Ultrafilters and Large Cardinals

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Abstract. This paper is a survey of basic large cardinal notions, and applications of large cardinal ultrafilters in forcing. The main application presented is the consistent failure of the singular cardinals hypothesis. Other applications are mentioned that involve variants of Prikry forcing, over models of choice and models of determinacy.

My talk at the Ultramath conference was about ultrafilters and large cardinals. As we shall see shortly, many large cardinal axioms can be viewed as asserting the existence of ultrafilters with specific properties. In a sense then one can say that all theorems that use these large cardinals are theorems about ultrafilters. But the study of these large cardinal axioms is far too wide a subject for a single talk, or even for many. My talk concentrated on one aspect, the use of the large cardinal ultrafilters in forcing. This paper follows a similar path.

The paper is intended for non-specialists, and the material is presented in a way that minimizes any prerequisites. Forcing is explained in Section 2, and the basic definitions of large cardinals in Section 1. Section 3 includes one of the most celebrated results combining these two topics, namely the consistent failure of the singular cardinals hypothesis. The proof illustrates most vividly how large cardinal ultrafilters are used in forcing constructions. Finally Section 4 gives some concluding remarks, on later uses of ultrafilters in forcing, both under the axiom of choice and under the axiom of determinacy.

The paper is expository, and the results presented, with the exception of some theorems in the last section, are not due to the author. It differs in some aspects from my talk at the Ultramath conference. In particular Theorem 4.2 was not yet known at the time of the conference.

1. Large cardinals

Recall that V denotes the universe of all sets. It is the union of the levels of the von Neumann hierarchy, V_α, $\alpha \in \mathrm{On}$.

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An embedding $\pi\colon V \to M$ is *elementary* if it preserves truth, meaning that for any formula φ and any $a_1, \ldots, a_n \in V$, $\varphi(\pi(a_1), \ldots, \pi(a_n))$ holds in M iff $\varphi(a_1, \ldots, a_n)$ holds in V.

Of course the identity map from V to V is elementary, but it is of no interest, and all maps below are assumed to be non-trivial. In all embeddings $\pi\colon V \to M$ below, π is a definable class, and $M \subseteq V$, as V is the universe of all sets. We often do not mention this explicitly. (There is another type of embeddings, called *generic* elementary embeddings, with M contained in a generic extension of V, and π definable in the extension.) Again in all embeddings $\pi\colon V \to M$ below, M is assumed to be transitive, though we do not mention this assumption explicitly.

The large cardinal axioms we work with in this paper assert the existence of (non-trivial) elementary embeddings $\pi\colon V \to M$. The strength of an axiom depends on the amount of agreement it gives between M and V. For example: the embedding is δ-*strong* if all bounded subsets of δ belong to M, namely $\bigcup_{\xi<\delta} \mathcal{P}(\xi) \subseteq M$. It is δ-supercompact if all subsets of M of size δ belong to M, namely $M^\delta \subseteq M$.

Axioms asserting the existence of such embeddings are called large cardinal axioms because they can be phrased as making assertions about very large cardinals. The next few claims isolate the most relevant cardinal to an embedding π.

Claim 1.1. *Let α be an ordinal. If $\pi \restriction \alpha$ is the identity, then $\pi \restriction V_\alpha$ is the identity.*

Proof. An easy transfinite induction using elementarity. The limit case is clear. Suppose then that $\pi \restriction \alpha + 1$ is the identity. Fix $x \in V_{\alpha+1}$, namely $x \subseteq V_\alpha$. Then by elementarity, $w \in x$ iff $\pi(w) \in \pi(x)$. Since $\pi \restriction V_\alpha$ is the identity by induction, $w \in x$ iff $w \in \pi(x)$ for all $w \in V_\alpha$. To get $\pi(x) = x$, it is enough then to show that $\pi(x) \subseteq V_\alpha$. Since $x \subseteq V_\alpha$, by elementarity $\pi(x) \subseteq V^M_{\pi(\alpha)}$, where $V^M_{\pi(\alpha)}$ is the $\pi(\alpha)$-th level of the von Neumann hierarchy computed in M. Using now the assumption that $\pi(\alpha) = \alpha$ we get $\pi(x) \subseteq V^M_\alpha$. An absoluteness argument shows V^M_α is equal to $V_\alpha \cap M$, hence $\pi(x) \subseteq V_\alpha \cap M \subseteq V_\alpha$. □

It follows that for any (non-trivial) elementary $\pi\colon V \to M$, there exists some ordinal α so that $\pi(\alpha) \neq \alpha$. The least such ordinal is called the *critical point* of π, denoted $\mathrm{Crit}(\pi)$.

Claim 1.2. *For any ordinal α, $\pi(\alpha) \geq \alpha$. In particular $\pi(\mathrm{Crit}(\pi)) > \mathrm{Crit}(\pi)$.*

Proof. Again an induction. If α is least so that $\pi(\alpha) < \alpha$, then setting $\beta = \pi(\alpha)$ we have $\pi(\beta) \geq \beta = \pi(\alpha)$. But then $\beta < \alpha$ while $\pi(\beta) \geq \pi(\alpha)$, contradicting elementarity. □

Let $\kappa = \mathrm{Crit}(\pi)$. It is easy to see that κ must be a cardinal: If $f\colon \lambda \to \kappa$ is a surjection with $\lambda < \kappa$, then by elementarity $\pi(f)$ is a surjection of $\lambda = \pi(\lambda)$ onto $\pi(\kappa) > \kappa$. There is then $\alpha < \lambda$ so that $\pi(f)(\alpha) = \kappa$. But by elementarity and since $\alpha = \pi(\alpha)$, $\pi(f)(\alpha) = \pi(f(\alpha))$. Since $f(\alpha) < \kappa$, $\pi(f(\alpha)) = f(\alpha) < \kappa$, contradiction.

It is also easy to see that (1) $\kappa = \mathrm{Crit}(\pi)$ is not the (cardinal) successor of any $\lambda < \kappa$, hence it must be a limit cardinal; (2) κ does not have cofinality $\lambda < \kappa$, hence it must be regular; and (3) there is no bijection of κ and the powerset of λ for any $\lambda < \kappa$. The critical point of an elementary embedding must therefore be (strongly) inaccessible. In particular, ZFC does not prove the existence of non-trivial elementary embeddings. In fact, by elementarity, the critical point cannot even be the first inaccessible cardinal: κ is inaccessible both in V and in M. Thus in M there is an inaccessible cardinal below $\pi(\kappa)$, and by elementarity there is an inaccessible cardinal below κ in V.

Indeed κ cannot be the first cardinal with any property whose definition involves only quantifiers on subsets of κ. Any such definition is absolute between V and M. (V and M have the same subsets of κ, since for any $x \subseteq \kappa$, $x = \pi(x) \cap \kappa$, and $\pi(x) \cap \kappa$ belongs to M.) If the property holds of κ in V then it holds of $\kappa < \pi(\kappa)$ in M, and hence by elementarity it holds of some $\alpha < \kappa$ in V.

The critical point of an elementary embedding must therefore be a very large cardinal. Axioms asserting the existence of elementary embeddings in effect assert the existence of cardinals with sufficient reflection properties that they can serve as critical points. The properties can be captured using ultrafilters.

Consider an elementary $\pi \colon V \to M$, and let $\kappa = \mathrm{Crit}(\pi)$. Define a collection U of subsets of κ setting $X \in U$ iff $\kappa \in \pi(X)$.

Claim 1.3. *U is an ultrafilter on $\mathcal{P}(\kappa)$. U is non-principal, meaning that no singleton set belongs to U. U is $<\kappa$ complete, meaning that the intersection of any family of fewer than κ elements of U is itself in U. U is normal, meaning that any function $f \colon \kappa \to \kappa$ which is regressive on a set in U, is constant on a set in U. (f is* regressive *on $A \subseteq \kappa$ if $f(\alpha) < \alpha$ for all $\alpha \in A$.)*

Proof. It is easy to verify that U is a filter using the elementarity of π. For example the formula $(\forall\alpha)(\alpha \in X \wedge \alpha \in Y \to \alpha \in X \cap Y)$ is true in V. Hence by elementarity $(\forall\alpha)(\alpha \in \pi(X) \wedge \alpha \in \pi(Y) \to \alpha \in \pi(X \cap Y))$. Applying this with $\alpha = \kappa$ it follows that if X and Y both belong to U, then so does $X \cap Y$. A similar argument replacing the pair $\langle X, Y\rangle$ with a sequence $\langle X_\xi \mid \xi < \tau\rangle$ for $\tau < \kappa$ establishes that U is $<\kappa$ complete.

