FORCING WITH SEQUENCES OF MODELS OF TWO TYPES

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Abstract. We present an approach to forcing with finite sequences of models that uses models of two types. This approach builds on earlier work of Friedman and Mitchell on forcing to add clubs in cardinals larger than $\aleph_1$, with finite conditions. We use the two-type approach to give a new proof of the consistency of the proper forcing axiom. The new proof uses a finite support forcing, as opposed to the countable support iteration in the standard proof. The distinction is important since a proof using finite supports is more amenable to generalizations to cardinals greater than $\aleph_1$.
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1. Introduction

There is by now a long history, tracing back to the work of Todorčević [9], of using finite increasing sequences of countable models as side conditions in forcing notions, to ensure properness of the resulting poset, and in particular ensure that $\aleph_1$ is not collapsed.

More recently Friedman [1] and Mitchell [4] independently discovered forcing notions that add clubs in $\theta > \aleph_1$ using finite conditions, while preserving both $\aleph_1$ and θ. The Friedman and Mitchell posets use countable models as side conditions to ensure preservation of the two cardinals. The side conditions are no longer increasing sequences; rather they are sets, with various agreement and coherence conditions on the models in them.

In this paper we reformulate this approach using models of *two* types, countable and transitive, rather than just countable. This allows us to return to a situation where side conditions are increasing sequences, simplifying the definition of the poset of side conditions.

We show how the resulting poset can be used for the initial Friedman and Mitchell applications, for an additional application which involves collapsing cardinals in contexts where it is important not to add branches to certain trees in V, and most importantly for a new proof of the consistency of the proper forcing axiom.

The original proof of the consistency of PFA used preservation of properness under countable support iterations. The use of countable support makes it impossible to apply similar ideas for forcing axioms that involve meeting more than $\aleph_1$ sets (in posets that admit master conditions for more than countable structures). The proof we give uses *finite* support, and instead of appealing to preservation of properness, which fails for finite support iterations, it incorporates the two-type side conditions into the iteration, using them to ensure preservation of $\aleph_1$, and of a supercompact cardinal that becomes $\aleph_2$. This finite support proof has analogs

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that yield forcing axioms for meeting more than $\aleph_1$ sets, but these will be handled in a separate paper.

The poset of sequences of models of two types is presented in Section 2. We present it in greater generality than we need. For most applications we only need the following finite version. Let K be a structure satisfying a large enough fragment of ZFC. Let $\mathcal{S}$ be a collection of countable $M \prec K$ with $M \in K$. Let $\mathcal{T}$ be a collection of transitive $W \prec K$ with $W \in K$. Suppose that $M \cap W \in W$ and $M \cap W \in \mathcal{S}$ whenever $M \in \mathcal{S}$, $W \in \mathcal{T}$, and $W \in M$. (This can be arranged in many different settings, for example if all elements of $\mathcal{T}$ are countably closed, and $\mathcal{S}$ consists of all countable elementary substructures of K.) Conditions in the finite two-type model sequence poset associated to $\mathcal{S}$ and $\mathcal{T}$ are simply $\in$-increasing sequences of models from $\mathcal{S} \cup \mathcal{T}$, closed under intersections. More precisely a condition is a sequence s of models $M_0 \in M_1 \in \dots M_{n-1}$, where $M_i \in \mathcal{S} \cup \mathcal{T}$ for each $i < n$, and so that for every $i, j < n$, the intersection $M_i \cap M_j$ appears in the sequence. Conditions are ordered by reverse inclusion.

For Q a model that appears in s, the *residue* of s in Q, denoted $\text{res}_Q(s)$, is the subsequence of s consisting of models of s that belong to Q. We prove that the residue is itself a condition (meaning that it is $\in$-increasing and closed under intersections). We also prove that if $t \in Q$ is a condition that extends $\text{res}_Q(s)$, then s and t are compatible. This is the most important result proved in Section 2. It allows deducing that the poset of two-type model sequences is strongly proper, in a sense defined by Mitchell [4].

The basics of strong properness are presented in Section 3, and are connected to the poset of two-type model sequences in Section 4. Then in Section 5 we present the initial applications, yielding the models of Friedman and Mitchell, and the method for collapsing cardinals to κ^+ without adding branches of length κ^+ through trees in V.

Finally in Section 6 we use the two-type model sequences to prove the consistency of PFA with finite supports.

The main definitions and results in this paper, including the finite two-type model sequences and their use for a finite support proof of the consistency of PFA, were presented in Neeman [7], which also goes further and indicates how forcing with finite conditions helps in obtaining higher analogs of the proper forcing axiom. Since then there have already been some applications of the two-type model sequences, for example by Veličković–Venturi [10], using side conditions to obtain new proofs results of Koszmider, adding a chain of length ω_2 in $(\omega_1^{\omega_1}, <_{\text{Fin}})$, and Baumgartner–Shelah, adding a thin very tall superatomic Boolean algebra. Earlier applications of the Friedman and Mitchell side conditions include Friedman [2], showing that PFA does not imply that a model correct about $\aleph_2$ must contain all reals, and Mitchell [5], showing that $I(\omega_2)$ can be trivial.

2. The model sequence poset

Fix cardinals $\kappa < \lambda$. Typically in uses later on λ will be the successor of κ. Most often in fact we will take $\kappa = \omega$ and $\lambda = \omega_1$. Fix a set K so that $\kappa, \lambda \in K$ and $(K; \in)$ satisfies some large enough fragment of ZFC. Typically in applications we will take $K = H(\theta)$ for a regular cardinal θ, but the important properties of K are closure under various elementary set operations such as intersections, unions, and ordering by von Neumann rank, and for some claims, the ability to form transitive closures and to biject each set with an ordinal.

Remark 2.1. Our main applications of the results in this section will all be with $\kappa = \omega$ and $\lambda = \omega_1$. Many of the definitions and claims in this section are simpler in this case. We will point out some of the simplifications, mainly to the definitions, in remarks throughout the section. Most of the simplifications in case $\kappa = \omega$ are due to the fact that all models are closed under finite sequences. Similar simplifications hold for $\kappa > \omega$ if we restrict to models that are $<\kappa$ closed.

