UNRAVELING Π^1_1 SETS, REVISITED

ITAY NEEMAN

Abstract. We present a method of unraveling Π^1_1 sets which greatly simplifies the construction in Neeman [6]. Apart from adding elegance, this method is also useful for proofs of determinacy involving long games, see Neeman [5].

By a tree we mean a set of finite sequences, closed under initial segments. Given a tree S let $[S]$ denote the set of infinite branches through S. Given further a set $A \subset [S]$ let $G_S(A)$ be the game in which players I and II collaborate to create a branch $\langle a_0, a_1, \dots \rangle$ through S. The players take turn, I picking a_n for even n, and II picking a_n for odd n, subject to the rule that $\langle a_0, \dots, a_n \rangle \in S$. The first player to violate this rule loses. If the rule is maintained for ω steps, then player I wins just in case that $\langle a_i \mid i < \omega \rangle$ belongs to A. $G_S(A)$ is **determined** if one of the players has a winning strategy.

Definition 1 (Martin [3]). Let S be a tree. A triple (T, π, Ψ) is a **covering** of S just in case that:

1. T is a tree.
2. $\pi \colon [T] \to [S]$.
3. $\Psi \colon \mathrm{Strat}(T) \to \mathrm{Strat}(S)$ (where $\mathrm{Strat}(T)$ is the set of strategies on T, and similarly for S). Ψ sends strategies for I on T to strategies for I on S, and similarly for II.
4. Ψ and π are connected through the following *lifting* condition: Suppose that $\Sigma \in \mathrm{Strat}(T)$ and $y \in [S]$ is according to $\Psi(\Sigma)$. Then there is $x \in [T]$, according to Σ, so that $\pi(x) = y$.

The covering (T, π, Ψ) **unravels** a set $A \subset [S]$ just in case that $\pi^{-1}(A)$ is *clopen* in $[T]$.

Using condition (4) it is easy to check that a winning strategy in the game $G_T(\pi^{-1}(A))$ is sent by Ψ to a winning strategy in the game $G_S(A)$. If $\pi^{-1}(A)$ is clopen then $G_T(\pi^{-1}(A))$ is determined by a theorem of Gale–Stewart [1]. It follows that if A can be unraveled, then $G_S(A)$ is determined.

Covers and the property of unraveling were introduced by Martin [3], who went on to inductively unravel all Borel sets, thereby obtaining an inductive proof of Borel determinacy, simplifying his earlier proof of Borel determinacy in [2]. Martin [4] took matters a bit further, and unraveled all Δ^1_1 sets (in the case of uncountable trees this is a larger pointclass than Borel).

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For reasons concerning the lower bounds on the large cardinal strength of determinacy for levels of the Borel hierarchy on Π^1_1 sets (see Steel [7]), it seemed natural to conjecture that it should be possible to unravel Π^1_1 sets using the following large cardinal assumption: $(*)$ there is a cardinal κ, so that for every $Z \subset \mathcal{P}(\kappa)$, there is a measure μ on κ so that Z belongs to $\mathrm{Ult}(M, \mu)$.

Neeman [6] unraveled Π^1_1 subsets of $\mathbb{R} = [\omega^{<\omega}]$ using this assumption, and obtained precisely the determinacy results suggested by Steel [7]. The construction in Neeman [6] involved a challenge game, in which player I proposes a system of trees, whose inverse limit she claims is a cover for S. Player II can either accept, in which case the game proceeds on the inverse limit proposed by I; or else player II can reject one of the levels of the system, in which case I must present this level as an inverse limit over the previous level, II must reject a level of this new system, etc.

Overall the construction in Neeman [6] was quite complicated, but it turns out that many of these complications are not necessary. We present here a substantially simpler construction of a cover which unravels a given Π^1_1 subset of $\mathbb{R} = [\omega^{<\omega}]$.

The construction is based on "rank games," which we introduce in this paper. These games allow the players to pick an ordinal together, somehow dividing the task between them in such a way that *we* can (in ultrapowers by measures given by the large cardinal assumption $(*)$ above) manipulate a strategy for *either* player into picking specific ordinals.

In Section 1 we introduce rank games, and begin to define the cover (T, π, Ψ). More precisely we define T and π in that section. The definition of Ψ and the proof of the lifting condition (4) in Definition 1 are spread over Sections 2 and 3. Section 2 defines $\Psi(\Sigma)$ in the case that Σ is a strategy for I, and proves that condition (4) holds in this case. Section 3 handles the case that Σ is a strategy for II.

It turns out that the rank games we introduce here are useful also in proofs of determinacy for long games, and this matter is investigated in Neeman [5]. The paper uses these rank games in a proof of the determinacy of games ending at the first admissible relative to the play, from optimal large cardinal assumptions.

§1. Rank games. Fix throughout the paper a map $s \mapsto \preceq_s$ which associates to each node $s \in \omega^{<\omega}$ a linear order $\preceq_s$ on $\mathrm{lh}(s) + 1$, in such a way that if s extends t then $\preceq_s$ extends $\preceq_t$. For each $x \in \omega^\omega$ let $\preceq_x = \bigcup_{n<\omega} \preceq_{x \restriction n}$. This is then a linear order on ω. For simplicity suppose that 0 is the largest element in $\preceq_s$, for each s.

Let $h\colon \mathrm{V} \to \mathrm{V} - \{0\}$ be the injection defined by $h(x) = x$ if $x \notin \omega$ and $h(x) = 1 + x$ if $x \in \omega$.

Given a cardinal κ, a set $A \subset \mathrm{V}_{\kappa+1}$, a node $s \in \omega^{<\omega}$, and a some $w \in \mathrm{V}_\kappa$, define the (s, w)–**section of** A, denoted $A_{s,w}$, to be the set $\{U \subset \mathrm{V}_\kappa \mid \{\langle s, w\rangle\} \times (\{0\} \cup h''U) \in A\}$. The (s, w)–section of A is then a subset of $\mathrm{V}_{\kappa+1}$. The purpose of the definition is to allow coding V_κ many subsets of $\mathrm{V}_{\kappa+1}$ as one. The following claim phrases this precisely:

Claim 1.1. *Suppose that to each $s \in \omega^{<\omega}$ and each $w \in \mathrm{V}_\kappa$ we have associated some $B(s,w) \subset \mathrm{V}_{\kappa+1}$. Then there is a set $A \subset \mathrm{V}_{\kappa+1}$ which codes this association, in the sense that for every s and w, $A_{s,w}$ is precisely equal to $B(s,w)$.*

Proof. Set $A = \{\, \{\langle s,w\rangle\} \times (\{0\} \cup h''U) \mid U \in B(s,w)\}$. $\dashv$

Remark 1.2. The point of the move from U to $\{0\} \cup h''U$ is to make sure that $\langle s,w\rangle \times \emptyset$ never comes up when dealing with sections. (We have to avoid this set since it is impossible to recover s and w from it.) This is a minor technical point, and for the sake of notational clarity we ignore it below, writing U where we should be writing $\{0\} \cup h''U$.

Definition 1.3. The **basic rank game** associated to κ, A, s, and w is played according to the following rules:

- Player I plays $U \subset \mathrm{V}_\kappa$ which belongs to the (s,w)–section of A;
- Player II plays $\langle\bar\kappa, \bar A\rangle$ so that:
  1. $\bar\kappa < \kappa$, and $\bar\kappa$ is larger than the Von-Neumann rank of w,
  2. $\bar A \subset \mathrm{V}_{\bar\kappa+1}$, and
  3. $\langle\bar\kappa, \bar A\rangle \in U$.

This ends the game.

The basic rank game thus starts with the pair $\langle\kappa, A\rangle$, and through the moves in the game the players collaborate to choose a new pair $\langle\bar\kappa, \bar A\rangle$, with $\bar\kappa < \kappa$. The actual choice is made by player II, but notice that I may regulate this choice through the restriction that $\langle\bar\kappa, \bar A\rangle$ must belong to U. Player I's choice of U in turn is regulated by the initial set A.