We leave most other clauses to the reader, and continue only to show that U is an ultrafilter. Let $X \subseteq \kappa$. Then $X \cup (\kappa - X) = \kappa$. Hence by elementarity $\pi(X) \cup \pi(\kappa - X) = \pi(\kappa)$. Since $\pi(\kappa) > \kappa$ it follows that κ belongs to $\pi(X) \cup \pi(\kappa - X)$, hence one of X, $\kappa - X$ is in U. □

Claim 1.4. *Let U be a normal, $<\kappa$ complete non-principal ultrafilter on $\mathcal{P}(\kappa)$. Then there is an elementary embedding $\pi \colon V \to M$, with $\mathrm{Crit}(\pi) = \kappa$ and $X \in U$ iff $\kappa \in \pi(X)$ for all $X \subseteq \kappa$.*

Proof sketch. Let $M^* = V^\kappa/U$; elements of M^* are equivalence classes of functions $f \colon \kappa \to V$ under the relation $f \sim g$ iff $\{\alpha \mid f(\alpha) = g(\alpha)\}$ belongs to U, and M^* is endowed with one binary relation $\in^*$ given by $f \in^* g$ iff $\{\alpha \mid f(\alpha) \in g(\alpha)\}$ belongs to U. Define $\pi^* \colon V \to M^*$ by $\pi(x) = [\mathrm{Cnst}_x]$, where $\mathrm{Cnst}_x \in V^\kappa$ is the function with constant value x. A standard result, called Łos's theorem, states that formula $\varphi([f_1], \ldots, [f_n])$ holds in M^* iff the set $\{\alpha \mid \varphi(f_1(\alpha), \ldots, f_n(\alpha))\}$ belongs to U. It follows from this that π^* is elementary from V to M^*. (In particular, M^* satisfies the axiom of extensionality.)

Using $<\kappa$ completeness of U, indeed countable completeness is enough, one can reduce the wellfoundedness of $\in^*$ to the wellfoundedness of $\in$, proving that $\in^*$ is wellfounded. It is easy to see that $\in^*$ is set-like, meaning that every element of M^* has only a set (not a proper class) of inequivalent $\in^*$ predecessors; indeed, all $\in^*$ predecessors of f have representatives whose rank in the von Neumann hierarchy is no larger than the rank of f. As a wellfounded set-like structure that satisfies the axiom of extensionality, M^* can be put in isomorphism with a transitive structure

M. Precisely, by $\in^*$ induction define $k(x) = \{k(z) \mid z \in^* x\}$, and let M be the image of k. Then $k\colon M^* \to M$ is an isomorphism, and M is transitive.

We now have an elementary embedding $\pi = k \circ \pi^*\colon V \to M$. The model M is called the *ultrapower* of V by the ultrafilter U, and $k \circ \pi^*$ is the *ultrapower embedding*.

Using the $<\kappa$ completeness of U and Łos's theorem, one can check that for any $\alpha < \kappa$, $f \in^* \mathrm{Cnst}_\alpha$ iff there exists $\beta < \alpha$ with $f \sim \mathrm{Cnst}_\beta$. It follows that $k([\mathrm{Cnst}_\alpha]) = \alpha$ for each $\alpha < \kappa$, and hence $k \circ \pi^* \restriction \kappa$ is the identity.

Let Id denote the identity function on κ. Then $\mathrm{Cnst}_\alpha \in^* \mathrm{Id}$ for every $\alpha < \kappa$, using completeness and the fact that U is non-principal. So $k([\mathrm{Id}]) \geq \kappa$. On the other hand, using normality any $f \in^* \mathrm{Id}$ is equivalent to Cnst_α for some $\alpha < \kappa$. So $k([\mathrm{Id}]) = \kappa$.

Note that $\mathrm{Id} \in^* \mathrm{Cnst}_\kappa$, so $k \circ \pi^*(\kappa) = k([\mathrm{Cnst}_\kappa]) > k([\mathrm{Id}]) = \kappa$. Since $k \circ \pi^* \restriction \kappa$ is the identity, it follows that $\mathrm{Crit}(k \circ \pi^*) = \kappa$. Finally, for any $X \subseteq \kappa$, $\kappa \in k \circ \pi^*(X)$ iff $\mathrm{Id} \in^* \mathrm{Cnst}_X$ iff $\{\alpha \mid \mathrm{Id}(\alpha) \in \mathrm{Cnst}_X(\alpha)\}$ belongs to U iff $X = \{\alpha \mid \alpha \in X\} \in U$. $\square$

A cardinal κ is *measurable* if it is the critical point of an elementary embedding. Equivalently by the last two claims, κ is measurable iff there is a normal, $<\kappa$ complete non-principal ultrafilter on $\mathcal{P}(\kappa)$. The characteristic function of such an ultrafilter is called a *measure* on κ.

The definition of measurable cardinals is due to Ulam (see [**43**]) who characterized them using measures. The modern approach through elementary embeddings which we followed above is largely due to Scott, who used it in [**34**] to prove that there are no measurable cardinals in the constructible universe L.

In modern view, measures extract a certain *canonical* content from an elementary embedding. The first indication of this was a theorem of Kunen [**19**], that if U and U' are two measures on κ, then $L[U] = L[U']$ and $U \cap L[U] = U' \cap L[U']$. This result was the seed for a development of a theory of canonical inner models for large cardinal axioms. These are built from sequences of measures, and more generally objects called extenders that extract more from an embedding than a single ultrafilter. Using results similar to Kunen's theorem one can show that the inner models constructed are canonical, in much the same way that L is. One then obtains canonical inner models for large cardinal axioms. The use of ultrafilters (and more generally extenders) in characterizing the large cardinals embeddings is essential for canonicity, and indeed Kunen's theorem and its generalizations to stronger large cardinal axioms rely on iterated constructions of ultrapowers, much like the one in Claim 1.4

It is still open whether all large cardinal axioms admit canonical inner models. To give the reader an impression of the reach of current constructions, we list a few more axioms in the large cardinal hierarchy.

A cardinal κ is δ*-strong* if there is a δ-strong elementary embedding $\pi\colon V \to M$ with critical point κ and $\pi(\kappa) > \delta$. (Recall π is δ-strong if all bounded subsets of δ belong to M.) κ is δ-strong *with respect to* H if in addition $\pi(H) \cap \delta = H \cap \delta$. As $\pi(\kappa) > \delta$ this is equivalent to $\pi(H \cap \kappa) \cap \delta = H \cap \delta$, so the condition asserts that π stretches $H \cap \kappa$ to agree with $H \cap \delta$.

A cardinal τ is *Woodin* if for every $H \subseteq \tau$, there exists $\kappa < \tau$ which is $<\tau$-strong with respect to H (meaning δ-strong with respect to H for all $\delta < \tau$).

A cardinal κ is λ-*supercompact* if there is a λ-supercompactness elementary embedding $\pi\colon V \to M$ with critical point κ and $\pi(\kappa) > \lambda$. (Recall π is λ-supercompact if $M^\lambda \subseteq M$.) κ is *supercompact* if it is λ-supercompact for all λ.

Supercompactness can be characterized using ultrafilters. Let $\mathcal{P}_\kappa(\lambda)$ denote the set $\{a \subseteq \lambda \mid \mathrm{Card}(a) < \kappa\}$. A function $f\colon \mathcal{P}_\kappa(\lambda) \to \lambda$ is *regressive* on X if for any $a \in X$, $f(a) \in a$. A filter U on $\mathcal{P}_\kappa(\lambda)$ is *normal* if any function regressive on a set in U is constant on a set in U. The filter is *fine* if for any $\xi \in \lambda$ the set $\{a \in \mathcal{P}_\kappa(\lambda) \mid \xi \in a\}$ belongs to U.

It is easy to see that a λ-supercompactness embedding $\pi\colon V \to M$ with critical point κ and $\pi(\kappa) > \lambda$ gives rise to a $<\!\kappa$ complete, fine, normal ultrafilter U on $\mathcal{P}_\kappa(\lambda)$. To see this note that by supercompactness, $\pi''\lambda$ belongs to M. Let $s = \pi''\lambda$, and define U by setting $X \in U$ iff $s \in \pi(X)$ for each $X \subseteq \mathcal{P}_\kappa(\lambda)$. Arguments similar to the ones in the proof of Claim 1.3 show that U is a $<\!\kappa$ complete, fine, normal ultrafilter. It is also true that such an ultrafilter gives rise to a λ-supercompactness embedding. The arguments for this direction are similar to these in the proof of Claim 1.4. This characterization of supercompactness embeddings using ultrafilters is due to Solovay. The ultrafilters, namely $<\!\kappa$ complete, fine, normal ultrafilters measuring subsets of $\mathcal{P}_\kappa(\lambda)$, are called *supercompactness* ultrafilters.

Measurable, Woodin, and supercompact are three of the most important large cardinal notions. Measurable cardinals are the starting point for the hierarchy of elementary embeddings, Woodin cardinals are closely connected with axioms of determinacy, and supercompact cardinals are used in proving the consistency of various forcing axioms.

Currently the construction of canonical inner models reaches levels that involve Woodin cardinals and interactions between Woodin cardinals and strong embeddings. A recent theorem of Woodin shows that once the construction is brought to a specific level somewhat beyond supercompact cardinals, the resulting inner model in fact captures practically all large cardinal properties in V. But the construction of canonical inner models for supercompact cardinals is still very much an open question. It is one of the major research projects involving large cardinals.