Definition 2.2. $\mathcal{S}$ and $\mathcal{T}$ are *appropriate* for κ, λ, and K if:

(1) $\mathcal{T}$ is a collection of transitive $W \prec K$, and $\mathcal{S}$ is a collection of $M \prec K$ with $\kappa \subseteq M$ and $|M| < \lambda$. All elements of $\mathcal{S}$ and $\mathcal{T}$ belong to K, and contain $\{\kappa, \lambda\}$.
(2) If $M_1, M_2 \in \mathcal{S}$ and $M_1 \in M_2$, then $M_1 \subseteq M_2$.
(3) If $W \in \mathcal{T}$, $M \in \mathcal{S}$, and $W \in M$, then $M \cap W \in W$ and $M \cap W \in \mathcal{S}$.
(4) Each $W \in \mathcal{T}$ is closed under sequences of length $< \kappa$ in K.

Remark 2.3. In case $\kappa = \omega$ and $\lambda = \omega_1$, conditions (2) and (4) are vacuous. Condition (1) simplifies to simply requiring that elements of $\mathcal{T}$ are transitive elementary submodels of K that belong to K, and elements of $\mathcal{S}$ are countable elementary submodels of K that belong to K.

Suppose that $\mathcal{S}$ and $\mathcal{T}$ are appropriate. We define a poset $\mathbb{P} = \mathbb{P}_{\kappa,\mathcal{S},\mathcal{T},K}$ that we call the poset of *two-type model sequences*, or simply the *sequence poset*, associated to κ, $\mathcal{S}$, $\mathcal{T}$, and K. Conditions are sequences of models in $\mathcal{S} \cup \mathcal{T}$, satisfying certain requirements that we specify in the next definition.

Definition 2.4. A condition in $\mathbb{P}$ is a sequence $\langle M_\xi \mid \xi < \gamma \rangle$ of length $\gamma < \kappa$, that belongs to K, and so that:

(1) For each ξ, M_ξ is either an element of $\mathcal{T}$ or an element of $\mathcal{S}$.
(2) The sequence is increasing in the following sense: for each $\zeta < \gamma$, the set $\{\xi < \zeta \mid M_\xi \in M_\zeta\}$ is cofinal in ζ. In particular for successor ordinals $\zeta < \gamma$, $M_{\zeta-1} \in M_\zeta$.
(3) For each $\zeta < \gamma$, the sequence $\langle M_\xi \mid \xi < \zeta \wedge M_\xi \in M_\zeta \rangle$ belongs to M_ζ.
(4) The sequence is closed under intersections, meaning that for all $\zeta, \xi < \gamma$, $M_\zeta \cap M_\xi$ is on the sequence.

Note that condition (3) is vacuous for transitive nodes M_ζ. It follows for such nodes from the closure of M_ζ in K, the fact that the entire sequence $\langle M_\xi \mid \xi < \gamma \rangle$ belongs to K, and enough of ZFC in K to identify $\langle M_\xi \mid \xi < \zeta \wedge M_\xi \in M_\zeta \rangle$.

Remark 2.5. In case $\kappa = \omega$ and $\lambda = \omega_1$, condition (3) of Definition 2.4 is vacuous, the requirement of membership in K is vacuous, and condition (2) requires simply that $M_{\zeta-1} \in M_\zeta$ for each $\zeta > 0$. Thus the definition in this case simplifies to the following: $\langle M_\xi \mid \xi < \gamma \rangle$ is a finite $\in$-increasing sequence of elements of $\mathcal{S} \cup \mathcal{T}$, and is closed under intersections.

Claim 2.6. *If $\xi < \zeta$, then M_ξ has smaller von Neumann rank than M_ζ.*

Proof. Immediate by induction on ζ using condition (2). □

Abusing notation slightly we often refer to a condition as a set $\{M_\xi \mid \xi < \gamma\}$ rather than a sequence. There is no loss of information in talking about the set rather than the sequence, since by Claim 2.6 the sequence order is determined uniquely from the elements of the sequence.

Definition 2.7. Conditions in $\mathbb{P}$ are ordered by reverse inclusion (with conditions viewed as sets). In other words, $\{M_\xi \mid \xi < \gamma\} \leq \{N_\xi \mid \xi < \delta\}$ iff $\{M_\xi \mid \xi < \gamma\} \supseteq \{N_\xi \mid \xi < \delta\}$.

Claim 2.8. *Suppose that κ is regular, let $\tau \leq \kappa$ be regular, and suppose that K and all models in $\mathcal{S}$ are $<\tau$ closed in V. Then $\mathbb{P}$ is $<\tau$ closed.*

Proof. If κ is regular and K and all models in $\mathcal{S}$ are $<\tau$ closed, then the union of a decreasing set of fewer than τ conditions in $\mathbb{P}$ is itself, when ordered by von Neumann rank, a condition in $\mathbb{P}$. This is immediate from the definitions. The closure is needed for condition (3) in Definition 2.4. □

We refer to elements of $\mathcal{T}$ and $\mathcal{S}$ as *nodes*. Elements of $\mathcal{T}$ are *transitive nodes*, also called nodes of *transitive type*. Elements of $\mathcal{S}$ are *small nodes*, or nodes of *small type*. We say that M is of *the same or smaller type* than N if the two nodes are of the same type, or M is of small type and N is of transitive type. Given a condition $s = \{M_\xi \mid \xi < \gamma\}$ and nodes $M = M_\xi$ and $N = M_\zeta$ which belong to s, we use interval notation in the natural way, for example (M, N) is the interval of nodes strictly between M and N, namely the interval of nodes M_ι, $\xi < \iota < \zeta$.

Definition 2.9. s is a *precondition* if it satisfies conditions (1) and (2) in Definition 2.4. s is a *nice precondition* if it also satisfies condition (3).

In case $\kappa = \omega$ and $\lambda = \omega_1$, a precondition is a finite $\in$-increasing sequence from $\mathcal{S} \cup \mathcal{T}$, and niceness holds vacuously.

Claim 2.10. *Let s be a precondition.*

(1) *If M and Q are nodes in s, with $M \in Q$ and M of the same or smaller type than Q, then $M \subseteq Q$.*
(2) *If W is a node in s of transitive type, and M is a node in s occurring below W, then $M \in W$ and hence by (1) also $M \subseteq W$.*
(3) *If $Q \in s$ is of small type, $M \in s$ occurs before Q, and there are no nodes of transitive type in s between M and Q, then $M \in Q$. If in addition M is of small type then by (1) also $M \subseteq Q$.*

Proof. The first condition is clear, by conditions (1) and (2) in Definition 2.2.