To a large extent our interest is in the new move $\bar\kappa$. The basic rank game lets the players choose this ordinal *together*. We now use the basic rank game to define a game in which players I and II collaborate to play $x \in \omega^\omega$, and in addition produce (among other things) an embedding of $\preceq_x$ into the ordinals. Each of the ordinals in the range of this embedding is chosen through a collaboration between the two players, using a basic rank game.

$$\begin{array}{c|l} \mathrm{I} & \cdots\cdots\cdots\ x(n-1)\ \ U_n \qquad\qquad\qquad \cdots\cdots \\ \hline \mathrm{II} & \qquad\qquad\qquad\qquad\qquad\quad \langle\kappa_n, A_n\rangle \end{array}$$

Diagram 1. Round n in the repeated rank game.

Definition 1.4. Fix a cardinal κ and a set $A \subset \mathrm{V}_{\kappa+1}$. In the **repeated rank game** associated to κ and A, players I and II collaborate to produce $x \in \omega^\omega$ and a sequence of pairs $\langle\kappa_n, A_n\rangle$ $(n < \omega)$ so that the map $i \mapsto \kappa_i$ embeds $\preceq_x$ into the ordinals. We set $\kappa_0 = \kappa$ and $A_0 = A$ to begin with. The game proceeds according to Diagram 1 and the following format, beginning with round 1:

- At the start of round n we have $x \restriction n-1$ and the pairs $\langle\kappa_i, A_i\rangle$ for $i < n$. We know inductively that $i \preceq_{x\restriction n-1} j$ iff $\kappa_i \le \kappa_j$ for $i, j < n$.
- The appropriate player—I if $n-1$ is even and II if $n-1$ is odd—plays $x(n-1) \in \omega$.

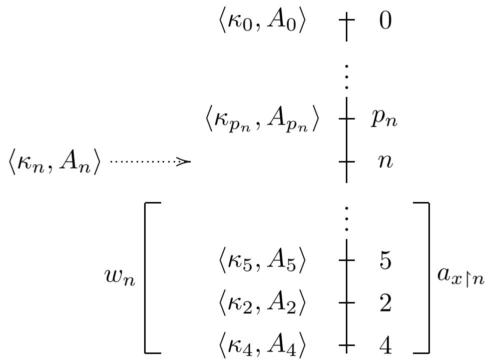

Diagram 2. Configuration at round n of the repeated rank game.

- Let p_n be the successor of n in the order $\preceq_{x\restriction n}$. Let $w_n = \{\langle\kappa_j, A_j\rangle \mid j \prec_{x\restriction n} n\}$. (Both assignments are illustrated in Diagram 2.)
- Players I and II now pick U_n and $\langle\kappa_n, A_n\rangle$ subject to the rules of the basic rank game associated to κ_{p_n}, A_{p_n}, $x\restriction n$, and w_n.

In the case of the last item notice that, by the rules of the basic rank game, κ_n is smaller than κ_{p_n} and larger than the Von-Neumann rank of w_n, hence larger than κ_j for each $j \prec_{x\restriction n} n$. It follows that $\kappa_j \leq \kappa_n$ iff $j \preceq_{x\restriction n} n$ for each $j \leq n$. By induction then, the map $i \mapsto \kappa_i$ embeds $\preceq_x$ into the ordinals, and we get the following claim:

Claim 1.5. *Suppose that $x \in \omega^\omega$ and $\langle\kappa_n, A_n\rangle$ ($n < \omega$) are part of an infinite play of the repeated rank game. Then $\preceq_x$ is wellfounded.* $\dashv$

It is worthwhile abstracting some of the properties of the objects which come up during round n of the repeated rank game.

Let $s \in \omega^{<\omega}$. A sequence $s^* \in \omega^{<\omega}$ is a **suitable extension** of s if it extends s and if in addition $\mathrm{lh}(s)$ is the successor of $\mathrm{lh}(s^*)$ in $\preceq_{s^*}$. In the context of the repeated rank game, displayed in Diagram 2, $x\restriction n$ is a suitable extension of $x\restriction p_n$.

Let a_s denote the set $\{j < \mathrm{lh}(s) \mid j \prec_s \mathrm{lh}(s)\}$. Diagram 2 illustrates $a_{x\restriction n}$. We say that w is **suitable** for s if it has the form $\{\langle\kappa_j, A_j\rangle \mid j \in a_s\}$ with the map $j \mapsto \kappa_j$ order preserving from $\preceq_s$ (more precisely its restriction to a_s) into the ordinals. Notice that this map is then determined uniquely by w and s. In the context of the repeated rank game, w_n is suitable for $x\restriction n$.

Let $s^* \in \omega^{<\omega}$ be an extension of s. Let w^* have the form $\{\langle\kappa^*_j, A^*_j\rangle \mid j \in a^*\}$ with a^* an initial segment in $\preceq_{s^*}$ and with the map $j \mapsto \kappa^*_j$ order preserving from $\preceq_{s^*}\restriction a^*$ into the ordinals. (For example any w^* which is suitable for s^* has this form.) Let w be suitable for s and let $j \mapsto \kappa_j$ be the unique map witnessing this. We say that w^* **extends** w (wrt s, s^*), or that w is an **initial segment** of w^*, just in case that the map $j \mapsto \kappa^*_j$ extends the map $j \mapsto \kappa_j$.

Claim 1.6. *Suppose that s_n, w_n ($n < \omega$) are such that $s_0 = \emptyset$, s_{n+1} is a suitable extension of s_n for each n, each w_n is suitable for s_n, and w_{n+1} extends w_n for each n. Let $x = \bigcup_{n<\omega} s_n$. Then:*

1. *$\preceq_x$ is illfounded; and*
2. *The wellfounded part of $\preceq_x$ is precisely $\bigcup_{n<\omega} a_{s_n}$.*

Proof. Let $k_n = \mathrm{lh}(s_n)$. From the assumption that s_{n+1} is a suitable extension of s_n it follows that $k_{n+1} \preceq_x k_n$. So $\preceq_x$ is illfounded.

Let $j \mapsto \kappa^n_j$ witness that w_n is suitable for s_n. The domain of this map is the set a_{s_n}, and the map embeds the restriction of $\preceq_x$ to a_{s_n} into the ordinals. Since w_{n+1} extends w_n for each n, the union of the maps makes sense, witnessing that $\preceq_x$ is wellfounded on $\bigcup_{n<\omega} a_{s_n}$. It's easy to check that $\bigcup_{n<\omega} a_{s_n}$ is closed downward in $\preceq_x$, and that every $j \in \omega - \bigcup_{n<\omega} a_{s_n}$ sits above some k_n in $\preceq_x$ and therefore belongs to the illfounded part of $\preceq_x$. It follows that the wellfounded part is precisely equal to $\bigcup_{n<\omega} a_{s_n}$. ⊣

Suppose now that κ is measurable. For each measure μ on κ let i_μ denote the ultrapower embedding of V by μ.

Definition 1.7. Let $A \subset \mathrm{V}_{\kappa+1}$. In the **inverted rank game** associated to κ and A players I and II collaborated to create, among other things, a sequence of objects s_n, w_n $(n < \omega)$ satisfying the assumptions of the previous claim. We set $A_0 = A$, $s_0 = \emptyset$, and $w_0 = \emptyset$ to begin with. The game proceeds according to Diagram 3 and the following format, beginning with round 1:

- At the start of round n we have s_{n-1}, w_{n-1}, and a set $A_{n-1} \subset \mathrm{V}_{\kappa+1}$.
- Player II plays $s_n \in \omega^{<\omega}$, $w_n \in \mathrm{V}_\kappa$, and $U_n \subset \mathrm{V}_\kappa$ so that s_n is a suitable extension of s_{n-1}, w_n is suitable for s_n, w_n extends w_{n-1}, and U_n belongs to the (s_n, w_n)–section of A_{n-1}.
- Player I plays μ_n and A_n so that μ_n is a measure on κ, $A_n \subset \mathrm{V}_{\kappa+1}$, and $\langle \kappa, A_n \rangle$ belongs to $i_{\mu_n}(U)$.

$$
\begin{array}{c|ccc}
\mathrm{I} & \cdots\cdots\cdots & \qquad\qquad \mu_n, A_n & \cdots\cdots \\
\hline
\mathrm{II} & & s_n, w_n, U_n \qquad\qquad &
\end{array}
$$

Diagram 3. Round n in the inverted rank game.