Other research projects deal with uses of large cardinal axioms. Some of the earliest uses involved partition properties. For example:

Claim 1.5. *Suppose κ is measurable, and let U be a normal, $<\!\kappa$ complete non-principal ultrafilter on $\mathcal{P}(\kappa)$. Let c be a $<\!\kappa$-coloring of finite subsets of κ. (Precisely, c is a function defined on finite subsets of κ, with range of cardinality $\lambda < \kappa$.) Then there is a set $H \in U$ which is* homogeneous *for c, meaning that for each n, the restriction of c to subsets of H of size n is constant.*

Proof. We adapt the standard proof of the infinite Ramsey's theorem on ω, replacing the largeness notion used there, namely "unbounded in ω", with "element of U".

For each $n < \omega$ let c_n be the restriction of c to sets of size n. It is enough to find a homogeneous set $H_n \in U$ for each c_n. Then $H = \bigcap_{n<\omega} H_n$ belongs to U by countable completeness, and is homogeneous for c by definition.

We prove by induction on n that there is a homogeneous set in U for any $<\!\kappa$-coloring c_n of subsets of κ of size n. The case of $n = 1$ is immediate using the $<\!\kappa$-completeness of U. So let us handle the inductive case.

Fix c_{n+1}. For each $s \subseteq \kappa$ of size n, there is $H_s \in U$ so that c_{n+1} is constant on $\{s \cup \{\alpha\} \mid \alpha \in H_s\}$. Let q_s be the color taken by c_{n+1} on this set. By induction, there is a homogeneous set $\bar{H} \in U$ for the coloring $s \mapsto q_s$. Let q be the constant value of q_s on $s \subseteq \bar{H}$.

Set $H = \{\alpha \in \bar{H} \mid \alpha \in H_s$ for all $s \subseteq \alpha \cap \bar{H}\}$. Clearly c_{n+1} is homogeneous on H, taking only the value q. And H belongs to U using normality: For every $\alpha \in \bar{H} - H$ there is $s \subseteq \alpha \cap \bar{H}$ so that $\alpha \notin H_s$. Let $f_1(\alpha), \ldots, f_n(\alpha)$ list the elements of such an s. Then the functions f_i are regressive on $\bar{H} - H$. If H does not belong to U then $\bar{H} - H$ belongs to U, and by normality there is a fixed s so that $\{\alpha \mid \{f_1(\alpha), \ldots, f_n(\alpha)\} = s\}$ belongs to U. But this is impossible, since this set is disjoint from H_s which belongs to U. □

The property of κ given by the claim, restricted to 2-colorings, is called the *Ramsey property.* The claim establishes that measurable cardinals are Ramsey. In fact measurability is stronger, for example giving a set $A \in U$ so that all δ in A are themselves Ramsey. For more on infinitary partition properties see Kanamori [**18**, Chapter 2]. Here let us mention only one more property, the *tree property.* A cardinal κ has this property if every tree T of height κ and width $< \kappa$ (meaning that every level of the tree has size $< \kappa$), has a cofinal branch. Note that the tree property for ω is simply König's lemma: every finitely branching tree of infinite height has an infinite branch.

Claim 1.6. *If κ is measurable then κ has the tree property.*

Proof. Let T be a tree of height κ and width $< \kappa$. Let A_α consist of the nodes on level α of T. Let $\pi\colon V \to M$ be elementary with $\mathrm{Crit}(\pi) = \kappa$. Re-naming the nodes of T, we may assume that $T \subseteq \kappa$, and A_α is a bounded subset of κ. Since $\pi \restriction \kappa$ is the identity, it follows that $\pi(A_\alpha) = A_\alpha$.

The height of $\pi(T)$ is $\pi(\kappa) > \kappa$, so we may fix a node u of level κ in $\pi(T)$. Consider the branch of $\pi(T)$ leading to u, namely $Z = \{x \mid x <_{\pi(T)} u\}$. Each $x \in Z$ belongs to $\pi(A_\alpha)$ for some $\alpha < \kappa$, and since $\pi(A_\alpha) = A_\alpha$ it follows that $Z \subseteq \kappa$ and hence $<_T \restriction Z = <_{\pi(T)} \restriction Z$. From this and the fact that Z is a branch of $\pi(T)$ of height κ, it follows that Z is also a branch of T of height κ. □

The tree property fails at the successor of κ, for any regular κ so that $2^{<\kappa} = \kappa$, and hence certainly for any measurable cardinal. For singular κ, the tree property may hold at κ^+, but the known examples require substantial large cardinal strength. For example it is shown in Magidor–Shelah [**24**] that if $\mathrm{Cof}(\kappa) = \omega$ and κ is a limit of supercompact cardinals, then the tree property holds at κ^+.

2. Forcing

Let M be a transitive model of ZFC. *Forcing* is a technique introduced by Paul Cohen [**5, 6**], for adjoining an external set G to M, to obtain an extension $M[G]$ of the initial model M. Our exposition of the technique follows the treatment in Shoenfield [**39**].

The new set G is a subset of some $\mathbb{P} \in M$. $M[G]$ consists of all sets which can be constructed from elements of M using the new set G. One way to make this precise is the following. For each $\tau \in M$ define $\tau[G] = \{\sigma[G] \mid (\exists p \in \mathbb{P})\langle \sigma, p\rangle \in \tau \wedge p \in G\}$. This definition is made by induction on the rank of τ. Then define

$M[G] = \{\tau[G] \mid \tau \in M\}$. In forcing lingo, elements p of $\mathbb{P}$ are called *conditions*, and $\tau \in M$ is said to *name* $\tau[G]$.

CLAIM 2.1. $M \cup \{G\} \subseteq M[G]$. *In particular, if* $G \notin M$, *then* $M[G]$ *is a proper extension of* M.

PROOF. By induction on rank define $\check{x} = \{\langle \check{y}, p\rangle \mid y \in x \wedge p \in \mathbb{P}\}$. Then for every $x \in M$, $\check{x} \in M$ and $\check{x}[G] = x$. So $M \subseteq M[G]$.

Let $\dot{G} = \{\langle \check{p}, p\rangle \mid p \in \mathbb{P}\}$. Then $\dot{G}[G] = \{\check{p}[G] \mid p \in G\} = \{p \mid p \in G\} = G$, and therefore $G \in M[G]$. $\square$

The fundamental theorem of forcing allows reasoning about $M[G]$ from within M. Precisely, there is a relation $\Vdash^M_{\mathbb{P}}$ on $\mathbb{P} \times \{\text{formulas with parameters}\}$, so that (with a certain restriction on $\mathbb{P}$ and G, see below):

(1) If $M[G] \models \varphi(\tau_1[G], \dots, \tau_n[G])$, then there exists $p \in G$ so that $p \Vdash^M_{\mathbb{P}} \varphi(\tau_1, \dots, \tau_n)$.
(2) If $p \Vdash^M_{\mathbb{P}} \varphi(\tau_1, \dots, \tau_n)$, then $M[G] \models \varphi(\tau_1[G], \dots, \tau_n[G])$ for all G with $p \in G$.
(3) For each formula φ, the relation $p \Vdash^M_{\mathbb{P}} \varphi(\tau_1, \dots, \tau_n)$ is uniformly definable from $\mathbb{P}$ over M.

$p \Vdash^M_{\mathbb{P}} \varphi(\tau_1, \dots, \tau_n)$ is read p *forces* $\varphi(\tau_1, \dots, \tau_n)$ in $\mathbb{P}$ over M.

Because of the definability requirement (3) one cannot expect the conditions to hold without some restriction on $\mathbb{P}$ and G. After all, conditions (1) and (2) involve quantification over G, which is not an element of M, and at face value may lead to a relation which is not definable over M. One of Cohen's great insights is that nonetheless, conditions (1) and (2) *do* hold for a relation with the definability in (3), provided one places the restriction that $\mathbb{P}$ is a partially ordered set (*poset*), and G ranges over M-generic filters on $\mathbb{P}$.

DEFINITION 2.2. A subset G of a poset $\mathbb{P}$ is a *filter* on $\mathbb{P}$ if (a) it is closed upward under the order of $\mathbb{P}$, and (b) every two elements p_1, p_2 of G are compatible within G, meaning that there is $r \in G$ which is below both p_1 and p_2 in the order of $\mathbb{P}$.

An *antichain* of $\mathbb{P}$ is a set of pairwise incompatible elements. G is *generic* over M if it meets every maximal antichain of $\mathbb{P}$ that belongs to M.

A subset D of $\mathbb{P}$ is *dense* if $(\forall p \in P)(\exists r < p) r \in D$. (By $r < p$ we mean that r is below p in the order of $\mathbb{P}$. We also say that r is *stronger* than p.) Genericity over M is equivalent to meeting every dense subset of $\mathbb{P}$ that belongs to M.

In general there is no guarantee that generic subsets of $\mathbb{P}$ must exist. But if M is countable then M generic sets can be constructed by a simple induction, using an enumeration in order type ω of the dense sets which belong to M. Typically they cannot exist inside M, and indeed this is one of the points of the method of forcing: to adjoin a set G that does not belong to M.

We assume for the rest of the section that $\mathbb{P}$ is a poset in M. We also assume that for every $p \in \mathbb{P}$, there is a generic G with $p \in G$. (This holds for example if M is countable.)

THEOREM 2.3 (The fundamental theorem of forcing). *For every formula* φ *there is a relation* $p \Vdash^M_{\mathbb{P}} \varphi(\tau_1, \dots, \tau_n)$, *uniformly definable from* $\mathbb{P}$ *over* M, *so that conditions (1) and (2) above hold when* G *is restricted to range over* M *generic filters on* $\mathbb{P}$.