For conditions (2), let $W \in s$ be of transitive type, and let M occur below W. By condition (2) of Definition 2.4, there are nodes of s at or above M which belong to W. Let M^* be the least one. By condition (1) of the claim, $M^* \subseteq W$. By minimality of M^* and condition (2) of Definition 2.4 it follows that M^* must be equal to M, and hence $M = M^* \in W$.

Fix finally Q and M as in condition (3). Again there are nodes of s at or above M which belong to Q. Let M^* be the first one. If M^* is M then $M \in Q$, and we are done, so suppose M^* occurs above M. In this case by assumption of the condition, M^* is of small type, and hence by condition (1), $M^* \subseteq Q$. As in the previous paragraph, the minimality of M^* now implies that M^* is M. □

Claim 2.11. *Let s be a precondition. Let M_1 and M_2 be nodes of s with M_1 occurring before M_2. Suppose that there are nodes of transitive type between M_1 and M_2. Then there are nodes of transitive type between them, that belong to M_2.*

Proof. Work by induction on M_2. Let W be a node of transitive type between M_1 and M_2. Using condition (2) of Definition 2.4, there is W^* occurring at or above W, that belongs to M_2. If W^* is of transitive type, we are done. Otherwise by Claim 2.10, $W^* \subseteq M_2$. By induction there is a node of transitive type between M_1 and W^*, that belongs to W^*. Since $W^* \subseteq M_2$, the node belongs to M_2. □

Claim 2.12. *Let s be a precondition. Suppose that s satisfies the following weak closure under intersections:*

(w4) *If W and M are nodes in s of transitive and small type respectively, and $W \in M$ (in particular W occurs before M), then $M \cap W$ is a node in s.*

Then s is closed under intersections, in other words it satisfies condition (4) in Definition 2.4.

Proof. Suppose s is not closed under intersection, and let Q and M witness this, with Q occurring before M, and M minimal.

If M is of transitive type, then by Claim 2.10, $Q \subseteq M$, hence $M \cap Q = Q$ is a node of s. So we may assume that M is of small type.

If there are no nodes of transitive type between Q and M, then by Claim 2.10, $Q \in M$. If Q is of small type then by the same claim $Q \subseteq M$ so $M \cap Q = Q$ is a node of s. If Q of transitive type then by condition (w4) $M \cap Q$ is a node of s.

So we may assume that there are nodes of transitive type between Q and M. By Claim 2.11 there is such a node W with $W \in M$. By condition (w4) then $M \cap W$ is a node of s. It must occur before W, and therefore before M. By induction then $(M \cap W) \cap Q$ is a node of s. Since W is a node of transitive type above Q, we have $Q \subseteq W$ by Claim 2.10. Hence $M \cap Q = (M \cap W) \cap Q$, so $M \cap Q$ is a node of s. □

Definition 2.13. Let s be a precondition and let Q be a node of s. The *residue* of s in Q, denoted $\text{res}_Q(s)$, is the set $\{M \in s \mid M \text{ belongs to } Q\}$.

Claim 2.14. *If s is a nice precondition then $\text{res}_Q(s)$ belongs to Q.*

Proof. Immediate from the definition of a nice precondition. □

Claim 2.15. *Let s be a condition. Let $Q \in s$ be of small type, and let $W \in \text{res}_Q(s)$ be of transitive type. Then there are no nodes of $\text{res}_Q(s)$ in the interval $[Q \cap W, W)$ of s. ($Q \cap W$ belongs to s by closure of s under intersections.)*

Proof. Let N be a model of $\text{res}_Q(s)$ that occurs before W in s. Then since W is of transitive type, $N \in W$ by Claim 2.10. Since $N \in \text{res}_Q(s)$, $N \in Q$ by definition of the residue. So $N \in Q \cap W$, and this implies that N can only occur before $Q \cap W$ in s. □

Definition 2.16. Let s be a condition and let $Q \in s$ be of small type. Let W be a node of transitive type in $\text{res}_Q(s)$. Then the interval $[Q \cap W, W)$ of s is called a *residue gap* of s in Q.

Claim 2.17. *Let s be a condition and let Q be a node of s. If Q is of transitive type, then $\text{res}_Q(s)$ consists of all nodes of s that occur before Q. If Q is of small type then $\text{res}_Q(s)$ consists of all nodes of s that occur before Q and do not belong to residue gaps of s in Q.*

Proof. If Q is of transitive type then by Claim 2.10 all nodes of s before Q belong to Q, and are therefore nodes of the residue. Nodes from Q upward have equal or higher von Neumann ranks than Q, and therefore cannot belong to Q, so they are not nodes of the residue.

Suppose Q is of small type. Again nodes occurring from Q upward cannot belong to Q because they have the same or higher von Neumann rank, so they are not nodes of the residue. Nodes below Q that belong to residue gaps are not in the residue by Claim 2.15. It remains to prove that all nodes below Q that are outside residue gaps, belong to the residue.

Let N be a node of s below Q, outside all residue gap. If there are no transitive nodes between N and Q, then by Claim 2.10, N belongs to Q and therefore N belongs to the residue. Suppose that there are transitive nodes between N and Q. By Claim 2.11 there are such nodes that belong to Q. Let W be the first one. Then $[Q \cap W, W)$ is a residue gap. Since N occurs below W and outside all residue gaps, it must occur below $Q \cap W$. By minimality of W there are no transitive nodes between N and $Q \cap W$ that belong to $Q \cap W$, and hence by Claim 2.11 there are outright no transitive nodes between N and $Q \cap W$. By Claim 2.10 then $N \in Q \cap W$, hence in particular $N \in Q$ and therefore N belongs to the residue. $\square$

Lemma 2.18. *Let s be a condition, and let Q be a node of s. Then $\text{res}_Q(s)$ is a condition.*

Proof. If Q is of transitive type then $\text{res}_Q(s)$ is an initial segment of s, and is easily seen to be a condition. Suppose then that Q is of small type. We prove that the residue satisfies the conditions in Definition 2.4.