Notice the reversal of roles in this game compared to the basic rank game. Here it is player II that picks U_n, and player I that picks A_n. Note further that I is accorded better freedom here than was given to II in the basic rank game. She is *not* asked to play $\bar{\kappa}$ *below* κ with $\langle \bar{\kappa}, A_n \rangle \in U_n$. Instead she gets to push the universe up using an ultrapower embedding i_{μ_n} by her choice of measure μ_n, refer to $i_{\mu_n}(U_n)$ instead of U_n itself, and continue to use the same κ. This is illustrated in Diagram 4.

Suppose now that κ satisfies the following assumption:

(∗) For every $Z \subset \mathrm{V}_{\kappa+1}$ there exists a measure μ on κ so that Z belongs to the ultrapower $\mathrm{Ult}(\mathrm{V}, \mu)$.

For the rest of the paper we work with a fixed κ satisfying this assumption.

Definition 1.8. Define T be the following game tree: In round 0 player I picks a set $A \subset \mathrm{V}_{\kappa+1}$. Player II can *accept*, or *reject* this set. This completes round 0. If player II accepts then the two players continue by playing the *repeated* rank

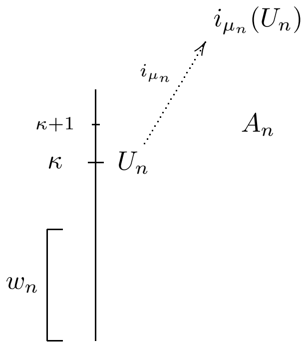

Diagram 4. Pushing U_n up.

game associated to κ and A. If player II rejects then the two players continue by playing the *inverted* rank game associated to κ and A.

T should be viewed as a game where player I proposes an outline of a division of work between herself and player II, with the aim of producing $x \in \mathbb{R}$ and witnessing that $\preceq_x$ is wellfounded. This is the set A, and the division of work is the repeated rank game associated to κ and A, which the players undertake if II accepts. If II rejects, then she gets to test player I's fairness through the reversal of roles in the inverted rank game.

Definition 1.9. Let $\pi\colon T \to \omega^{<\omega}$ be the natural projection: If $v \in T$ is a position in which II accepts, covering rounds 0 through $n-1$ say, then $\pi(v)$ is the sequence $x\restriction n-1$ constructed through the moves in the repeated rank game made in v. If $v \in T$ is a position in which II rejects then $\pi(v) = s_n$ where s_n is the last sequence played by II through her moves in v for the inverted rank game. (If v only involves moves in round 0 of T then $\pi(v) = \emptyset$.)

For an infinite branch $\vec{v} \in [T]$ let $\pi(\vec{v}) = \bigcup_{n<\omega} \pi(\vec{v}\restriction n)$.

Claim 1.10. *Let $C \subset \mathbb{R}$ be the set of x so that $\preceq_x$ is wellfounded. Then $\pi^{-1}(C)$ is a clopen subset of $[T]$.*

Proof. $\pi^{-1}(C)$ consists precisely of those plays in which II accepts. $\dashv$

For each $x \in \omega^\omega$ so that $\preceq_x$ is illfounded, let $\vec{k}(x)$ be the **left-most** infinite descending chain in $\preceq_x$. Precisely this is the chain $\langle k_n \mid n < \omega\rangle$ determined by: $k_0 = 0$; and for each n, k_{n+1} is equal to the least $k > k_n$ which (a) belongs to the illfounded part of $\preceq_x$, and (b) sits below k_n in $\preceq_x$

Claim 1.11. *The function $\vec{v} \mapsto \vec{k}(\pi(\vec{v}))$ is Lipschitz continuous on the set $\{\vec{v} \in T \mid$ II rejects in $\vec{v}\}$. More precisely, for $\vec{v} \in T$ in which II rejects, $k_0(\pi(\vec{v})), \dots, k_n(\pi(\vec{v}))$ depend only the moves in rounds 0 through n in $\vec{v}$.*

Proof. Let $\vec{v}$ be a run of T in which II rejects. Let s_n, w_n, U_n, μ_n, and A_n denote the moves in round n of $\vec{v}$. Let $x = \bigcup_{n<\omega} s_n$. The left-most infinite descending chain in $\preceq_x$ is then precisely the sequence $\langle k_n = \mathrm{lh}(s_n) \mid n < \omega\rangle$. (This uses Claim 1.6 and the fact that for each n, s_{n+1} is a suitable extension of s_n.) The claim follows. $\dashv$

Our plan is to expand T and π to a cover of $\omega^{<\omega}$. Once we do this we'll be done: By Claim 1.10 that cover unravels the Π^1_1 set $C = \{x \mid \preceq_x \text{ is wellfounded}\}$. Since $s \mapsto \preceq_s$ is arbitrary it follows that any Π^1_1 set can be unraveled.

Moreover, Neeman [6, §6] shows that for any countable collection of Π^1_1 sets D_i, there is a map $s \mapsto \preceq_s$ so that any cover with the property given by Claim 1.11 unravels each of the sets D_i. Once we expand T and π to a cover it will therefore follow that any countable collection of Π^1_1 sets can be unraveled, by a cover using T and π. From this one can obtain a wide array of determinacy results, see Neeman [6, §7].

§2. Strategies for player I. Let $s \in \omega^{<\omega}$. An s–**iteration** consists of sequences $\langle M_j \mid j \leq \mathrm{lh}(s)\rangle$ and $\langle \mu_j \mid 0 < j \leq \mathrm{lh}(s)\rangle$ so that:

- For each $0 < j \leq \mathrm{lh}(s)$, μ_j is a measure in M_j;
- $M_k = \mathrm{Ult}(M_j, \mu_j)$ where k is the $\preceq_s$ successor of j; and
- $M_k = \mathrm{V}$ in the case that k is smallest in $\preceq_s$.

This is simply an iteration of V, of order type $\preceq_s$. For $j \preceq_s k$ we use $i_{j,k} \colon M_j \to M_k$ to denote the embedding induced by the iteration. We use i_n to denote the embedding $i_{k,n}$ where k is smallest in $\preceq_s$. This is an embedding from V into M_n.

Recall that T is the tree of Definition 1.8, and π is the projection of Definition 1.9. We call a position v in T **whole** if it ends with a complete round (as opposed to just the first move in that round for example). Otherwise v is **medial**. Note that if v is a whole position in which II rejects, then the first player to move after v is player II, see Diagram 3. If v is a medial position in which II rejects, then the first player to move after v is player I.

Definition 2.1. A **cluster** for $s \in \omega^{<\omega}$ consists of an s–iteration together with positions $a \in i_0(T)$ and $r_j \in i_j(T)$ for $j \leq \mathrm{lh}(s)$, so that: a is a whole position in which II accepts, covering rounds 0 through $\mathrm{lh}(s)$, with $i_0(\pi)(a)$ equal to s; and each r_j is a position (either whole or medial) in which II does not accept, with $i_j(\pi)(r_j)$ equal to $s \restriction j$. (For $j = 0$ we allow r_j to be the empty position. Except for this case, r_j must be a non-empty position in which II rejects. Note that the position a is in the shift of T to M_0, and the positions r^n_j are in the shifts of T to the models M_j.)

A cluster is **according** to Σ, where Σ is a strategy for one of the players in T, just in case that the position a is consistent with $i_0(\Sigma)$, and for each $j \leq \mathrm{lh}(s)$, the position r_j is consistent with $i_j(\Sigma)$.

Definition 2.2. Let $n = \mathrm{lh}(s)$, and let $\bar{s} = s \restriction n-1$. Let p be the successor of n in $\preceq_s$. (These settings are related to the situation illustrated in Diagram 2.) Let $\{M_j, \mu_j, r_j, a\}$ be a cluster for s, and let $\{\bar{M}_j, \bar{\mu}_j, \bar{r}_j, \bar{a}\}$ be a cluster for $\bar{s}$. We say that $\{M_j, \mu_j, r_j, a\}$ **extends** $\{\bar{M}_j, \bar{\mu}_j, \bar{r}_j, \bar{a}\}$ just in case that the following conditions hold:

1. $M_j = \bar{M}_j$, $\mu_j = \bar{\mu}_j$, and $r_j = \bar{r}_j$ for each $j \prec_s n$;
2. $M_n = \bar{M}_p$, and r_n strictly extends $\bar{r}_p$;[1]

[1]The requirement $M_n = \bar{M}_p$ is in fact implied by condition (1). We have $M_n = \mathrm{Ult}(M_j, \mu_j) = \mathrm{Ult}(\bar{M}_j, \bar{\mu}_j) = \bar{M}_p$ where j is the $\preceq_s$ predecessor of n, or equivalently the

3. For $j \succeq_s p$, $M_j = i(\bar{M}_j)$, $\mu_j = i(\bar{\mu}_j)$, and $r_j = i(\bar{r}_j)$, where $i = i^{M_n}_{\mu_n}$ is the ultrapower embedding of M_n $(= \bar{M}_p)$ by μ_n; and
4. $M_0 = i(\bar{M}_0)$ and a strictly extends $i(\bar{a})$, where again $i = i^{M_n}_{\mu_n}$.