We refer the reader to Kunen [**20**] and Jech [**16**] for a proof of the fundamental theorem. The theorem allows for a very powerful analysis of the extension model $M[G]$. For example, if the poset $\mathbb{P}$ is countable in M, then using the theorem one can show that every cardinal of M remains a cardinal in $M[G]$. This is Claim 2.8 below.

REMARK 2.4. By extending the relation $\Vdash^M_{\mathbb{P}}$ if needed, we may assume that if p forces $\varphi(\tau_1,\dots,\tau_n)$ and $q<p$, then q also forces $\varphi(\tau_1,\dots,\tau_n)$. The extended relation still satisfies (1) and (2). That (1) continues to hold is trivial. The proof of (2) for the extended relation uses the upward closure of filters.

REMARK 2.5. Again by extending $\Vdash^M_{\mathbb{P}}$ if needed, we may assume that if no $q<p$ forces the negation of $\varphi(\tau_1,\dots,\tau_n)$, then p forces $\varphi(\tau_1,\dots,\tau_n)$. The extended relation trivially continues to satisfy (1). To see that (2) continues to hold note that if there is a generic G with $p\in G$ such that $M[G]\not\models\varphi(\tau_1[G],\dots,\tau_n[G])$, then there is $q<p$ forcing $\neg\varphi(\tau_1,\dots,\tau_n)$. (There is some $\bar{q}\in G$ forcing $\neg\varphi(\tau_1,\dots,\tau_n)$ by (1). Since p and $\bar{q}$ both belong to G, they are compatible in G, so there is $q\in G$ stronger than both. By the previous remark and since $q<\bar{q}$, q forces $\neg\varphi(\tau_1,\dots,\tau_n)$.)

CLAIM 2.6. *For any $p\in\mathbb{P}$ and any $\varphi(\tau_1,\dots,\tau_n)$, there exists $q<p$ which either forces $\varphi(\tau_1,\dots,\tau_n)$, or forces $\neg\varphi(\tau_1,\dots,\tau_n)$. (Such q is said to* decide *$\varphi(\tau_1,\dots,\tau_n)$.)*

PROOF. This is immediate from the previous remark, but for clarity we give a direct proof.

Let G be M-generic on $\mathbb{P}$. Then $\varphi(\tau_1[G],\dots,\tau_n[G])$ either holds or fails in $M[G]$. If it holds, then by condition (2) there is $q\in G$ forcing $\varphi(\tau_1,\dots,\tau_n)$, and using Remark 2.4 and compatibility of conditions in G we may take $q<p$. A similar argument produces a condition forcing the negation of $\varphi(\tau_1,\dots,\tau_n)$ in the case that $\varphi(\tau_1[G],\dots,\tau_n[G])$ fails in $M[G]$. $\square$

REMARK 2.7. Suppose p forces "τ is a function with domain $\check{\lambda}$". Then for every $\alpha\in\lambda$ there is β and $q<p$ so that q forces "τ applied to $\check{\alpha}$ is equal to $\check{\beta}$". (We write this simply as $q\Vdash^M_{\mathbb{P}}\tau(\check{\alpha})=\check{\beta}$.) The proof is similar to that of the previous claim. Let G be M-generic with $p\in G$. Then $f=\tau[G]$ is a function. Let $\beta=f(\alpha)$, and find $q\in G$ below p forcing $\tau(\check{\alpha})=\check{\beta}$.

CLAIM 2.8. *Suppose $\mathbb{P}$ is countable in M, and G is an M-generic filter on $\mathbb{P}$. Then all cardinals of M remain cardinals in $M[G]$.*

PROOF. Let κ be a cardinal of M and suppose for contradiction that there is $\lambda<\kappa$ and a surjective function $f\colon\lambda\to\kappa$ in $M[G]$. Let $\tau\in M$ be such that $f=\tau[G]$.

Fix $p_0\in G$ forcing that τ is a function. Such p_0 exists using (1). By Remark 2.4, every $p<p_0$ forces τ to be a function. For each $p<p_0$ and each $\alpha<\lambda$, there exists at most one ordinal β so that $p\Vdash^M_{\mathbb{P}}\tau(\check{\alpha})=\check{\beta}$. For otherwise, taking a generic H with $p\in H$, we would get that $\tau[H]$ applied to α gives two distinct values, contradicting the fact that $\tau[H]$ is a function.

Define $\beta_{p,\alpha}$ to be this unique ordinal when it exists, and otherwise leave $\beta_{p,\alpha}$ undefined. Since the relation $p\Vdash^M_{\mathbb{P}}\tau(\check{\alpha})=\check{\beta}$ is definable over M, the assignment $p,\alpha\mapsto\beta_{p,\alpha}$ ($p\in\mathbb{P}$, $\alpha\in\lambda$) is definable over M, and by the Replacement axiom it belongs to M. Let A be the range of the assignment. Then A belongs to M and the cardinality of A in M is at most $\mathrm{Card}^M(\mathbb{P}\times\lambda)$. Since $\lambda<\kappa$ and $\mathbb{P}$ is countable in M, this cardinality is smaller than κ. Hence, $M \models \mathrm{Card}(A) < \kappa$.

But since f is a surjection onto κ, for every $\beta<\kappa$ there is some α so that $f(\alpha)=\beta$. Using (1), there is $p\in G$ which forces $\tau(\check{\alpha})=\check{\beta}$. Since both p and p_0 belong to G they are compatible in G, and so strengthening p we may assume $p<p_0$. But then p and α are such that $\beta_{p,\alpha}=\beta$, and so $\beta\in A$.

We have shown on the one hand that $M\models \mathrm{Card}(A)<\kappa$, and on the other hand that $\kappa\subseteq A$. This contradiction completes the proof of the claim, showing that $f\in M[G]$ cannot embed $\lambda<\kappa$ onto κ. The driving force in the proof is our ability to estimate f inside M using (1) and (2) for a relation definable over M. The estimate is sufficiently close to derive a contradiction, because $\mathbb{P}$ is countable. □

Continuing to work with the definitions of the last proof, it is easy to see that if $p<q$ and $\beta_{q,\alpha}$ is defined, then $\beta_{p,\alpha}$ is defined and equal to $\beta_{q,\alpha}$. It follows that the size of $\{\beta_{p,\alpha} \mid p\in\mathbb{P}\}$ is the size of a maximal antichain in $\mathbb{P}$. Thus we could replace the assumption in the last claim that $\mathbb{P}$ is countable in M, with the assumption that, in M, all antichains of $\mathbb{P}$ are countable.

Definition 2.9. A poset $\mathbb{P}$ has the *countable chain condition* if all antichains of $\mathbb{P}$ are countable.

Claim 2.10. *Suppose $\mathbb{P}$ has the countable chain condition in M, and G is an M-generic filter on $\mathbb{P}$. Then all cardinals of M remain cardinals in $M[G]$. Moreover cofinalities are preserved between M and $M[G]$: If $M\models \mathrm{Cof}(\kappa)=\delta$ then $M[G]\models \mathrm{Cof}(\kappa)=\delta$ too.*

Proof. Similar to the proof of Claim 2.8. In the second part, adapt the argument to work with cofinal $f\colon\lambda\to\kappa$ in $M[G]$ for $\lambda<\delta$, to obtain an approximation in M which on the one hand is cofinal in κ and on the other has size λ, contradicting the fact that $\mathrm{Cof}(\kappa)>\lambda$ in M. □

Claim 2.11. *Suppose M is a model of* ZFC*, and G is M-generic for $\mathbb{P}$. Then $M[G]$ is also a model of* ZFC

Proof sketch. Use the forcing relation in M and conditions (1) and (2) to reduce ZFC axioms in $M[G]$ to statements in M that are provable from ZFC. Let us explicitly handle only one axiom here. We show that Pairing holds in $M[G]$.

Fix $x,y\in M[G]$. We show that $\{x,y\}\in M[G]$. Let τ and σ be such that $x=\tau[G]$ and $y=\sigma[G]$. Let $\rho=\{\langle\tau,p\rangle \mid p\in\mathbb{P}\}\cup\{\langle\sigma,p\rangle\mid p\in\mathbb{P}\}$. Then using the axioms of ZFC in M (for example Union, Pairing, Comprehension), ρ is a set that belongs to M. It is clear that $\rho[G]=\{\tau[G],\sigma[G]\}=\{x,y\}$, so $\{x,y\}\in M[G]$.

For some other axioms the arguments are more involved. For example in the case of Comprehension, one is given a set $x=\dot{x}[G]\in M[G]$, and a formula φ, and must show that the set $\{u\in x \mid M[G]\models\varphi(u)\}$ belongs to $M[G]$. The proof uses the forcing relation to define in M a name $\dot{z}$ for this set. Briefly, $\dot{z}=\{\langle\dot{u},p\rangle\in V^M_\alpha \mid p\Vdash^M_{\mathbb{P}} \dot{u}\in\dot{x}\wedge\varphi(\dot{u})\}$ (for $\alpha\in M$ large enough that $\dot{x}\in V^M_\alpha$). $\dot{z}$ belongs to M since the forcing relation is definable in M, and M satisfies Comprehension. Using (1) and (2) one can show that $\dot{z}[G]=z$, so $z\in M[G]$ as needed. We refer the reader to Kunen [**20**] for a full proof, and proofs of the other axioms of ZFC in $M[G]$. □

Forcing was first used by Cohen to obtain a model where the continuum hypothesis fails, that is a model where there are more than $\aleph_1$ subsets of $\aleph_0=\omega$. This is done by using a poset G which "adds" subsets of ω.