The residue belongs to K since it can be obtained from Q and s using simple set operations. Condition (1) of Definition 2.4 for the residue is immediate from the same condition for s. Condition (4) for the residue is again immediate, from the same condition for s and the elementarity of Q in K (which implies that the intersection of two nodes that belong to Q is itself an element of Q). Conditions (2) and (3) are clear if $M_\zeta \in \text{res}_Q(s)$ is of small type, since in this case $M_\zeta \subseteq Q$ and any node of s that belongs to M_ζ is also a node of the residue. Finally, if $M_\zeta \in \text{res}_Q(s)$ is of transitive type, then all nodes of s (and of $\text{res}_Q(s)$) that occur before M_ζ are elements of M_ζ, by Claim 2.10. Condition (2) for the residue at ζ follows immediately from this. Condition (3) is vacuous since M_ζ is of transitive type. $\square$

Definition 2.19. Two conditions s and t are compatible in $\mathbb{P}$ if there is a condition r such that $r \supseteq s \cup t$. The conditions are *directly compatible* if r is exactly the closure of $s \cup t$ under intersections.

Lemma 2.20. *Let s be a condition, and let $Q \in s$ be a transitive node. Suppose that t is a condition that belongs to Q and extends $\text{res}_Q(s)$. Then $s \cup t$ is a condition, and in particular s and t are directly compatible.*

Proof. K satisfies enough of ZFC that $s \cup t \in K$. Condition (1) of Definition 2.4 is immediate for $s \cup t$. Condition (2) too is clear, using the fact that $s \cup t$ is the same as s above Q, and the same as t below Q. For condition (4), it is enough by Claim 2.12 to verify (w4) of the claim. Fix nodes $W \in M$ of transitive and small type respectively in $s \cup t$. If M occurs below Q then both M and W are nodes of t, and closure of t under intersections implies that $M \cap W$ is a node of t hence of $s \cup t$. If M and W both occur at or above Q then both are nodes of s, and closure of s implies $M \cap W$ is a node of s hence of $s \cup t$. If M occurs above Q and W occurs below Q, then by Claim 2.10, $W \subseteq Q$, so $M \cap W = M \cap Q \cap W$. $M \cap Q$ is a node of s below Q, hence a node of $\text{res}_Q(s)$, hence a node of t. W is also a node of t. By closure of t under intersections it follows that $M \cap Q \cap W$ is a node of t, hence a node of $s \cup t$.

Finally, condition (3) for $s \cup t$ is clear if M_ζ occurs below Q, since the part of $s \cup t$ below Q is simply t. The condition is also clear if $M_\zeta = Q$, since $t \in Q$. If M_ζ occurs above Q, then $\{N \in s \cup t \mid N \in M_\zeta\} = \{N \in s \mid N \in M_\zeta\} \cup \{N \in t \mid N \in M_\zeta \cap Q\}$. The left part of the union belongs to M_ζ since s is a condition, and the right part of the union belongs to M_ζ (in fact to $M_\zeta \cap Q$) since t is a condition and $M_\zeta \cap Q$ is a node of t. The entire union then belongs to M_ζ by elementarity of M_ζ in K. $\square$

Lemma 2.21. *Let s be a condition and let $Q \in s$ be a small node. Suppose that t is a condition that belongs to Q and extends* $\text{res}_Q(s)$*. Then s and t are directly compatible.*

Proof. We first show that $s \cup t$ is increasing. (The proof of this will use the assumption that s is closed under intersections.) We then add nodes to $s \cup t$ that close it under intersections, and show that the sequence resulting from the addition of these nodes is a condition. All the sequences we generate during the proof are obtained from s and t using simple set operations, and therefore belong to K. So we will only have to worry about the other clauses in Definition 2.4.

Claim 2.22. *$s \cup t$ is increasing, and is therefore a precondition.*

Proof. Let u consist of the nodes of s above Q. It is clear that $t \cup \{Q\} \cup u$ is increasing, or more precisely that it satisfies condition (2) of Definition 2.4, when ordered in the natural way, namely the nodes of t ordered as they are in t, followed by Q, followed by the nodes of u ordered as they are in s. The reason is that t is a condition and hence increasing, $t \in Q$ hence $t \subseteq Q$ by condition (1) in Definition 2.2 and the fact that $|t| < \kappa$, and $\{Q\} \cup u$ is a tail-end of the condition s and hence increasing.

Since t extends $\text{res}_Q(s)$, by Claim 2.17, the only nodes of s that do not belong to $t \cup \{Q\} \cup u$ are the nodes in residue gaps of s in Q. Recall that residue gaps are intervals in s of the form $[Q \cap W, W)$ where W is a transitive node of s that belongs to Q. In particular W belongs to t and hence to $t \cup \{Q\} \cup u$.

We prove that the sequence obtained from $t \cup \{Q\} \cup u$ by adding the nodes of each residue gap $[Q \cap W, W)$, immediately before W and ordered inside the interval according to their ordering in s, is increasing. Since the resulting sequence has all nodes of $s \cup t$, this establishes the claim.

Since $t \cup \{Q\} \cup u$ is increasing, and each residue gap is increasing (being a segment of a condition), it is enough to check condition (2) of Definition 2.4 at the borders of each residue gap $[Q \cap W, W)$. At the higher border the condition follows from the same condition for s at W, since the residue gap includes a tail-end of nodes of s below W. At the lower end the condition follows from the fact that t is increasing and contained in Q. Since t is increasing, the set of nodes of t which belong to W is cofinal below W. Since t is contained in Q, all these nodes belong to $Q \cap W$. $\square$

Remark 2.23. It follows from the proof that for any residue gap $[Q \cap W, W)$ of s in Q, no nodes of t occur in the interval $[Q \cap W, W)$ of $s \cup t$.

Let W be a transitive node of t which does not belong to s. Suppose there are transitive nodes of s in the interval (W, Q) of $s \cup t$, and let W^* be the first one.

Let E_W list, in order, the small nodes of s starting from $Q \cap W^*$, up to but not including the first transitive node of s above $Q \cap W^*$.

Note that $W \in M$ for each $M \in E_W$. This is certainly the case for the first element of E_W, namely $Q \cap W^*$, since $W \in W^*$ by Claim 2.10 and $W \in t \subseteq Q$. For M occurring above $Q \cap W^*$ in E_W, $W \in Q \cap W^* \subseteq M$, where the final inclusion holds by Claim 2.10 as all nodes of s between $Q \cap W^*$ and M are small.