This situation is illustrated in Diagram 5.

Notice that for each $k < n$ there is an elementary embedding $h\colon \bar{M}_k \to M_k$. h is equal to the identity if $k \prec_s p$, and equal to $i^{M_n}_{\mu_n}$ if $k \succeq_s p$. Notice that in both cases, $\mu_k = h(\bar{\mu}_k)$. We refer to h as the **extension embedding** associated to $\bar{M}_k$ and the two clusters.

$$
\begin{tikzcd}
 & r_j = \bar{r}_j & & r_n \supsetneq \bar{r}_p & r_j = i^{M_n}_{\mu_n}(\bar{r}_j) & a \supsetneq i^{M_n}_{\mu_n}(\bar{a}) \\
M_4 & M_2 & M_5 \arrow[r, dotted] & M_n \arrow[r, "i^{M_n}_{\mu_n}"] & M_p \arrow[r, dotted] & M_0 \\
\bar{M}_4 \arrow[u, equal] \arrow[r, "\bar{\mu}_4"'] & \bar{M}_2 \arrow[u, equal] \arrow[r, "\bar{\mu}_2"'] & \bar{M}_5 \arrow[u, equal] \arrow[r, dotted] & \bar{M}_p \arrow[u, equal] \arrow[r, dotted] \arrow[ur, "i^{M_n}_{\mu_n}"'] & \bar{M}_0 &
\end{tikzcd}
$$

Diagram 5. $\{M_j, \mu_j, r_j, a\}$ extends $\{\bar{M}_j, \bar{\mu}_j, \bar{r}_j, \bar{a}\}$.

Lemma 2.3. *Fix a strategy Σ for one of the players in T. Let $x \in \omega^\omega$. Suppose that there is a sequence of clusters $\{M^n_j, \mu^n_j, r^n_j, a^n\}$ so that:*

1. *For each $n < \omega$, $\{M^n_j, \mu^n_j, r^n_j, a^n\}$ is a cluster for $x \restriction n$;*
2. *Each of the clusters $\{M^n_j, \mu^n_j, r^n_j, a^n\}$ is according to Σ; and*
3. *For each $n > 0$, $\{M^n_j, \mu^n_j, r^n_j, a^n\}$ extends $\{M^{n-1}_j, \mu^{n-1}_j, r^{n-1}_j, a^{n-1}\}$.*

Then there exists a branch $\vec{v} \in [T]$ so that $\vec{v}$ is consistent with Σ and $\pi(\vec{v}) = x$.

Remark 2.4. Clusters satisfying the conditions of the lemma are naturally created by strategies for player I in T, as we shall see later on, in Lemma 2.6.

Remark 2.5. The conclusion of the lemma fits with the requirements for covers, and we shall use this later on.

Proof of Lemma 2.3. For each k and each $n > k$ let $h^{n-1,n}_k\colon M^{n-1}_k \to M^n_k$ be the extension embedding associated to M_k and the clusters $\{M^n_j, \mu^n_j, r^n_j, a^n\}$ and $\{M^{n-1}_j, \mu^{n-1}_j, r^{n-1}_j, a^{n-1}\}$. Let M^∞_k be the direct limit of the models M^n_k under the embeddings $h^{n-1,n}_k$. Let $h^{n,\infty}_k\colon M^n_k \to M^\infty_k$ be the direct limit embeddings. This is illustrated in Diagram 6.

Case 1: If $\preceq_x$ is wellfounded. We work in this case, essentially using the positions a^n (which are increasing, modulo some elementary embeddings, by condition (4) in Definition 2.2) to construct an infinite branch through the shift

$\preceq_{\bar{s}}$ predecessor of p. The first and last equalities come from the definition of an iteration, and the middle equality is by condition (1)

$$\begin{tikzcd}
 & M_2^\infty & & M_p^\infty & & M_0^\infty \\
 & & & M_p^{n+1} \arrow[u, dotted, "h_p^{n+1,\infty}"] & & \\
M_4^n \arrow[r] & M_2^n \arrow[r, dotted] \arrow[uu, dotted, "h_2^{n,\infty}"] & M_n^n \arrow[r] & M_p^n \arrow[rr, dotted] \arrow[u, "h_p^{n,n+1}"] & & M_0^n \arrow[uu, dotted] \\
M_4^{n-1} \arrow[u, "h_4^{n-1,n}"] \arrow[r, "\mu_4^{n-1}"'] & M_2^{n-1} \arrow[u, "h_2^{n-1,n}"] \arrow[r, dotted] & M_p^{n-1} \arrow[ur, "h_p^{n-1,n}"'] \arrow[rr, dotted] & & M_0^{n-1} \arrow[ur, "h_0^{n-1,n}"'] & \\
M_0^0 \arrow[urrrr, dotted] & & & & &
\end{tikzcd}$$

Diagram 6. The direct limits M_k^∞.

of T to M_0^∞. Then using absoluteness and elementarity we will pull the existence of a suitable branch back to V.

For $k \neq 0$ let $\mu_k^\infty = h^{n,\infty}(\mu_k^n)$ for some/any $n \geq k$. (It doesn't matter which n is used, since $\mu_k^n = h_k^{n-1,n}(\mu_k^{n-1})$ for each $n > k$.) Notice then that $\langle M_k^\infty \mid k < \omega \rangle$, $\langle \mu_k^\infty \mid 0 \neq k < \omega \rangle$ is an iteration of V of order type $\preceq_x$. This means that: for k the smallest in $\preceq_x$, $M_k^\infty = \mathrm{V}$; for each successor k in $\preceq_x$, M_k^∞ is the ultrapower of M_j^∞ by μ_j^∞, where j is the $\preceq_x$ predecessor of k; and for each limit k in $\preceq_x$, M_k^∞ is the direct limit of the models M_j^∞ for $j \prec_x k$. All these properties can be verified easily using the various definitions and the commutativity of Diagram 6 (with the horizontal embeddings coming from the iterations of the relevant clusters).

Since $\preceq_x$ is wellfounded, $\langle M_k^\infty \mid k < \omega \rangle$, $\langle \mu_k^\infty \mid 0 \neq k < \omega \rangle$ is an iteration of V of wellfounded order type. It follows that M_0^∞, the final model of this iteration, is *wellfounded*. We will use the wellfoundedness of M_0^∞ later on.

For $n < \omega$ let $i^n \colon \mathrm{V} \to M_0^n$ be the iteration embedding induced by the cluster $\{M_j^n, \mu_j^n, r_j^n, a^n\}$. The definition of a cluster is such that a^n is a position in $i^n(T)$, with $i^n(\pi)(a^n)$ compatible with x. From the fact that the clusters here are according to Σ it follows further that a^n is consistent with $i^n(\Sigma)$. It is easy to check that the map $i^n \colon \mathrm{V} \to M_0^n$ (horizontal in Diagram 6) is equal to the map $h_0^{0,n} = h_0^{n-1,n} \circ \cdots \circ h_0^{0,1}$ (diagonal from M_0^0 to M_n^n in Diagram 6). So a^n is a position in $h_0^{0,n}(T)$, consistent with $h_0^{0,n}(\Sigma)$, and such that $h_0^{0,n}(\pi)(a^n)$ is compatible with x. Using the elementarity of $h_0^{n,\infty}$ to transfer this to M_0^∞ we get that $h_0^{n,\infty}(a^n)$ is a position in $h_0^{0,\infty}(T)$, consistent with $h_0^{0,\infty}(\Sigma)$, and such that $h_0^{0,\infty}(\pi)(h_0^{n,\infty}(a^n))$ is compatible with x.