Definition 2.12. $\mathrm{Add}(\omega, \kappa)$ is the poset $\mathbb{P}$ of finite partial functions from $\kappa \times \omega$ into $\{0, 1\}$, ordered by reverse inclusion, meaning $p < q$ if $p \supseteq q$.

Fact 2.13. $\mathrm{Add}(\omega, \kappa)$ has the countable chain condition. This is a combinatorial claim. We omit the proof, and refer the reader to [**20**, Chapter 2 exercise 20].

Claim 2.14. *Let $\kappa = \aleph_2^M$ and let $\mathbb{P} = \mathrm{Add}(\omega, \kappa)$. Let G be M-generic on $\mathbb{P}$. Then $M[G]$ satisfies "there are at least κ subsets of ω" and "κ is $\aleph_2$". In particular $M[G]$ is a model of the failure of the continuum hypothesis.*

Proof. Each element of $G \subseteq \mathbb{P}$ is a finite partial function from $\kappa \times \omega$ into $\{0, 1\}$. Since all elements of G are compatible, these finite partial functions agree with each other on any common part of their domain. So $f = \bigcup\{p \mid p \in G\}$ is itself a function from $\kappa \times \omega$ into $\{0, 1\}$. Using genericity it is easy to see that f is total. (For any $\langle \alpha, n\rangle \in \kappa \times \omega$, the set $\{p \mid \langle \alpha, n\rangle \in \mathrm{Dom}(p)\}$ is dense and belongs to M. The fact that G meets this set implies that $\langle \alpha, n\rangle \in \mathrm{Dom}(f)$.)

Again using genericity, it is easy to see that for every $\alpha \neq \beta$, there is n so that $f(\alpha, n) \neq f(\beta, n)$. (Since all elements of $\mathbb{P}$ are finite, the set $\{p \mid (\exists n)\ p(\alpha, n)$ and $p(\beta, n)$ are both defined and take different values$\}$ is dense.) Letting $x_\alpha = \{n \mid f(\alpha, n) = 1\}$ it follows that $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$.

The sequence $\langle x_\alpha \mid \alpha < \kappa\rangle$ is built from G and it is not hard to show that it belongs to $M[G]$. Since the x_αs are distinct subsets of ω it follows that in $M[G]$ there are at least κ subsets of ω.

It remains to show that κ is the second uncountable cardinal of $M[G]$. But this is immediate since M and $M[G]$ have exactly the same cardinals, by Claim 2.10. $\square$

More generally forcing can be used to increase the powerset of any regular cardinal δ.

A poset $\mathbb{P}$ has the δ^+ *chain condition* if every antichain of $\mathbb{P}$ has size at most δ. For $\delta < \kappa$ let $\mathrm{Add}(\delta, \kappa)$ be the poset of partial functions of size $< \delta$, from $\kappa \times \delta$ into $\{0, 1\}$. The arguments above generalize to give the following facts:

- $\mathrm{Add}(\delta, \kappa)$ has the $(2^{<\delta})^+$ chain condition. ($2^{<\delta}$ is $\sup\{2^\lambda \mid \lambda < \delta\}$.) In particular, if $2^\lambda = \lambda^+$ for all $\lambda < \delta$, then the poset has the δ^+ chain condition.
- If $\mathbb{P}$ has the δ^+ chain condition in M, and G is M-generic on $\mathbb{P}$, then M and $M[G]$ have the same cardinals above δ. They also agree on cofinalities of cardinals whose cofinality in M is at least δ^+.
- If $\mathbb{P} = \mathrm{Add}(\delta, \kappa)^M$ and G is M-generic for $\mathbb{P}$ then in $M[G]$ there are at least κ subsets of δ.

It follows from the above facts that if we start from a model M that satisfies the generalized continuum hypothesis ($2^\lambda = \lambda^+$ for all λ), and pass to a generic extension $M[G]$ using the poset $\mathbb{P} = \mathrm{Add}(\delta, \kappa)^M$ for $\kappa = (\delta^{++})^M$, then $M[G]$ satisfies "$2^\delta \geq \delta^{++}$". However this by itself does not mean that we managed to increase the powerset of a cardinal greater than $\aleph_0$, since we have not yet ruled out that $M[G] \models$"δ is countable". The next two claims rule this out.

Definition 2.15. A poset $\mathbb{P}$ is $<\delta$-closed if for any $\lambda < \delta$ and any decreasing sequence $\langle p_\xi \mid \xi < \lambda\rangle$ of elements of $\mathbb{P}$, there is $q \in \mathbb{P}$ below all p_ξ.

CLAIM 2.16. *If δ is regular then* $\mathrm{Add}(\delta,\kappa)$ *is* $<\!\delta$ *closed.*

PROOF. Fix a decreasing sequence $\langle p_\xi \mid \xi < \lambda\rangle$, with $\lambda < \delta$, and let $q = \bigcup_{\xi<\lambda} p_\xi$. Since the sequence is decreasing, all functions p_ξ agree with each other, and q is therefore also a function. q is a union of λ sets each of size $< \delta$. Since δ is regular, q too has size $< \delta$. So q is an element of $\mathbb{P}$. It is clear that $q < p_\xi$ for each ξ. $\square$

CLAIM 2.17. *Let $\mathbb{P}$ be a poset which is $<\!\delta$ closed in M. Let G be M-generic on $\mathbb{P}$. Then M and $M[G]$ agree on cardinals below δ, and on cofinalities of cardinals whose cofinality in $M[G]$ is $< \delta$.*

PROOF. We show that any sequence of ordinals of length $< \delta$ that belongs to $M[G]$ in fact belongs to M. This implies the statement in the claim.

Let $f \in M[G]$ be a sequence of ordinals of length $\lambda < \delta$. Let τ be such that $f = \tau[G]$ and let $p \in G$ force that τ is a function with domain $\check{\lambda}$, taking ordinal values.

Working in M, and using the closure of $\mathbb{P}$, construct sequences $\langle p_\xi \mid \xi \leq \lambda\rangle$ and $\langle \beta_\xi \mid \xi < \lambda\rangle$ so that:

- $p_0 = p$.
- For each ξ, $p_{\xi+1} < p_\xi$ and $p_{\xi+1}$ forces $\tau(\check{\xi}) = \check{\beta}_\xi$.
- For limit α, p_α is a lower bound in $\mathbb{P}$ for the sequence $\langle p_\xi \mid \xi < \alpha\rangle$.

For the second item, $p_{\xi+1}$ and β_ξ can be chosen using Remark 2.7. The third item uses the closure of $\mathbb{P}$.

Since $p_\lambda < p_{\xi+1}$, p_λ forces $\tau(\check{\xi}) = \check{\beta}_\xi$ for each ξ. Let h be the function $\xi \mapsto \beta_\xi$. Since the entire construction was done in M, we have $h \in M$. We also have $p_\lambda \Vdash^M_{\mathbb{P}} \tau = \check{h}$.

Working in M, let $D = \{q \mid (\exists h) q \Vdash^M_{\mathbb{P}} \tau = \check{h}\}$. We showed that D is non-empty. The same argument, starting with an arbitrary $p_0 < p$, produces an element of D below p_0. So $D \in M$ is dense below p. Since $p \in G$, this together with M-genericity implies that G meets D. Fix then a condition $q \in D \cap G$. From the definition of D and the fact that $q \in G$ it now follows that $f = \tau[G]$ is equal to $h = \check{h}[G]$ for some $h \in M$, and therefore $f \in M$. $\square$

Forcing can thus be used to violate the continuum hypothesis at any regular cardinal. For example, if M is a model of the generalized continuum hypothesis, and α is such that in M, $\aleph_\alpha$ is regular, then there is a generic extension $M[G]$ of M so that the two models agree on all cardinals, yet in $M[G]$, $2^{\aleph_\alpha} \geq \aleph_{\alpha+2}$. Furthermore, with posets $\mathrm{Add}(\delta,\kappa)$ for $\kappa = (\aleph_\beta)^M$, possibly much bigger than $(\aleph_{\alpha+2})^M$, we could make $2^{\aleph_\alpha} \geq \aleph_\beta$ hold in $M[G]$ for our favorite $\beta \in M$, no matter how large.

To increase the powerset of $\aleph_\alpha$ we had to assume it is regular; the assumption was used in Claim 2.16. It had been expected initially after Cohen's introduction of forcing that more sophisticated forcing constructions would allow similar manipulations of the powerset of singular cardinals. But it turned out that manipulating the powerset function at singular cardinals is much harder, and indeed in many cases it is outright impossible.