Define F_W to be $\{M \cap W \mid M \in E_W\}$, with the ordering induced by the ordering of nodes in E_W. Since M is small and $W \in M$ for each $M \in E_W$, each element $M \cap W$ of F_W is a small node by Definition 2.2. It is easy to check that F_W is increasing, and indeed $M \cap W \in M' \cap W$ whenever M occurs below M' in E_W. The lowest node of F_W is $(Q \cap W^*) \cap W = Q \cap W$, and by Definition 2.2, all nodes of F_W belong to W.

Let W be a transitive node of t which does not belong to s, and suppose that there are no transitive nodes of s in the interval (W, Q). Let E_W list, in order, the small nodes of s starting from Q, up to but not including the first transitive node of s above Q if there is one, and all nodes of s starting from Q if there are no transitive nodes above Q.

Again, $W \in M$ for each $M \in E_W$, since $W \in Q \subseteq M$, with the final inclusion using Claim 2.10. Again define F_W to be $\{M \cap W \mid M \in E_W\}$, with the ordering induced by the ordering of the nodes in E_W. Again by Definition 2.2, all elements of F_W are small nodes that belong to W. The first node of F_W is $Q \cap W$, and again F_W is increasing.

Let r be obtained from $s \cup t$ by adding all nodes in F_W, for each transitive W that belongs to $t - s$, placing the nodes of F_W in order, right before W.

Remark 2.24. Note that every node of r that belongs to Q is a node of t. This is certainly the case for nodes of s, since t extends $\mathrm{res}_Q(s)$. The only other nodes of r which need to be checked are the nodes in F_W for transitive $W \in t - s$, but these do not belong to Q: They have the form $M \cap W$ with $M \in E_W$, which implies that M is a small node that contains either Q or $Q \cap W^*$, for W^* transitive above W. Either way $M \cap W$ contains $Q \cap W$, which is impossible when $M \cap W \in Q$ (since then $M \cap W \in Q \cap W$).

Claim 2.25. *r is increasing, and is therefore a precondition.*

Proof. Since $s \cup t$ is increasing, and each added interval F_W is increasing, it is enough to verify condition (2) of Definition 2.4 at the borders of each added interval F_W.

At the upper border, every element of F_W belongs to W. For the lower border, we have to show that cofinally many elements of r below $Q \cap W$ belong to $Q \cap W$. Since all elements of t below W belong to $Q \cap W$ (as $t \subseteq Q$), it is enough to check that t is cofinal in r below $Q \cap W$. To check this, note that the only elements of r below $Q \cap W$ which do not belong to t are either in (a) residue gaps $[Q \cap \bar{W}, \bar{W})$ of s in Q, or (b) added intervals $F_{\bar{W}}$. The gap or added interval cannot overlap W, since W itself is a node of t. (Residue gaps do not include nodes of t by Remark 2.23.) So in both (a) and (b), $\bar{W}$ occurs before W. Since $\bar{W}$ is a node of t in both cases, each of the intervals in (a) and (b) is capped by a node of t below $Q \cap W$. It follows that the nodes of t are cofinal in r below $Q \cap W$. □

Claim 2.26. *r is closed under intersections.*

Proof. It is enough to verify condition (w4) of Claim 2.12. Fix then $W \in M$ of transitive and small type respectively, both nodes of r. We prove that $M \cap W$ is a node of r.

Since the intervals added to $s \cup t$ to form r consist only of small nodes, W must be a node of $s \cup t$. Suppose first that it is a node of s.

If W occurs above Q, then M must also occur above Q, hence M is a node of s, and $M \cap W$ is a node of s by closure of s under intersections. Suppose then that W occurs below Q. If M is a node of s then $M \cap W$ is a node of s hence of r. If M is a node of one of the intervals added to $s \cup t$ to form r, then it is of the form $M' \cap W^*$ for some $M' \in s$ and $W^* \in t$. W^* is above M, hence above W, so $W \subseteq W^*$. Then $M \cap W = (M' \cap W^*) \cap W = M' \cap W \in s$ where membership of $M' \cap W$ in s follows from the closure of s, as both M' and W are nodes of s. The last remaining possibility (in the case that W is a node of s) is that M is a node of t. Then $M \in Q$ and since M is small it follows that $M \subseteq Q$, so $W \in Q$. W is then a node of $\text{res}_Q(s)$, hence a node of t. By closure of t, $M \cap W$ is a node of t, and hence of r.

Consider next the case that W is not a node of s. In other words it is a node of $t - s$. If M is a node of t, then $M \cap W$ is a node of t hence of r. If M is a node of one of the intervals added to $s \cup t$ to form r, then it is of the form $M' \cap W'$ for some $M' \in s$ and W' above M and hence above W. In this case $M \cap W = M' \cap W$, so that membership of $M \cap W$ in r reduces to membership of $M' \cap W$ in r, for a node M' of s. Thus it is enough to consider the case that M is a node of s. If there are transitive nodes W' of s between W and M, we may replace M with $M \cap W'$, since $M \cap W'$ is also a node of s, and $M \cap W = (M \cap W') \cap W$. So, in sum, we may assume that W is a node of $t - s$, M is a node of $s - t$, and there are no transitive nodes of s between W and M. It is easy to check in this case that M belongs to E_W, hence $M \cap W$ belongs to F_W and is a node of r by definition. □

To complete the proof of the lemma it remains to verify condition (3) of Definition 2.4 for r. The condition holds trivially at nodes of transitive type. For a small node N of r that belongs to t, we have $N \subseteq Q$ and therefore by Remark 2.24, $\{M \in r \mid M \in N\} = \{M \in r \mid M \in N \wedge M \in t\}$. Condition (3) for r at N therefore follows from the same condition for t. We must check the condition for the other small nodes of r, namely small nodes that belong to $s - t$, and small nodes in the added intervals F_W.

Claim 2.27. *Let N be a small node of r that belongs to s. Then every node of r that belongs to N is either a node of s in N, or a node of t in N, or the intersection of a small node of s in N with a transitive node of t in N.*

Proof. Suppose not. Since the only nodes of r that do not belong to $s \cup t$ are nodes in added intervals F_W, this means that there is a transitive node W of t and a node $M \in F_W$, equal to $M^* \cap W$ say with $M^* \in E_W$, so that $M = M^* \cap W \in N$, but M^* and W do not both belong to N.