Let $\vec{a}^\infty = \bigcup_{n<\omega} h_0^{n,\infty}(a^n)$. Definition 2.2 is such that for each n, a^{n+1} strictly extends $h^{n,n+1}(a^n)$. So $\vec{a}$ is infinite. Using the conclusion of the previous paragraph it follows that:

1. $\vec{a}$ is an infinite branch through $h_0^{0,\infty}(T)$,
2. $\vec{a}$ is consistent with $h_0^{0,\infty}(\Sigma)$, and
3. $h_0^{0,\infty}(\pi)(\vec{a}) = x$.

Since M_0^∞ is *wellfounded*, the existence of a sequence $\vec{a}$ satisfying these conditions reflects from V to M_0^∞. Thus $M_0^\infty \models$"there exists a sequence $\vec{a}$ satisfying conditions (1)–(3)." Pulling this statement back using the elementary embedding $h_0^{0,\infty}\colon \mathrm{V} \to M_0^\infty$ it follows that there is an infinite branch $\vec{a}$ through T, so that $\vec{a}$ is consistent with Σ and $\pi(\vec{a}) = x$. $\dashv$ (Case 1)

Case 2: If $\preceq_x$ is illfounded.

For each n let e^n be the $\preceq_x$ least $e \leq n$ which belongs to the illfounded part of $\preceq_x$. For each n let M^n denote $M^n_{e^n}$, let i^n denote the embedding from V into $M^n_{e^n}$ given by the cluster $\{M_j^n, \mu_j^n, r_j^n, a^n\}$, and let r^n denote $r^n_{e^n}$. r^n is then a position in $i^n(T)$, consistent with $i^n(\Sigma)$, and such that $i^n(\pi)(r^n)$ is compatible with x.

We wish to accumulate the positions r^n to obtain an infinite branch through a shift of T to some direct limit, just as we accumulated the positions a^n to obtain the branch $\vec{a}$ in case 1 above.

Notice that the series $\langle e^n \mid n < \omega\rangle$ is increasing, and grows in jumps: e^n is either equal to e^{n-1}, or else it jumps to equal n.

Consider first the case that $e^n = n$. It is easy to check that e^{n-1} is the $\preceq_x$ successor of n in this case, so that, referring to the notation of Diagram 5, e^{n-1} is equal to p. Continuing with the reference to Diagram 5 (with the upper cluster there standing for $\{M_j^n, \mu_j^n, r_j^n, a^n\}$ and the lower cluster standing for $\{M_j^{n-1}, \mu_j^{n-1}, r_j^{n-1}, a^{n-1}\}$) we see that M_n^n is *equal* to $M^{n-1}_{e^{n-1}}$, and (by condition (2) in Definition 2.2) r_n^n strictly extends $r^{n-1}_{e^{n-1}}$. In other words M^n is equal to M^{n-1} and r^n strictly extends r^{n-1}. Let $h^{n-1,n}\colon M^{n-1} \to M^n$ in this case be the identity embedding.

Consider next the case that $e^n = e^{n-1}$. Let $h^{n-1,n}\colon M^{n-1} \to M^n$ be the extension embedding associated to $M^{n-1}_{e^{n-1}}$ and the clusters $\{M_j^n, \mu_j^n, r_j^n, a^n\}$ and $\{M_j^{n-1}, \mu_j^{n-1}, r_j^{n-1}, a^{n-1}\}$. Definition 2.2 is such that $r^n = h^{n-1,n}(r^{n-1})$ in this case.

Now let M^* be the direct limit of the models M^n under the embeddings $h^{n-1,n}$. Let $h^{n,*}\colon M^n \to M^*$ be the direct limit embeddings. Let $\vec{r} = \bigcup_{n<\omega} h^{n,*}(r^n)$. From the two paragraphs above it follows that:

1. $\vec{r}$ is an infinite branch through $h^{0,*}(T)$;
2. $\vec{r}$ is consistent with $h^{0,*}(\Sigma)$; and
3. $h^{0,*}(\pi)(\vec{r}) = x$.

(To relate $\vec{r}$ with $h^{0,*}(T)$, $h^{0,*}(\Sigma)$, and $h^{0,*}(\pi)$, we are using the observation above that each r^n is a position in $i^n(T)$, consistent with $i^n(\Sigma)$, and such that $i^n(\pi)(r^n)$ is compatible with x. We are using also the commutativity of Diagram

6, to switch from the embedding i^n to the embedding $h^{0,n} = h^{n-1,n} \circ \cdots \circ h^{0,1}$, which we then compose with $h^{n,*}$.)

For each n let $u^n = \{j \le n \mid j \prec_x e^n\}$. These are the numbers corresponding to models to the left of M^n in the format of Diagram 6. Let $u = \bigcup_{n<\omega} u^n$. Using the commutativity of Diagram 6 one can check that M^* is precisely equal to the direct limit of the iteration $\langle M^\infty_k, \mu^\infty_k \mid k \in u\rangle$. (Notice the restriction to $k \in u$, that is to k which are to the left of the models leading to M^*.) This is an iteration of V of order type $\preceq_x \restriction u$. Now our definition of $\langle e^n \mid n < \omega\rangle$ is such that u is precisely equal to the wellfounded part of $\preceq_x$. So $\langle M^\infty_k, \mu^\infty_k \mid k \in u\rangle$ is an iteration of V of wellfounded order type. It follows that the iteration has a wellfounded direct limit. So M^* is wellfounded. We now continue as in case 1, to reflect the existence of a branch $\vec{r}$ satisfying conditions (1)–(3) into M^*, and then pull back to V using the elementary embedding $h^{0,*}$, to get an infinite branch $\vec{r} \in [T]$, consistent with Σ, and such that $\pi(\vec{r}) = x$. $\dashv$ (Case 2, Lemma 2.3)

LEMMA 2.6. *Let Σ be a strategy for player* I *in T. Then there is a strategy σ for player* I *in $\omega^{<\omega}$, so that for every $x \in \omega^\omega$: if x is according to σ then there is a sequence of clusters $\{M^n_j, \mu^n_j, r^n_j, a^n\}$ satisfying the assumptions in Lemma 2.3.*

PROOF. We intend to define σ by describing how to play for I on $\omega^{<\omega}$. The description will take the form of a construction, joint with an opponent who plays II's part in x. We will construct $x \in \omega^\omega$, and in addition we will construct a sequence of clusters $\{M^n_j, \mu^n_j, r^n_j, a^n\}$ satisfying the assumptions in Lemma 2.3. We intend to arrange things so that *all* the work will be done by Σ and its shifts to the various models in our clusters. For this we will use the reversal or roles in the two parts of T—the part where II accepts and the part where II rejects. We will pit Σ's actions in one part against its actions in the other part.

Let us first isolate the basic case. Recall that T is the tree of Definition 1.8, and κ is the cardinal fixed just before that definition.

DEFINITION 2.7. Let r be a whole position in T in which II rejects, covering rounds 0 through j say. The position leads to the objects s_j, w_j, and A_j (see the first item in Definition 1.7). We refer to the tuple $\langle s_j, w_j, A_j\rangle$ as the **ending** of r.

DEFINITION 2.8. Let M be an iterate of V with $i\colon \mathrm{V} \to M$ the iteration embedding. Let r be a whole position in T in which II rejects, and let $\langle \bar{s}, \bar{w}, \bar{A}\rangle$ be the ending of r. Let a be a whole position in $i(T)$ (note the shift by i) in which II accepts, consisting of rounds 0 through $n-1$, and leading to $x\restriction n-1$ and the pairs $\langle \kappa_j, A_j\rangle$ for $j < n$ (see the first item in Definition 1.4). We say that r is **compatible** with a (over V, relative to the embedding i) just in case that:

1. $\bar{s} \subset s$;
2. $\kappa_p = \kappa$ and $A_p = \bar{A}$ where $p = \mathrm{lh}(\bar{s})$; and
3. $\bar{w}$ is an initial segment of $\{\langle \kappa_j, A_j\rangle \mid i \prec_{x\restriction n-1} p\}$.