The *singular cardinals hypothesis* (SCH) asserts that for every singular κ, if $2^{<\kappa} = \kappa$, then $2^\kappa = \kappa^+$. One example of a theorem showing that some manipulations of the powerset function at singular cardinals are outright impossible is a result of Silver [**40**], that the smallest failure of the singular cardinals hypothesis cannot occur at a cardinal of uncountable cofinality. Another example is Shelah's famous result [**37**] that if $2^{\aleph_n} < \aleph_\omega$ for all n, then $2^{\aleph_\omega} < \aleph_{\omega_4}$. This is in stark contrast to the situation at a regular $\aleph_\alpha$, where $2^{\aleph_\alpha}$ can be made arbitrarily large.

Even in cases where the singular cardinals hypothesis can be violated, matters are complicated, and in fact the forcing notions involved make use of large cardinal ultrafilters.

3. Forcing with large cardinal ultrafilters

The basic idea for forcing a failure of the singular cardinals hypothesis is simple: start with a regular cardinal κ and force to increase its powerset, then force to make κ singular. The end result is a cardinal κ where $2^\kappa > \kappa^+$ through the first stage, and κ is singular through the second.

It is essential of course that κ remains a cardinal through both stages. But in general there is no guarantee that κ can be singularized without being collapsed. This is where large cardinals come in. We shall see that if κ is measurable, then there is a forcing notion that singularizes it without collapsing any cardinals. This will take care of the second stage, provided we can ensure that κ is measurable following the first stage.

The first stage involves iterated forcing. One could attempt to simply use the methods of the previous section, to construct a model $M[G]$ where $2^\kappa \geq \kappa^{++}$. But then κ may fail to be measurable in $M[G]$. Indeed, if κ is to be measurable, then it cannot be the *first* cardinal δ of $M[G]$ so that $2^\delta \geq \delta^{++}$. (The proof of this is similar to the arguments in Section 1 that a measurable cardinal cannot be the first α with a property whose definition involves quantifiers only over subsets of α.) But if M satisfies $2^\delta = \delta^+$ for all $\delta < \kappa$, then the preservation theorems in Section 2 show that $M[G]$ satisfies the same. It follows in this case that κ cannot be measurable in $M[G]$.

To get to a situation where in $M[G]$, κ is measurable and $2^\kappa = \kappa^{++}$, we must force to increase not only the powerset of κ, but also the powerset of many $\delta < \kappa$. This is where iterated forcing comes in. Through an iteration whose basic step is the forcing $\mathrm{Add}(\delta, \delta^{++})$, one creates an extension $M[G]$ satisfying $2^\delta = \delta^{++}$ for all inaccessible $\delta \leq \kappa$ in M.

One then argues that κ is measurable in $M[G]$, using a large cardinal assumption on κ in M. It is not enough to assume that κ is measurable in M; greater large cardinal strength is needed in M to ensure that measurability survives to $M[G]$. The first theorem along these lines was due to Silver, and the assumption needed for the proof was that κ is supercompact in M. The assumption of supercompactness was reduced by Woodin to the existence of an elementary embedding j with critical point κ so that $j(\kappa) > \kappa^{++}$ and the target model of j is closed under sequences of length κ. The existence of a model with such an embedding was proved by Gitik [**10**] to follow from the existence of many measures on κ, and all components combined led to Theorem 3.2 below:

DEFINITION 3.1. The Mitchell relation $\triangleleft$ on ultrafilters is given by $U \triangleleft U^*$ iff U belongs to the ultrapower of V by U^*. This relation is wellfounded. The *Mitchell order* of U, denoted $o(U)$, is its rank in the relation $\triangleleft$. The Mitchell order of a cardinal κ is $\sup\{o(U) \mid U \text{ a measure on } \kappa\}$.

THEOREM 3.2. *Let M be a countable model of* ZFC. *Suppose that in M, κ has Mitchell order κ^{++}. Then there is a countable model M^* where κ remains* *measurable, and $2^\kappa = \kappa^{++}$. All cardinals of M remain cardinals of M^*. (Indeed, M^* is a cardinals and cofinalities preserving generic extension of an inner model of M.)*

Let us now consider the second stage, that is singularizing a measurable cardinal κ. Fix a measurable cardinal κ in a model M. We attempt to find a forcing notion $\mathbb{P}$ so that in the resulting generic extension $M[G]$, $\mathrm{Cof}(\kappa) = \omega$.

Consider first the poset of finite partial increasing functions from ω to κ. If G is a generic filter on this poset, then $\bigcup G$ is a total increasing function from ω to κ. Using genericity it is easy to see that the function is cofinal in κ. So in $M[G]$ the cofinality of κ is ω. Alas, this forcing notion has no property that implies preservation of cardinals $\leq \kappa$. Indeed it is easy to see that it collapses κ (and all cardinals below it) to ω, meaning that in $M[G]$, κ is countable. So in $M[G]$, κ is not a cardinal at all, let alone a singular cardinal. We need a different poset, one that adds a cofinal function from ω into κ, but does not collapse cardinals below κ.

Definition 3.3. Let κ be measurable, and let U be a normal, $<\kappa$ complete non-principal ultrafilter on $\mathcal{P}(\kappa)$. *Prikry forcing* using U is the poset whose elements are pairs $\langle s, A\rangle$, where s is a finite increasing sequence of ordinals in κ, and A is a subset of κ that belongs to U. s is called the *stem* of $\langle s, A\rangle$. The ordering of $\mathbb{P}$ is the relation $\langle t, B\rangle < \langle s, A\rangle$ iff t extends s, and all elements of $t - s$ belong to A.

Prikry forcing is similar to the simple forcing of increasing finite functions, but it includes restrictions on the possible growths of the function: an element $\langle s, A\rangle$ of the poset not only specifies an initial segment s of a function from ω into κ, but also a set A from which all elements in extensions of s must be taken. The set A is required to be large, in the sense that it must belong to the ultrafilter U.

Lemma 3.4. *Let κ be measurable in M and let U be a normal, $<\kappa$ complete non-principal ultrafilter on $\mathcal{P}(\kappa)$ in M. Let $\mathbb{P}$ be Prikry forcing defined from U in M. Let G be M-generic on $\mathbb{P}$.*

Then $M[G] \models \mathrm{Cof}(\kappa) = \omega$, and every cardinal of M is a cardinal in $M[G]$. Moreover if $\mathrm{Cof}^M(\lambda) \neq \kappa$, then $\mathrm{Cof}^{M[G]}(\lambda) = \mathrm{Cof}^M(\lambda)$.

Proof. Let $f = \bigcup\{s \mid (\exists A)\langle s, A\rangle \in G\}$. Since all conditions in G are compatible, f is the increasing union of finite increasing sequences of ordinals in κ. Using genericity it is easy to see that there are conditions $\langle s, A\rangle \in G$ with s of arbitrarily large finite length. (For every $n < \omega$, the set $D = \{\langle s, A\rangle \mid \mathrm{lh}(s) \geq n\}$ is dense.) Hence f is an increasing length ω sequence of ordinals in κ. Again using genericity it is easy to see that f is cofinal in κ. (For every $\alpha < \kappa$, The set $D = \{\langle s, A\rangle \mid s \text{ has elements above } \alpha\}$ is dense. This is because all sets $A \in U$ are unbounded in κ.) Thus, in $M[G]$, f is a sequence of length ω cofinal in κ. In particular $\mathrm{Cof}^{M[G]}(\kappa) = \omega$.

It remains to show that cardinals are not collapsed in the move from M to $M[G]$, and indeed no cofinalities are changed other than the fact that cardinals of cofinality κ in M have cofinality ω in the extension. For cardinals and cofinalities above κ the next claim, and the results of the previous section, suffice.

Claim 3.5. *$\mathbb{P}$ has the κ^+ chain condition in M.*

Proof. It is enough to show that any two elements of $\mathbb{P}$ with the same stem are compatible. Since there are only $\kappa^{<\omega} = \kappa$ possible stem it follows that antichains cannot have size greater than κ.

Fix two elements $\langle s, A\rangle$ and $\langle s, B\rangle$ of $\mathbb{P}$ with the same stem s. To see that the elements are compatible, simply note that $\langle s, A\cap B\rangle$ is an element of $\mathbb{P}$ ($A\cap B\in U$ since U is closed under intersections), and it is below both $\langle s, A\rangle$ and $\langle s, B\rangle$. $\square$

Given the last claim, we know that M and $M[G]$ have the same cardinals above κ, and agree on cofinalities of cardinals whose cofinalities in M are greater than κ. It is for cardinals below κ that we really need the ultrafilter U. The methods of the previous section would allow us to prove preservation of cardinals below κ if $\mathbb{P}$ were $<\kappa$ closed. Of course Prikry forcing is not $<\kappa$ closed, as the stems are required to be finite. But for any stem s, the restriction of $\mathbb{P}$ to elements with stem equal to s *is* $<\kappa$ closed, because of the completeness of U. We intend to mirror a preservation proof from the previous section, using this restriction. The following claim provides the first step. It mirrors Claim 2.6.