Suppose for simplicity that there is a transitive node of s in the interval (W, Q) of $s \cup t$, and let W^* be the least such. The case that there are no transitive nodes of s in that interval is similar.

Replacing N with $N \cap W^*$ if needed, we may assume that N occurs before W^* in s. N cannot occur at or before M in r, since $M \in N$. And it cannot occur in the interval (M, W) of r, since this interval is contained in F_W which is disjoint from

$s \cup t$. So N must occur above W in r. Thus, both M^* and N occur between W and W^* in r, and by minimality of W^* it follows that there are no transitive nodes of s between them.

If N occurs before M^*, then by Claim 2.10, $N \in M^*$. Then $N \cap W \in M^* \cap W$ by a simple calculation using the fact that $W \in M^*$. On the other hand $M^* \cap W \in N$, and this implies that $M^* \cap W \in N \cap W$. Altogether then $N \cap W \in M^* \cap W \in N \cap W$, contradiction. A similar argument leads to a contradiction in case $N = M^*$.

So it must be that N occurs above M^*. By Claim 2.10 it follows that $M^* \in N$, and that $M^* \subseteq N$. Since $W \in M^*$ it follows further that $W \in N$. □

Claim 2.28. *Let N be a small node of r that belongs to s. Then $\{M \in r \mid M \in N\}$ belongs to N.*

Proof. By the previous claim, and the closure of $r \supseteq s \cup t$ under intersections, $\{M \in r \mid M \in N\}$ consists precisely of the nodes of s that belong to N, the nodes of t that belong to N, and all intersections of small nodes of s that belong to N with transitive nodes of t that belong to N. Thus it is enough to prove that both $\{M \in s \mid M \in N\}$ and $\{M \in t \mid M \in N\}$ belong to N. $\{M \in r \mid M \in N\}$ is the closure of the union of these two sequences under intersections of small nodes in the former with transitive nodes in the latter, and belongs to N by elementarity of N.

That $\{M \in s \mid M \in N\}$ belongs to N is clear by condition (3) of Definition 2.4 for s. We prove that $\{M \in t \mid M \in N\}$ belong to N.

We can assume that there are no transitive nodes of s above N: otherwise letting W be the first such node, we can replace s, Q, and t by $s \cap W$, $Q \cap W$, and $t \cap W$ respectively. We can also assume that there are no transitive nodes of s above Q: otherwise letting W be the first such node, we can replace s and N by $s \cap W$ and $N \cap W$ respectively.

We now divide into two cases. If N occurs at or above Q, then (using the assumptions above, and Claim 2.10) $Q \subseteq N$, and since $t \subseteq Q$ it follows that $\{M \in t \mid M \in N\} = t \in Q \subseteq N$. If N occurs below Q, then (again using the assumptions above, and Claim 2.10) $N \in Q$, and since $t \supseteq \text{res}_Q(s)$ it follows that $N \in t$, hence $\{M \in t \mid M \in N\} \in N$ by condition (3) of Definition 2.4 for t. □

Claim 2.29. *Let N be a small node of r that belongs to an added interval F_W. Then $\{M \in r \mid M \in N\}$ belongs to N.*

Proof. Let $N^* \in E_W$ be such that $N = N^* \cap W$. Then by the previous claim $\{M \in r \mid M \in N^*\}$ belongs to N^*. It is shown in the paragraphs defining E_W that $W \in N^*$. Thus by elementarity of N^*, $\{M \in r \mid M \in N^*\} \cap W$ belongs to N^*. By closure of W, $\{M \in r \mid M \in N^*\} \cap W$ belongs also to W. Thus $\{M \in r \mid M \in N\} = \{M \in r \mid M \in N^* \cap W\} = \{M \in r \mid M \in N^*\} \cap W \in N^* \cap W = N$. □

The last two claims complete the proof of condition (3) of Definition 2.4 for r, and with it the proof that r is a condition. Since $r \supseteq s \cup t$, and all nodes of r are in the closure of $s \cup t$ under intersections, this completes the proof of Lemma 2.21. □

Remark 2.30. In case $\kappa = \omega$ and $\lambda = \omega_1$, condition (3) of Definition 2.4 is vacuous, and therefore Claims 2.27 through 2.29 in the proof of Lemma 2.21 are not necessary.

Corollary 2.31. *Let s be a condition and let Q be a node of s. Suppose t is a condition that belongs to Q and extends $\text{res}_Q(s)$. Then:*

(1) *s and t are directly compatible.*
(2) *Let r witness that s and t are directly compatible, meaning that r is the closure of $s \cup t$ under intersections. Then $\text{res}_Q(r) = t$.*
(3) *The small nodes of r outside Q are all of the form N or $N \cap W$ where N is a small node of s and W is a transitive node of t.*

Proof. Condition (1) is immediate form Lemma 2.20 if Q is of transitive type, and from Lemma 2.21 if Q is of small type.

If Q is of transitive type, then the proof of Lemma 2.20 shows that r is simply $s \cup t$, and since t extends $\text{res}_Q(s)$ it is clear that $\text{res}_Q(r) = t$.

If Q is of small type, then by Remark 2.24, every node of r that belongs to Q is a node of t, in other words $\text{res}_Q(r) \subseteq t$. Since $t \subseteq r$ and $t \subseteq Q$ the reverse inclusion holds trivially, completing the proof of condition (2).

Condition (3) is clear if Q is transitive, since then all nodes of r outside Q are nodes of s. If Q is small, then r consists of $s \cup t$ together with the added intervals F_W from the proof of Lemma 2.21. Nodes in t belong to Q, and nodes in the added intervals are of the form $N \cap W$, for small $N \in s$ and transitive $W \in t$, by the definition of the added intervals. □

Corollary 2.32. *Let $M \in \mathcal{S} \cup \mathcal{T}$, and let t be a condition that belongs to M. Then there is a condition $r \leq t$ with $M \in r$. Moreover r is simply the closure of $t \cup \{M\}$ under intersections.*

Proof. Let $s = \{M\}$. It is clear that s is a condition, and that $\text{res}_M(s) = \emptyset$. Since $t \leq \emptyset$, by Corollary 2.31 the conditions s and t are directly compatible. Let r witness this. Then $r \leq t$, $M \in r$, and r is the closure of $t \cup \{M\}$ under intersections. □

Claim 2.33. *Let s and t be conditions, and let W be a transitive node that belongs to both. Suppose that s and t are directly compatible and let r witness this. Then $\text{res}_W(r)$ is the closure of $\text{res}_W(s) \cup \text{res}_W(t)$ under intersections.*

Proof. Since $\text{res}_W(s) \cup \text{res}_W(t) \subseteq (s \cup t) \cap W$, the closure of $\text{res}_W(s) \cup \text{res}_W(t)$ under intersections is contained in r, and in W, hence in $\text{res}_W(r)$.