The point of the definition is this: Work in the settings of Definition 2.8 and suppose further that $x(n-1)$ is given and that $p = \mathrm{lh}(\bar{s})$ is the successor of n in

$\preceq_{x\restriction n}$. (Note in this case that $x\restriction n$ is a suitable extension of $x\restriction p$.) Suppose that r is compatible with a. Let $s = x\restriction n$ and let $w = \{\langle\kappa_j, A_j\rangle \mid j \prec_{x\restriction n} n\}$. Then:

(a) U is a legal move for I in $i(T)$ following $a^\frown\langle x(n-1)\rangle$ iff the triple consisting of s, w, and U is legal for II in T following r; and
(b) μ and A (where μ is a measure on κ and $A \subset \mathrm{V}_{\kappa+1}$) form a legal move for I in T following $r^\frown\langle s, w, U\rangle$ iff the pair $\langle\kappa, A\rangle$ is legal for II in $(i^* \circ i)(T)$ following $i^*(a^\frown\langle x(n-1), U\rangle)$, where i^* is the ultrapower embedding by μ.

These two conditions can be verified easily from the definitions, using the reversal of roles in the inverted rank game (compared to the repeated rank game). Indeed, the reversal of roles was specifically tailored to result in conditions (a) and (b).

The conditions show that moves for I following a and r respectively double as moves for II following r and (a shift of) a. A strategy for player I is therefore enough to generate *all* the moves necessary for extending both r and (a shift of) a by one round. This is made precise in the following claim:

Claim 2.9. *Work in the settings above and suppose further that r is consistent with Σ and that $a^\frown\langle x(n-1)\rangle$ is consistent with $i(\Sigma)$. (Recall that Σ is a strategy for player* I *in T, see the assumptions in Lemma 2.6.) Then there is an extension r^* of r by one round in T, a measure μ^* on κ, and an extension a^* of $i^*(a)$ by one round in $(i^* \circ i)(T)$, where i^* is the ultrapower embedding by μ^*, so that:*

- *r^* is according to Σ and a^* is according to $(i^* \circ i)(\Sigma)$;*
- *$\pi(r^*) = x\restriction n$ and $(i^* \circ i)(\pi)(a^*) = x\restriction n$; and*
- *r^* is compatible with a^* (over* V*, and relative to the embedding $i^* \circ i$).*

Proof. Use Σ to obtain a move U for I in $i(T)$ following $a^\frown\langle x(n-1)\rangle$. Then use condition (a) to transfer U to a move for II in T following r, and use Σ's reply to the move in condition (a) to obtain μ and A. Set $r^* = r^\frown\langle s, w, U, \mu, A\rangle$, $\mu^* = \mu$, and $a^* = i^*(a)^\frown\langle x(n-1), i^*(U), \kappa, A\rangle$. It is easy to check that these objects satisfy the conditions in the conclusion of the claim. ⊣

Equipped with the last claim we can begin the construction which describes the strategy σ of Lemma 2.6. We work in stages. At the start of stage n we will have $x\restriction n$ and the clusters $\{M^k_j, \mu^k_j, r^k_j, a^k\}$ for $k \le n$. We will make sure that:

(i) $\{M^n_j, \mu^n_j, r^n_j, a^n\}$ is a cluster for $x\restriction n$,
(ii) it is according to Σ, and
(iii) it extends $\{M^{n-1}_j, \mu^{n-1}_j, r^{n-1}_j, a^{n-1}\}$.

From these conditions it follows that every x according to our construction has associated to it a sequence of clusters as in Lemma 2.3, and this will prove Lemma 2.6.

For each $j \preceq_{x\restriction n} l$ let $i^n_{j,l}\colon M^n_j \to M^n_l$ be the appropriate iteration embedding given by the cluster $\{M^n_j, \mu^n_j, r^n_j, a^n\}$. Let $i^n_l\colon \mathrm{V} \to M^n_l$ be the embedding $i^n_{j,l}$ where j is least in $\preceq_{x\restriction n}$.

We intend to maintain two additional conditions:

(iv) The positions r^n_j are whole; and
(v) For each $j \le n$, r^n_j is compatible with a^n (over M^n_j and relative to the embedding $i^n_{j,n}$).

Let A be Σ's first move in T. A is then a subset of $\mathrm{V}_{\kappa+1}$, which player II can either accept or reject. (We intend to do both: accept in the positions a^n, and reject in the positions r^n_j.)

To start the construction let $\{M^0_j, \mu^0_j, r^0_j, a^0\}$ be the cluster consisting of the model $M^0_0 = \mathrm{V}$, the position a^0 given by Σ's first move A followed by "accept" for II, and the position r^0_0 given by Σ's first move A followed by "reject" for II. Conditions (i)–(v) for $n = 0$ hold trivially with these assignments.

Suppose that we reached stage $n-1$, and conditions (i)-(v) hold for $n-1$. We describe how to construct $x(n-1)$ and the cluster $\{M^n_j, \mu^n_j, r^n_j, a^n\}$.

If $n-1$ is even, let $x(n-1)$ be the move that $i^{n-1}_0(\Sigma)$ plays following the position a^{n-1}. If $n-1$ is odd let $x(n-1)$ be played by the opponent. We now have $x\restriction n$, and in both of the cases above $a^{n-1}{}^\frown\langle x(n-1)\rangle$ is consistent with $i^{n-1}_0(\Sigma)$.

Let p be the successor of n in $\preceq_{x\restriction n}$. Apply Claim 2.9 over M^{n-1}_p and relative to $i^{n-1}_{p,0}\colon M^{n-1}_p \to M^{n-1}_0$, with the positions r^{n-1}_p and a^{n-1}. (It's easy to check that the current settings fit the claim, using, among other things, condition (v) for $n-1$ with $j = p$.) Let r^*, μ^*, and a^* be given by the claim. Define the cluster $\{M^n_j, \mu^n_j, r^n_j, a^n\}$ by setting $r^n_n = r^*$, $\mu^n_n = \mu^*$, and $a^n_0 = a^*$. These assignments and condition (iii) determine the cluster completely. It is easy to check that the new cluster satisfies conditions (i)–(v).

The inductive construction above can be formalized into a strategy σ that plays for I in $\omega^{<\omega}$, and makes sure that every x it produces comes equipped with a sequence of clusters as in Lemma 2.3. This completes the proof of Lemma 2.6. $\dashv$

Corollary 2.10. *Let Σ be a strategy for player* I *in T. Then there is a strategy σ for player* I *in $\omega^{<\omega}$ so that: for every play x according to σ, there exists an infinite branch $\vec{v} \in [T]$ according to Σ, with $\pi(\vec{v}) = x$.*

Proof. This is a direct combination of Lemmas 2.6 and 2.3. $\dashv$

Remark 2.11. The strategy σ in the last corollary is obtained through the construction of Lemma 2.6. That construction is continuous in Σ, in the sense that the restriction of σ to positions of length at most m depends only on the restriction of Σ to positions of $m+1$ rounds. This is not quite the same as the Lipschitz continuity of Martin [3]. We could obtain that Lipschitz continuity if we adjusted the definition of the repeated rank game, to include two moves on $\omega^{<\omega}$ in each round (including round 0), instead of just one in each round (and none in round 0). But this would have complicated the indexing, already in the inverted rank game, and in all the subsequent proofs.

§3. Strategies for player II. Suppose now that Σ is a strategy for player II. We wish to prove a parallel of Corollary 2.10. Many of the ingredient for the proof we can simply take from the previous section: Lemma 2.3, and the scheme of the construction in Lemma 2.6 apply just as well when Σ is a strategy for II. But Claim 2.9 does not. Its proof rested on the fact that, by both rejecting and accepting an offer made by player I, we can force player I to produce all the moves that come up in the construction. This is convenient when we are dealing with a given strategy for player I. It becomes a burden when the strategy is for II, and *we* have to ascribe moves for I.

We obtain here a substitute for Claim 2.9. It is for this substitute that we finally use the large cardinal assumption $(*)$, that for every $Z \subset \mathrm{V}_{\kappa+1}$ there exists a measure μ on κ so that Z belongs to $\mathrm{Ult}(\mathrm{V}, \mu)$.

Fix a strategy Σ for player II in T. We work with this fixed strategy.

