some h so that

- h is $\mathrm{col}(\omega, j_b(\delta))$–generic/$M_b$; and
- $x \in j_b(\dot{A})[h]$.

Σ^1_2 determinacy:

Fix $A \subset \mathbb{R}$, a Σ^1_2 set (say the set of reals which satisfy a given Σ^1_2 statement ϕ).

Suppose there is an iterable class model M with a Woodin cardinal δ. Suppose that (in V) there is g which is $\mathrm{col}(\omega, \delta)$–generic/$M$.

We intend to prove that (in V) $G_\omega(A)$ is determined.

Let $\dot{A} \in M$ name A. More precisely, $\dot{A}$ names the set of reals of $M^{\mathrm{col}(\omega,\delta)}$ which satisfy ϕ in $M^{\mathrm{col}(\omega,\delta)}$.

We have $x \mapsto \mathcal{A}[x]$, $x, g \mapsto \sigma_{\mathrm{gen}}[x, g]$, etc. as before.

Let G be the following game, defined and played inside M:

$$\begin{array}{c|cccccccc} \mathrm{I} & x_0 & a_{0-\mathrm{I}} & & & a_{1-\mathrm{I}} & & x_2 & \ldots \\ \hline \mathrm{II} & & & a_{0-\mathrm{II}} & x_1 & & a_{1-\mathrm{II}} & & \end{array}$$

I and II alternate playing natural numbers, producing together $x = \langle x_0, x_1, \ldots \rangle \in \mathbb{R}$. In addition they play moves $a_{0-\mathrm{I}}, a_{0-\mathrm{II}}, \ldots$ in $\mathcal{A}[x]$.

II is the closed player; she wins if she can last all ω moves. Otherwise I wins.

G is a closed game, hence determined. A winning strategy exists in M.

Case 1: I wins G. Fix $\Sigma \in M$ a winning strategy for I (the open player).

We wish to show that I wins $G_\omega(A)$ in V. Let us play $G_\omega(A)$ against an imaginary opponent. We describe how to play, and win.

We construct a run $x \in \mathbb{R}$ of $G_\omega(A)$. At the same time we construct $\mathcal{T}$, $\vec{a}$, a run of $\mathcal{A}^*[x]$.

The participants in our construction are:

- The imaginary opponent: playing x_n for odd n.

- The strategy $\sigma_{\mathsf{piv}}[g, x]$: playing for II in $\mathcal{A}^*[x]$.

- The strategy Σ and its shifts along the even branch of $\mathcal{T}$: playing x_n for even n and playing for I in $\mathcal{A}^*[x]$ (i.e. playing for I in shifts of $\mathcal{A}[x]$).

We obtain $x \in \mathbb{R}$ and $\mathcal{T}$, $\vec{a}$ a run of $\mathcal{A}^*[x]$ according to $\sigma_{\mathsf{piv}}[x, g]$.

We must check that x belongs to A.

$$
\begin{tikzcd}[column sep=small]
M = M_0 \arrow[rr, bend left] & M_1 & M_2 \arrow[rr, bend left] & M_3 & M_4 \arrow[rr, bend left, dotted] & M_5 & M_6 & \cdots
\end{tikzcd}
$$

Σ	x_0		
Σ	l_0 $\mathcal{X}_0 \rightsquigarrow$ p_0		
σ_{piv}		$\mathcal{F}_0$ $\mathcal{D}_0$	
Oppnt		x_1	
$j_{0,2}(\Sigma)$		l_1 $\mathcal{X}_1 \rightsquigarrow$ p_1	
σ_{piv}			$\mathcal{F}_1$ $\mathcal{D}_1$
$j_{0,4}(\Sigma)$			x_2
$j_{0,4}(\Sigma)$			l_2 $\mathcal{X}_2 \rightsquigarrow$ p_2

Note that $x, \vec{a}$ is an infinite run of $j_{\text{even}}(G)$ according to $j_{\text{even}}(\Sigma)$.

Now Σ is a strategy for the open player in G. So there are no infinite runs according to Σ. But there is an infinite run according to $j_{\text{even}}(\Sigma)$. Thus M_{even} is **illfounded**.

M is iterable. So there exists some branch b of $\mathcal{T}$ so that M_b is wellfounded. b must be an odd branch.

By Lemma 2, $\mathcal{T}$, $\vec{a}$ is a pivot for x. Thus there is h so that

- h is col$(\omega, j_b(\delta))$–generic/M_b and
- $x \in j_b(\dot{A})[h]$.

This means that in $M_b[h]$, x satisfies the Σ^1_2 statement ϕ.

By absoluteness, x satisfies ϕ in V. (This uses the wellfoundedness of M_b.)

So $x \in A$ as required. $\square$(Case 1.)