Every node M in r is of the form $\bigcap_{i<k} M_i$ where $M_i \in s \cup t$. If M belongs to W then $M = M \cap W = \bigcap_{i<k} M_i \cap W$. By closure of s and t under intersections, and since W belongs to both s and t, $M_i \cap W \in s \cup t$. Since $M_i \cap W \in W$, this implies that $M_i \cap W \in \text{res}_W(s) \cup \text{res}_W(t)$. So M belongs to the closure of $\text{res}_W(s) \cup \text{res}_W(t)$ under intersections. □

We end this section with a slight modification of the poset $\mathbb{P} = \mathbb{P}_{\kappa, \mathcal{S}, \mathcal{T}, K}$, which we call the *decorated* version. We prove that this modified version satisfies a parallel of Corollary 2.31.

Claim 2.34. *Let s be a condition and let M and N be nodes of s so that $M \in N$. Let M^* be the successor of M in s. Then $M^* \subseteq N$.*

Proof. If N is of transitive type, or there are no nodes of transitive type between M and N, then $M^* \subseteq N$ by Claim 2.10. Suppose that N is of countable type and there are nodes of transitive type between M and N. Let W be the least one. By Claim 2.10, $M \in W$. Since $M \in N$ it follows that $M \in N \cap W$, hence in particular $N \cap W$

occurs above M. Since $N \cap W$ occurs below W, there are no nodes of transitive type between M and $N \cap W$. By Claim 2.10 it follows that $M^* \subseteq N \cap W$. □

Definition 2.35. Define $\mathbb{P}^{\text{dec}} = \mathbb{P}^{\text{dec}}_{\kappa,\mathcal{S},\mathcal{T},K}$ to be the poset consisting of pairs $\langle s, f \rangle$ where:

(1) $s \in \mathbb{P}_{\kappa,\mathcal{S},\mathcal{T},K}$.
(2) f is a function on the nodes of s. For every $M \in s$, $f(M)$ is a set of size $< \kappa$.
(3) If $M \in s$ is not the largest node in s, then $f(M)$ is an element of the successor of M in s. If M is the largest node of s, then $f(M) \in K$.
(4) f belongs to K, and for every node $N \in s$, the restriction of f to $\text{res}_N(s)$ belongs to N.

Note that condition (4) is vacuous for transitive N; it follows for such N using the closure of N given by Definition 2.2.

The ordering on $\mathbb{P}^{\text{dec}}$ is the following: $\langle s^*, f^* \rangle \leq \langle s, f \rangle$ iff $s^* \leq s$, and $f^*(M) \supseteq f(M)$ for every $M \in s$.

Remark 2.36. In case $\kappa = \omega$, conditions (3) is equivalent to the requirement that $f(M) \subseteq M^*$ when M^* is the successor of M in s, and $f(M) \subseteq K$ when M is the largest node of s. Moreover condition (4) is vacuous in this case: By Claim 2.34, $f(M) \subseteq N$ for every $M \in \text{res}_N(s)$, and together with the fact $\text{res}_N(s)$ and $f(M)$ for $M \in \text{res}_N(s)$ are all finite, this automatically gives $f \restriction \text{res}_N(s) \in N$. Finiteness also automatically gives $f \in K$.

Thus, in case $\kappa = \omega$, the definition simplifies to the following: $s \in \mathbb{P}$, f is a function defined on the nodes of s, and for each $M \in s$, $f(M)$ is a finite subset of the successor of M in s if there is one, and of K if M is the largest node.

Let $\langle s, f \rangle \in \mathbb{P}^{\text{dec}}$, and let Q be a node of s. Then the residue of $\langle s, f \rangle$ in Q, denoted $\text{res}_Q(s, f)$, is defined to be $\langle \text{res}_Q(s), f \restriction \text{res}_Q(s) \rangle$.

Claim 2.37. *$\text{res}_Q(s, f)$ is a condition in $\mathbb{P}^{\text{dec}}$, and belongs to Q.*

Proof. Condition (1) and (2) for $\text{res}_Q(s, f)$ are clear from the same conditions for $\langle s, f \rangle$. Since $\text{res}_Q(s) \in Q$, condition (4) for $\langle s, f \rangle$, used with $N = Q$, directly implies that $\text{res}_Q(s, f)$ belongs to Q. Condition (4) transfers from $\langle s, f \rangle$ to $\text{res}_Q(s, f)$: closure of K implies $f \restriction \text{res}_Q(s) \in K$, and for any small node $N \in \text{res}_Q(s)$, $f \restriction \text{res}_Q(s) \restriction \text{res}_N(\text{res}_Q(s)) = f \restriction \text{res}_N(s) \in N$, using the fact that $N \subseteq Q$.

Finally condition (3) for $\text{res}_Q(s, f)$ is immediate from the same condition for s, in all instances except when the successor of M in $\text{res}_Q(s)$ is the bottom node of a residue gap of s in Q. Let $[Q \cap W, W)$ be the gap. By condition (3) for s, $f(M) \in Q \cap W$. In particular $f(M) \in W$. Since W is the successor of M in $\text{res}_Q(s)$ this establishes the M instance of condition (3) for $\text{res}_Q(s, f)$. □

Two conditions $\langle s, f \rangle, \langle t, g \rangle \in \mathbb{P}^{\text{dec}}$ are *directly compatible* in $\mathbb{P}^{\text{dec}}$ if there is $\langle r, h \rangle$ witnessing their compatibility, with all nodes of r obtained by intersections from nodes of $s \cup t$.

Claim 2.38. *Let $\langle s, f \rangle \in \mathbb{P}^{\text{dec}}$, let Q be a node of s, and let $\langle t, g \rangle \in \mathbb{P}^{\text{dec}}$ be an element of Q that extends $\text{res}_Q(s, f)$. Then $\langle s, f \rangle$ and $\langle t, g \rangle$ are directly compatible in $\mathbb{P}^{\text{dec}}$.*

Proof. By Corollary 2.31, s and t are directly compatible in $\mathbb{P}$. Let r witness this. It is enough now to find h so that $\langle r, h\rangle \in \mathbb{P}^{\text{dec}}$, for every $M \in s$, $h(M) \supseteq f(M)$, and for every $M \in t$, $h(M) \supseteq g(M)$.