Definition 3.1. Let r be a medial position in T in which II rejects, covering rounds 0 through the first half of round $n > 0$ say, and ending with the move s_n, w_n, U_n by II. (See Diagram 3 for the format of the game.) We say that a triple $\langle s, w, U\rangle$ is **reachable** from r if there exists a move μ_n, A_n for I in T following r that would cause Σ to reply with the move consisting of s, w, and U. Otherwise we say that $\langle s, w, U\rangle$ is **unreachable** from r. We use $\mathrm{unrch}(r)$ to denote the set

$$\{\ \{\langle s, w\rangle\} \times U \quad | \quad s \in \omega^{<\omega},\ w \in \mathrm{V}_\kappa,\ U \subset \mathrm{V}_\kappa, \text{ and } \langle s, w, U\rangle \text{ is unreachable from } r.\ \}$$

$\mathrm{unrch}(r)$ thus codes those triples $\langle s, w, U\rangle$ which have format suitable for moves by player II in the inverted rank game, but are *not* played by Σ in response to any move by I following r.

The definition of course depends on Σ, but we suppress this in the notation. When we wish to emphasize the dependence we talk about reachable and unreachable relative to Σ.

Claim 3.2. *Let r be a medial position in T in which* II *rejects. Suppose that r is according to Σ. Then there does* not *exist any measure μ on κ so that the move $\langle \mu, \mathrm{unrch}(r)\rangle$ is legal for player* I *in T following r.*

Proof. Suppose for contradiction that μ and $\mathrm{unrch}(r)$ form a legal move for I following r in T. Let $\langle s, w, A\rangle$ be Σ's reply to $r^\frown\langle \mu, \mathrm{unrch}(r)\rangle$. The rules of the inverted rank game (see Definition 1.7) are such that U must belong to the (s, w)–section of $\mathrm{unrch}(r)$. In other words $\{\langle s, w\rangle\} \times U$ must belong to $\mathrm{unrch}(r)$. But this contradicts Definition 3.1, since there *is* a move for player I following r that makes Σ reply with $\langle s, w, U\rangle$, namely the move $\langle \mu, \mathrm{unrch}(r)\rangle$. ⊣

We will use Claim 3.2 later on. Let us for the moment expand our definition to the case of the empty position r, and see what becomes of the claim in that case.

Definition 3.3. Let r be the empty position in T. We say that $\langle s, w, U\rangle$ is **reachable** from r if there exists a move A for player I in round 0 of T that would cause Σ to *reject* and then play s, w, and U as a first move in round 1. Otherwise we say that $\langle s, w, U\rangle$ is **unreachable** from r. Again we define $\mathrm{unrch}(r)$ to be the set

$$\{\ \{\langle s, w\rangle\} \times U \quad | \quad s \in \omega^{<\omega},\ w \in \mathrm{V}_\kappa,\ U \subset \mathrm{V}_\kappa, \text{ and } \langle s, w, U\rangle \text{ is unreachble from } r.\ \}$$

Here too $\mathrm{unrch}(r)$ codes triples $\langle s, w, U\rangle$ which have the format suitable for moves by player II in the inverted rank game, but are *not* played by Σ, this time in response to any proposal by player I in round 0. Notice that $\mathrm{unrch}(r)$ is a subset of $\mathrm{V}_{\kappa+1}$. It therefore has the format suitable for a move by player I in T.

Claim 3.4. *If* I *plays* unrch($\emptyset$) *as her proposal in round* 0 *of* T, *then* Σ *replies with "*accepts*" for* II.

Proof. Suppose for contradiction that Σ rejects the proposal $A = \text{unrch}(\emptyset)$. It must then continue to play some triple $\langle s, w, U\rangle$ in round 1 of the inverted rank game associated to κ and A. The rules of the inverted rank game are such that U must belong to the (s, w)–section of $A = \text{unrch}(\emptyset)$. In other words $\{\langle s, w\rangle\} \times U$ must belong to unrch($\emptyset$). But this contradicts Definition 3.3, since there *is* a move A for player I in round 0 of T that causes Σ to reject and then play $\langle s, w, U\rangle$, namely the move $A = \text{unrch}(\emptyset)$. ⊣

Claim 3.4 provides a proposal that Σ cannot reject. We will use it later on. Let us now define a substitute for the notion of compatibility in Section 3, and prove the appropriate parallel of Claim 2.9 in that section.

Definition 3.5. Let M be an iterate of V with $i\colon \mathrm{V} \to M$ the iteration embedding. Let r be either a medial position in T in which II rejects, or the empty position. Let a be a whole position in $i(T)$ in which II accepts, consisting of rounds 0 through $n-1$, and leading to $x\restriction n-1$ and the pairs $\langle \kappa_j, A_j\rangle$ for $j < n$. We say that r is **compatible** with a (over V, relative to Σ and to the embedding i) just in case that:

- $\kappa_p = \kappa$, and $A_p = \text{unrch}(r)$ where $p = \text{lh}(\pi(r))$.

The condition here should be compared with condition (2) in Definition 2.8. In that definition we dealt with a whole position r and condition (2) referred to the move $\bar{A}$ played by I in the final round of r. Here we are dealing with a medial (or empty) position r. I has not yet played her move in the final round of r, and instead of referring to I's move we refer to unrch(r). (Note that the use of unrch(r) introduces a dependence on Σ into the definition.)

Lemma 3.6. *Work in the settings of Definition 3.5 (and under the assumption that r is compatible with a). Let $x(n-1)$ be given. Suppose that* lh$(\pi(r))$ *is the successor of n in $\preceq_{x\restriction n}$. Suppose that r is consistent with Σ, and $a^\frown\langle x(n-1)\rangle$ is consistent with $i(\Sigma)$.*

Then there is a proper, medial extension r^* of r in T, a measure μ^* on κ, and an extension a^* of $i^*(a)$ by one round in $(i^* \circ i)(T)$, where i^* is the ultrapower embedding by μ^*, so that:

- *r^* is according to Σ and a^* is according to $(i^* \circ i)(\Sigma)$;*
- *$\pi(r^*) = x\restriction n$ and $(i^* \circ i)(\pi)(a^*) = x\restriction n$; and*
- *r^* is compatible with a^* (over* V*, and relative to Σ and to the embedding $i^* \circ i$).*

Proof. Let p denote lh$(\pi(r))$. Let s^* denote $x\restriction n$, and let w^* denote $\{\langle \kappa_j, A_j\rangle \mid i \prec_{x\restriction n} n\}$.

Claim 3.7. *Let U be a subset of V_κ. Then U is a legal move for player* I *following $a^\frown\langle x(n-1)\rangle$ iff U belongs to the (s^*, w^*)–section of* unrch(r).

Proof. The rules of the repeated rank game, specifically the last item in Definition 1.4 and the first item in Definition 1.3, are such that U is legal for I in T following $a^\frown\langle x(n-1)\rangle$ iff U belongs to the (s^*, w^*)–section of A_p. A_p is equal to unrch(r) by the compatibility of r and a (see Definition 3.5). ⊣

Let Y be the set:

$$\{\ \langle\tau, B\rangle \quad|\quad \tau<\kappa,\ B \subset \mathrm{V}_{\tau+1}, \text{ and there does } \textit{not} \text{ exist any } U \text{ which is legal for player I following } a^\frown\langle x(n-1)\rangle \text{ and so that } i(\Sigma)\text{'s reply to } a^\frown\langle x(n-1)\rangle^\frown\langle U\rangle \text{ is } \langle\tau, B\rangle.\ \}$$

Claim 3.8. *Y does not belong to the (s^*, w^*)–section of* $\operatorname{unrch}(r)$. *(In other words $\{\langle s^*, w^*\rangle\} \times Y$ does not belong to* $\operatorname{unrch}(r)$.*)*

Proof. Suppose that it does. By the previous claim then, Y is legal for I in $i(T)$ following $a^\frown\langle x(n-1)\rangle$. Play Y for I. Let $\langle\tau, B\rangle$ be the reply given by $i(\Sigma)$. The rules of the basic rank game, specifically the rules in the second item of Definition 1.3, demand that $\langle\tau, B\rangle \in Y$. But this contradicts the definition of Y, since there *is* a legal move U for I which causes $i(\Sigma)$ to reply with $\langle\tau, B\rangle$, namely $U = Y$. ⊣

Corollary 3.9. *There is a proper, medial extension r^* of r so that r^* is according to Σ, and* II*'s (namely Σ's) final move in r^* is $\langle s^*, w^*, Y\rangle$.*

Proof. This is immediate from the last claim and the definition of $\operatorname{unrch}(r)$ (Definition 3.3 if r is the empty position, and Definition 3.1 otherwise): From the fact that $\{\langle s^*, w^*\rangle\} \times Y$ does not belong to $\operatorname{unrch}(r)$ it follows that $\langle s^*, w^*, Y\rangle$ *is* reachable from r, so there is a move for player I following r that makes Σ reply with $\langle s^*, w^*, Y\rangle$. ⊣

We have now the extension r^* of r. It remains to define μ^* and a^*.