Claim 3.6. *For any finite increasing sequence s of ordinals in κ, and any $\varphi(\tau_1,\dots,\tau_n)$, there is $A\in U$ so that either $\langle s, A\rangle \Vdash^M_{\mathbb{P}} \varphi(\tau_1,\dots,\tau_n)$ or $\langle s, A\rangle \Vdash^M_{\mathbb{P}} \neg\varphi(\tau_1,\dots,\tau_n)$.*

Proof. Fix s and $\varphi(\tau_1,\dots,\tau_n)$. Working in M using the relation $\Vdash^M_{\mathbb{P}}$, define a coloring c on finite $a\subseteq\kappa$ as follows. If a has elements below $\sup(s)$, then $c(a) =$"small". Otherwise let t be the extension of s obtained by appending the elements of a to s in increasing order. If for some $Y\in U$, $\langle t, Y\rangle$ forces either $\varphi(\tau_1,\dots,\tau_n)$ or $\neg\varphi(\tau_1,\dots,\tau_n)$, then set $c(a) =$"true" or $c(a) =$"false" depending on whether the formula forced is $\varphi(\tau_1,\dots,\tau_n)$ or its negation. (They cannot both be forced, since any two conditions with the same stem t are compatible.) Otherwise set $c(a) =$"undefined".

c is a 4-coloring of finite subsets of κ. Using Claim 1.5 in M, there is a set $A\in U$ so that c is homogeneous on A. We claim that $\langle s, A\rangle$ provides the conclusion of the claim.

Suppose not. Then by Remark 2.5 there are two extensions $\langle t_1, B_1\rangle$ and $\langle t_2, B_2\rangle$ of $\langle s, A\rangle$ forcing $\varphi(\tau_1,\dots,\tau_n)$ and $\neg\varphi(\tau_1,\dots,\tau_n)$ respectively. Extending the conditions if needed we may assume that their stems have the same length l. Since both conditions are below $\langle s, A\rangle$, all elements of $t_1 - s$ and $t_2 - s$ belong to A. So $c(t_1 - s) = c(t_2 - s)$. But $c(t_1 - s) =$"true" since $\langle t_1, B_1\rangle$ forces $\varphi(\tau_1,\dots,\tau_n)$, while $c(t_2 - s) =$"false" since $\langle t_2, B_2\rangle$ forces $\neg\varphi(\tau_1,\dots,\tau_n)$. This contradiction completes the proof of the claim. $\square$

Claim 3.7. *Let $f\colon\lambda\to\delta$ belong to $M[G]$, where both λ and δ are smaller than κ. Then f belongs to M.*

Proof. Let τ be such that $f=\tau[G]$. Let $p=\langle s, A\rangle$ force "τ is a function from $\check{\lambda}$ into $\check{\delta}$". We prove that there is a function $h\in M$ and $q<p$ so that $q\Vdash^M_{\mathbb{P}} \tau=\check{h}$. It then follows, as in Claim 2.17, that $f\in M$.

Let $\langle\alpha_\xi,\beta_\xi\rangle$ for $\xi<\mathrm{Card}(\lambda\times\delta)$ enumerate $\lambda\times\delta$. Working in M, fix for each ξ a set $A_\xi\in U$ so that $\langle s, A_\xi\rangle$ either forces $\tau(\check{\alpha})=\check{\beta}$, or forces $\tau(\check{\alpha})\neq\check{\beta}$. This is possible using the previous claim. Let A^* be the intersection of all the sets A_ξ, and the set A. Since $\mathrm{Card}(\lambda\times\delta)<\kappa$, and since U is $<\kappa$ complete, A^* belongs to U.

$\langle s, A^*\rangle$ is stronger than each $\langle s, A_\xi\rangle$, so for every $\alpha\in\lambda$ and $\beta\in\delta$, either $\langle s, A^*\rangle$ forces $\tau(\check{\alpha})=\check{\beta}$, or else it forces $\tau(\check{\alpha})\neq\check{\beta}$. Holding α fixed, the latter cannot hold for all β, since $\langle s, A^*\rangle$ is also stronger than $\langle s, A\rangle$ which forces $(\exists\beta\in\check{\delta})\tau(\check{\alpha})=\beta$.

So for each $\alpha \in \lambda$ there exists $\beta \in \delta$ so that $\langle s, A^* \rangle$ forces $\tau(\check{\alpha}) = \check{\beta}$. This β must be unique, since $\langle s, A^* \rangle < \langle s, A \rangle$ forces "τ is a function".

Continuing to work in M, define $h(\alpha)$ to be the unique β so that $\langle s, A^* \rangle \Vdash_{\mathbb{P}}^{M} \tau(\check{\alpha}) = \check{\beta}$. Then $h\colon \lambda \to \delta$ belongs to M, and it is clear that $\langle s, A^* \rangle \Vdash_{\mathbb{P}}^{M} \tau = \check{h}$. □

Recall that Claim 3.5 implies that M and $M[G]$ have the same cardinals above κ, and agree on cofinalities of cardinals whose cofinalities in M are greater than κ. Our latest claim, 3.7, implies that M and $M[G]$ have the same cardinals below κ, and agree on cofinalities of cardinals whose cofinalities in M are smaller than κ. This completes the proof of Lemma 3.4. □

Remark 3.8. Note that we did not parallel Remark 2.7 for the restriction of $\mathbb{P}$ to conditions with a fixed stem s. We only paralleled Claim 2.6. It is because of this difference that the argument above is not an exact parallel of Claim 2.17. It establishes preservation of cofinalities for cardinals whose cofinality is smaller than κ in M; Claim 2.17 applied more generally to cardinals whose cofinality is smaller than κ in $M[G]$. The more general preservation fails for Prikry forcing, since the cofinality of κ is not preserved.

Corollary 3.9. *Suppose M is a countable model of* ZFC *satisfying "there exists κ of Mitchell order κ^{++}". Then there is a countable model of* ZFC *where the singular cardinals hypothesis fails.*

Proof. Using Theorem 3.2 there is a model M^* in which κ is measurable and $2^\kappa = \kappa^{++}$. Now using Prikry forcing over M^* one obtains an extension $M^*[G]$ where κ is singular of cofinality ω, and such that all cardinals of M^* remain cardinals in $M^*[G]$.

Since κ is measurable in M^*, $2^\delta < \kappa$ for $\delta < \kappa$, and using preservation of cardinals the same holds in $M^*[G]$. Since $2^\kappa = \kappa^{++}$ in M^*, and κ^{++} is the same in M^* and $M^*[G]$, $2^\kappa = \kappa^{++}$ also in $M^*[G]$. Thus, in $M^*[G]$, the singular cardinals hypothesis fails at κ. □

4. Further results

Corollary 3.9 is one of the most spectacular uses of large cardinal ultrafilters in forcing. There are many later results that build on similar techniques. For example Magidor [**21, 22**] develops techniques that combine Prikry forcing at a cardinal κ, with collapses between the points of the cofinal sequence of length ω added by the generic. Enough cardinals are collapsed below κ, that κ becomes $\aleph_\omega$ in the extension. The result is a model in which the singular cardinals hypothesis fails at $\aleph_\omega$:

Theorem 4.1. *Suppose there is a countable model of* ZFC *satisfying "there exists a cardinal κ of Mitchell order κ^{++}". Then there is a countable model of* ZFC *satisfying failure of the singular cardinals hypothesis fails at $\aleph_\omega$.*

Additional methods to reach such a failure were discovered later, for example in Gitik–Magidor [**12, 13**].

In a different direction, Prikry forcing can be generalized to yield an arbitrary cofinality for κ in the generic extension. The method for doing this was developed by Magidor [**23**]. It has seen many applications since, including some that are not immediately related to singularizing cardinals. Among them for example is a recent theorem of Gitik–Neeman–Sinapova [**14**] that (assuming large cardinals) it is possible to construct models $M \subseteq M[G]$ which have the same reals and cardinals, yet the class of ordinals of countable cofinality in M is so thin in $M[G]$, that it is not stationary at any regular cardinal λ of M beyond a starting cardinal κ.

In yet another direction, Prikry forcing can be modified to use supercompactness measures. Some of the most recent applications include constructions by Gitik–Sharon [**15**], obtaining a model where failure of the singular cardinals hypothesis is combined with failures of various combinatorial properties. The model was analyzed further by Cummings–Foreman [**7**], and the construction was combined with Magidor forcing by Sinapova [**41**] to obtain similar results on singular cardinals of arbitrary cofinality. Later on it was combined in Neeman [**29**] with a result of Magidor–Shelah [**24**] that the tree property holds at the cardinal successor of a limit of supercompact cardinals, to yield the following theorem:

THEOREM 4.2. *Suppose M is a countable model with ω supercompact cardinals. Let κ be the first one. Then there exists a generic extension $M[G]$ of M in which* $\mathrm{Cof}(\kappa) = \omega$*,* $2^{<\kappa} = \kappa$*,* $2^\kappa = \kappa^{++}$*, and the tree property holds at* κ^+*.*

The theorem answers a question of Woodin and others on whether failure of the singular cardinals hypothesis at κ implies failures of the tree property at κ^+. More specifically Woodin asked whether this is the case for $\kappa = \aleph_\omega$, and this question remains open.

Note that the failure of the singular cardinals hypothesis given by the theorem is qualitatively different from that given by Corollary 3.9. The Corollary is proved by singularizing a measurable cardinal κ without collapsing any cardinals. The tree property *fails* at successors of measurable cardinals, and the manner of failure is sufficiently absolute that it continues to fail in cardinal preserving generic extensions. The theorem in contrast reaches a situation where the tree property holds at the successor of κ.