Note that $\operatorname{unrch}(r^*)$ is a subset of $\mathrm{V}_{\kappa+1}$. Using (at last!) the large cardinal assumption $(*)$ in Section 1, fix a measure μ^* on κ so that $\operatorname{unrch}(r^*)$ belongs to $\operatorname{Ult}(\mathrm{V}, \mu^*)$. Let i^* denote the ultrapower embedding by μ^*.

Claim 3.10. *$\langle\kappa, \operatorname{unrch}(r^*)\rangle$ does not belong to $i^*(Y)$.*

Proof. Suppose that it does. The rules of the inverted rank game are such that $\langle\mu^*, \operatorname{unrch}(r^*)\rangle$ is then a legal move for player I in T following r^*. But this is in contradiction to Claim 3.2. ⊣

Let A^* denote $\operatorname{unrch}(r^*)$. By the choice of μ^*, we know that $\langle\kappa, A^*\rangle$ belongs to $\operatorname{Ult}(\mathrm{V}, \mu^*)$. By the last claim though, $\langle\kappa, A^*\rangle$ does not belong to $i^*(Y)$. From the definition of Y, or more precisely its shift by i^*, it follows that is a legal move U^* for player I following $i^*(a)^\frown\langle x(n-1)\rangle$ which causes $(i^* \circ i)(\Sigma)$ to reply with $\langle\kappa, A^*\rangle$. Define a^* to be the extension of $i^*(a)$ by one round consisting of the moves $x(n-1)$, U^*, and $\langle\kappa, A^*\rangle$.

Remark 3.11. Note the use of the fact that A^* belongs to $\operatorname{Ult}(\mathrm{V}, \mu^*)$ in the previous paragraph. Without this fact we wouldn't be able to apply the condition of membership in $i^*(Y)$ to $\langle\kappa, A^*\rangle$; $\langle\kappa, A^*\rangle$ would fail to belong to $i^*(Y)$ simply because it fails to belong to the ultrapower. The fact that A^* belongs to $\operatorname{Ult}(\mathrm{V}, \mu^*)$ of course traces back to our use above of the large cardinal assumption $(*)$.

We have by now defined r^*, μ^*, and a^*. Our definitions are such that $\pi(r^*) = s^* = x{\restriction}n$, $\pi(a^*) = x{\restriction}n$, r^* is consistent with Σ, and a^* is consistent with $(i^* \circ i)(\Sigma)$.

Let $\langle \kappa^*_j, A^*_j \rangle$, $i \leq n$, denote II's moves in a^*. Our definition of a^* is such that $\kappa^*_n = \kappa$ and A^*_n is equal to A^*, namely to unrch(r^*). It follows from this that r^* is compatible with a^*. ⊣ (Lemma 3.6)

With Lemma 3.6 as a parallel of Claim 2.9 we can now adapt the work of the previous section to the case that Σ is a strategy for player II:

LEMMA 3.12. *Let Σ be a strategy for player* II *in T. Then there is a strategy σ for player* II *in $\omega^{<\omega}$, so that for every $x \in \omega^\omega$: if x is according to σ then there is a sequence of clusters $\{M^n_j, \mu^n_j, r^n_j, a^n\}$ satisfying the assumptions in Lemma 2.3.*

PROOF. We define σ by describing how to construct $x \in \omega^\omega$ and the necessary sequence of clusters (working with an opponent who provides I's moves in x). We use $i^n_{j,l} \colon M^n_j \to M^n_l$ to denote the iteration embeddings given by the cluster $\{M^n_j, \mu^n_j, r^n_j, a^n\}$. We use $i^n_l \colon \mathrm{V} \to M^n_l$ to denote the embedding $i^n_{j,l}$ where j is least in $\preceq_{x\restriction n}$. We will make sure that:

(i) $\{M^n_j, \mu^n_j, r^n_j, a^n\}$ is a cluster for $x \restriction n$,
(ii) it is according to Σ, and
(iii) it extends $\{M^{n-1}_j, \mu^{n-1}_j, r^{n-1}_j, a^{n-1}\}$.

These are the same conditions we had in the proof of Lemma 2.6. We will also make sure that:

(iv) r^n_0 is the empty position, and for $0 < j \leq n$ the position r^n_j is medial.
(v) For each $j \leq n$, r^n_j is compatible with a^n (over M^n_j, relative to $i^n_j(\Sigma)$ and to the embedding $i^n_{j,n}$).

Compatibility here is in the sense of Definition 3.5 of course.

Let $A_0 = $ unrch$(\emptyset)$, see Definition 3.3. By Claim 3.4, if I plays A_0 as her proposal in round 0 of T, then Σ replies with "accept."

Let r^0_0 be the empty position in T. Let a^0 be the position consisting of the moves A_0 for I and "accept" for II in round 0 of T. r^0_0 is clearly consistent with Σ. By the previous paragraph so is a^0. Since $A_0 = $ unrch(r^0_0), r^0_0 is compatible with a^0 (over V and relative to Σ and to $i = $ id).

Let $\{M^0_j, \mu^0_j, r^0_j, a^0\}$ be the cluster consisting of the model $M^0_0 = \mathrm{V}$ and the positions r^0_0 and a^0 defined above. It is clear, using the previous paragraph, that conditions (i)–(v) hold for $n = 0$ with these assignments.

Now continue to construct as in the proof of Lemma 2.6, only having the opponent provide $x(n-1)$ for even $n-1$ now rather than odd, having $i^{n-1}_0(\Sigma)$ provide $x(n-1)$ for odd $n-1$ rather than even, and, most importantly, using Lemma 3.6 instead of Claim 2.9. ⊣ (Lemma 3.12)

COROLLARY 3.13. *Let Σ be a strategy for player* II *in T. Then there is a strategy σ for player* II *in $\omega^{<\omega}$ so that: for every play x according to σ, there exists an infinite branch $\vec{v} \in [T]$ according to Σ, with $\pi(\vec{v}) = x$.*

PROOF. Immediate from Lemmas 3.12 and 2.3. ⊣

Corollaries 2.10 and 3.13 provide a function Ψ, acting on strategies in T (for either player) and producing strategies (for the same player) on $\omega^{<\omega}$, so that for any strategy Σ on T, if $x \in \omega^\omega$ is according to $\Psi(\Sigma)$, then there is a run $\vec{v} \in [T]$, according to Σ, with $\pi(\vec{v}) = x$. (T, π, Ψ) is therefore a cover of $\omega^{<\omega}$.

As indicated at the end of Section 1, covers of this kind can be made to unravel any given Π^1_1 set, and in fact any given countable collection of Π^1_1 sets.

REMARK 3.14. In both Section 2 and Section 3 we worked to obtain a play according to a strategy Σ (for I in Section 2 and for II in Section 3) on T. Such a play consists of a real x and, among other things, ordinals κ_n, for n in the wellfounded part of $\preceq_x$, embedding the wellfounded part of $\preceq_x$ into the ordinals.

We obtained the play by producing a wellfounded iterate M^* of M, with iteration embedding $j\colon M \to M^*$, and producing over M^* a play according to $j(\Sigma)$. (We then appealed to the elementarity of j to get a play in V.)

j and M^* were obtained through iterated ultrapowers by measures on κ and its images, and in the play that we produced over M^*, *the ordinals κ_n were precisely the images of κ in the iteration.* (This can be verified by going through the various constructions.) This is an indication of the careful balance in the choice of these ordinals through rank games. The choice is not made by either player; rather it is divided between the players in such a way that, ultimately, it can be made by us, regardless of which player we work against.

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