Despite this qualitative difference, the theorem too is proved using Prikry forcing. But this time the Prikry forcing is relative to supercompactness measures, and many cardinals are collapsed. Indeed, letting τ be the supremum of the ω supercompact cardinals of M, all cardinals between κ and τ are collapsed, and κ^+, κ^{++} of $M[G]$ are the cardinals τ^+, τ^{++} of M.

Among the many other applications of forcing with ultrafilters there are some that are done under the axiom of determinacy, rather than the axiom of choice. By $\mathbb{R}$ below we mean $\mathbb{N}^\omega$ with the product topology of the discrete topology on $\mathbb{N}$. (The space is isomorphic to the irrational numbers with the usual topology.) In a *game* with payoff $A \subseteq \mathbb{R}$, two players I and II alternate turns playing $x(i) \in \mathbb{N}$, $i < \omega$. Player I wins a run $x = \langle x(i) \mid i < \omega \rangle$ if $x \in A$, and otherwise player II wins. The game is *determined* if one of the players has a winning strategy. The set A is called the *payoff* for the game. Determinacy for a pointclass $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ is the statement that every such game with payoff in Γ is determined.

Perhaps surprisingly, determinacy turns out to provide the appropriate axiomatization for studying definable sets of reals, for example Borel sets, the sets generated by projections and complementations starting from Borel sets, all sets in $L(\mathbb{R})$, and more. Determinacy for all Borel sets is a theorem of ZFC, proved by Martin [**25, 26**]. For the larger pointclasses mentioned above, determinacy is provable using large cardinal axioms in the region of Woodin cardinals, see Martin–Steel [**27**] and Woodin [**45**] for the original proof, Neeman [**30, 31**] for a proof from optimal assumptions, and Neeman [**33**] for a self-contained proof.

The full axiom of determinacy (AD) asserts that all games of the kind described above, with any payoff set, are determined. Full determinacy contradicts the axiom of choice, and one can only expect it to hold in a submodel of V where choice fails. Under large cardinal axioms it holds in the model $L(\mathbb{R})$, consisting of all sets constructible from reals. Determinacy turns out to provide the correct axiomatization for studying this model.

One of the consequences of $\mathsf{AD}^{L(\mathbb{R})}$ is that there are many countably complete ultrafilters in $L(\mathbb{R})$. Even the first uncountable cardinal, $\aleph_1$, carries a measure. (Note that this does not mean there is an elementary embedding with critical point $\aleph_1$ definable over $L(\mathbb{R})$; such a map would contradict the discussion following Claim 1.2. It only means there is a countably complete, normal, non-principal ultrafilter on $\aleph_1$. The connection between these two statements breaks down without the axiom of choice.) These ultrafilters can be used for forcing over $L(\mathbb{R})$. We say more on this below.

A set A is $\boldsymbol{\Sigma}^1_1$ if it is the projection (for example from a plane to the line) of a Borel set. A set is $\boldsymbol{\Pi}^1_n$ if it is the complement of a $\boldsymbol{\Sigma}^1_n$ set, and $\boldsymbol{\Sigma}^1_{n+1}$ if it is the projection of a $\boldsymbol{\Pi}^1_n$ set. A set is $\boldsymbol{\Delta}^1_n$ if it is both $\boldsymbol{\Sigma}^1_n$ and $\boldsymbol{\Pi}^1_n$.

A *prewellorder* is a relation which is transitive, reflexive, linear, and wellfounded. Given a prewellorder R, define an equivalence relation $\sim$ by $x \sim y$ iff $x\, R\, y \wedge y\, R\, x$. Then R induces a wellorder on the equivalence classes of $\sim$. The *ordertype* of R is the unique ordinal isomorphic to this wellorder. We use prewellorders rather than wellorders below, since in contexts where the axiom of choice fails, for example in the context of definable sets of reals, wellorders are too restrictive. (Moving from prewellorders to wellorders requires choosing representatives for equivalence classes, and this cannot always be done in a definable way.)

For each n, $\boldsymbol{\delta}^1_n$ is the supremum of the ordertypes of $\boldsymbol{\Delta}^1_n$ prewellorders on $\mathbb{R}$. It is clear that every such ordertype is $< \mathrm{Card}(\mathbb{R})^+$, so $\boldsymbol{\delta}^1_n < (2^{\aleph_0})^+$. $\boldsymbol{\delta}^1_n$ is a measure of the size of the continuum in term of definable orders. We saw that the size of $2^{\aleph_0}$ can easily be increased beyond $\aleph_1$ by forcing. As one can imagine, it is much harder to change the size of the $\boldsymbol{\delta}^1_n$s. The ordinals are specified using definable orders, and the mere addition of reals will not change them. Nonetheless, it is consistent that some of these ordinals are cardinals greater than $\aleph_1$. For example by forcing over a model of determinacy, Steel–Van Wesep [**42**] showed it is consistent with ZFC that $\boldsymbol{\delta}^1_2 = \aleph_2$. They assumed more than determinacy, and forced over a model bigger than $L(\mathbb{R})$. But the assumption was reduced to AD for forcing over $L(\mathbb{R})$, by Woodin [**44**].

It is still open whether any $\boldsymbol{\delta}^1_n$ can be a cardinal greater than $\aleph_2$. However it is known that $\aleph_2$ can be realized by other $\boldsymbol{\delta}^1_n$s, not just $\boldsymbol{\delta}^1_2$.

Theorem 4.3. *It is consistent with* ZFC *that* $\boldsymbol{\delta}^1_3 = \aleph_2$.

The theorem is due independently to Neeman and Woodin, see Neeman [**32**]. It generalizes to $n > 3$. The proof is done by forcing over $L(\mathbb{R})$ assuming $\mathsf{AD}^{L(\mathbb{R})}$. The forcing notion uses a supercompactness measure on $\mathcal{P}_{\omega_1}(\lambda)$, where λ is the cardinal predecessor of $\boldsymbol{\delta}^1_3$, in $L(\mathbb{R})$. It is very similar to Prikry forcing, but the stems are countable rather than finite, and the measure used is not the supercompactness measure itself, but a certain product of the measure, that applies to sequences of variable countable length. This product is analyzed in [**32**] using the close connections between determinacy and inner model theory.

We concentrated above on uses of large cardinals in forcing, that directly involve the large cardinal ultrafilters. There are many other uses that involve reflection properties of the large cardinals, rather than the resulting ultrafilters. Perhaps the most famous are the proofs of the consistency of the *proper and semi-proper forcing axioms* (PFA and SPFA), and *Martin's maximum* (MM) from supercompact cardinals. The axioms were developed through work of Baumgartner, Shelah, and Foreman–Magidor–Shelah , see [**3, 4, 36, 35, 9**]. They assert the existence, in the ground model, of partial generics for all forcing notions in specific classes of posets, called proper, semi-proper, and stationary set (in ω_1) preserving. The third axiom follows from the second. The consistency of each of the first two is obtained by forming an iteration, composed of a selection of posets in the respective class, reaching up to a supercompact cardinal. The reflection properties of the supercompact cardinal are such that, with a careful selection of the posets via a mechanism due to Laver, the resulting model has partial generics for *all* posets in the class. For a survey of this subject we refer the reader to Abraham [**1**].

Let us now return to our original application of ultrafilters in forcing, Corollary 3.9. Its initial form, using the existence of a supercompact cardinal in the hypothesis, is due to Silver. The large cardinal assumption was brought down to the existence of a cardinal κ of Mitchell order κ^{++} through work of Woodin and Gitik [**10**]. This lower assumption is *necessary*:

THEOREM 4.4. *Suppose it is consistent that the singular cardinals hypothesis fails. Then it is consistent that there exists a cardinal κ of Mitchell order κ^{++}.*

The theorem is due to Gitik [**11**]. It is proved using two very deep subjects in set theory: PCF theory which we did not discuss and which the reader can find in Shelah [**38**] and Abraham–Magidor [**2**]; and inner model theory, which we discussed briefly in Section 1.

Inner model theory is concerned with the construction of canonical models for large cardinal axioms, and is used to extract large cardinal strength from statements about the universe. The application of inner model theory in the proof of Theorem 4.4 builds heavily on the development of inner models at the level of measures of high order, due to Mitchell [**28**]. One of the key results is a *covering lemma* for these models, due to Mitchell, that allows approximating elements of the actual universe V using elements of the core models and ordinals generated from the measures in the core model. The first result of this kind is due to Jensen [**8**] and deals with the model L. It states that either (a) every uncountable set of ordinals in the universe can be covered by a set of the same size that belongs to L; or (b) there is a class of ordinals that is homogeneous with respect to truth in L (such classes are called classes of indiscernibles for L, and they capture the content of a measurable cardinal, restricted to L). Covering lemmas continue to hold with L replaced by inner models for large cardinals, but their forms become more restricted as one allows greater large cardinals. One of the most recent examples can be found in Jensen–Schimmerling–Schindler–Steel [**17**], where it is used to derive large cardinal strength from the proper forcing axiom. But the exact strength of the proper forcing axiom remains open. It may well be at the level of supercompact cardinals.

Bringing inner model theory to this level is one of the longest standing open projects in set theory.

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