Set $h(M)$ to be equal to $g(M)$ for $M \in t$, equal to $f(M)$ for $M \in s - t$, and equal to $\emptyset$ for $M \in r - (s \cup t)$. Since $\langle t, g\rangle \leq \text{res}_Q(s, f)$, $g(M) \supseteq f(M)$ for $M \in s \cap t$. It follows that $h(M) \supseteq f(M)$ for $M \in s$. By definition, $h(M) \supseteq g(M)$ for $M \in t$. It remains to check that $\langle r, h\rangle$ satisfies the conditions in Definition 2.35.

Conditions (1) and (2) are clear. Fix $M \in r$ for condition (3). If M is the largest node of r then M is the largest node of s, and $h(M) = f(M) \in K$. If $M \in s - t$ then M is either at or above Q, or in a residue gap $[Q \cap W, W)$ of s in Q. Either way the successor M^* of M in r is the successor of M in s, and $h(M) = f(M) \in M^*$. If $M \in r - (s \cup t)$ then $h(M) = \emptyset$, which certainly belongs to M^*. It remains to consider $M \in t$. If Q if of transitive type, then t is an initial segment of r, M^* is either the successor of M in t or $M^* = Q$. $h(M) = g(M) \in M^*$ by condition (3) for $\langle t, g\rangle$ in the former case, and since $\langle t, g\rangle \in Q$ in the latter. Suppose Q is of small type. If the successor M^* of M in r is equal to the successor of M in t, or to Q, then $h(M) \in M^*$ as in the case of transitive type Q. If M^* is the bottom node of a residue gap $[Q \cap W, W)$ of s in Q, then condition (3) for $\langle t, g\rangle$ yields $g(M) \in W$. Since $\langle t, g\rangle \in Q$ certainly $g(M) \in Q$. So $h(M) = g(M) \in Q \cap W = M^*$. By the proof of Lemma 2.21, the only other option is that M^* is the lowest node of an added interval F_W (in the terminology of the lemma). This lowest node is equal to $Q \cap W$, where W is a transitive node of t above M. Then by Claim 2.10 and condition (3) for $\langle t, g\rangle$, $g(M) \in W$. $g(M)$ belongs to Q since $\langle t, g\rangle \in Q$. So $h(M) = g(M) \in Q \cap W = M^*$.

For condition (4), it is enough to prove that for small $N \in r$, $g{\restriction}N$ belongs to N, and if $N \not\subseteq Q$, then $f{\restriction}N$ belongs to N too. This allows constructing $h{\restriction}N$ inside N. (The full function h can be constructed inside K from f and g.) Consider first the case that $N \in Q$. Then $N \subseteq Q$ because N is small, and $N \in t$ by part (2) of Corollary 2.31. By condition (4) for $\langle t, g\rangle$, $g{\restriction}N \in N$ as required.

Suppose next that $N \notin Q$. We prove that $f{\restriction}N$ and $g{\restriction}N$ both belong to N. The proof in both cases is by induction on N.

If N is above Q, then $N \in s$ and this immediately implies $f{\restriction}N \in N$. If there are no nodes of transitive type between Q and N then $Q \subseteq N$ and since $\langle t, g\rangle \in Q$ it follows that $g{\restriction}N = g \in N$. If there are transitive nodes between Q and N, let R be such a node. By induction $g{\restriction}N \cap R \in N \cap R \subseteq N$. Since R is above Q and $g \in Q$, $g{\restriction}N \cap R = g{\restriction}N$, so $g{\restriction}N \in N$.

A similar argument applies if N belongs to a residue gap of s in Q, say $[Q \cap W, W)$, using the fact that N is above $Q \cap W$, and $g{\restriction}N = g{\restriction}Q \cap W$ in case $Q \cap W \subseteq N$.

The only remaining alternative is that Q is of small type and N belongs to an added interval F_W in the terminology of Lemma 2.21. In this case N contains the smallest node of the interval, $Q \cap W$, and $g{\restriction}N = g{\restriction}Q \cap W$. Since $g \subseteq Q$, $g{\restriction}Q \cap W = g{\restriction}W$, which belongs to W by condition (4) for $\langle t, g\rangle$ and to Q since $g, W \in Q$. So $g{\restriction}N = g{\restriction}Q \cap W \in Q \cap W \subseteq N$. N itself is equal to $N^* \cap W$ for some $N^* \in s$. By condition (4) for $\langle s, f\rangle$, $f{\restriction}N^* \in N^*$. Since $N = N^* \cap W$, $f{\restriction}N$ is an initial segment of $f{\restriction}N^*$, hence it too belongs to N^*. It belongs to W by closure of W under sequences of length $< \kappa$. So $f{\restriction}N \in N^* \cap W = N$. □

Remark 2.39. As usual, the part of the proof of Claim 2.38 handling condition (4) of Definition 2.35 is not necessary in case $\kappa = \omega$.

Corollary 2.40. *Let $M \in \mathcal{S} \cup \mathcal{T}$, and let $\langle t, g\rangle \in \mathbb{P}^{\text{dec}}$ be an element of M. Then there is $\langle r, h\rangle \in \mathbb{P}^{\text{dec}}$ extending $\langle t, g\rangle$, with $M \in r$.*

Proof. The proof of this, from Claim 2.38, is similar to the proof of Corollary 2.32 from Corollary 2.31. □

3. Strong properness

This section includes some basic results about strong properness. The notion and the results presented are due to Mitchell [4], except for Claim 3.8 which is due to Friedman [2]. We also include some well known results about properness and preservation of cardinals.

A condition p in a poset $\mathbb{Q}$ is a *strong master condition* for a model M if it forces that $\dot{G} \cap M$ is generic for $\mathbb{Q} \cap M$. In other words it forces the generic filter to meet every dense subset of $\mathbb{Q} \cap M$. The poset is *strongly proper* for M if every condition in M can be extended to a strong master condition for M.

Recall that p is an ordinary *master condition* for M if it forces the generic object to meet, inside M, every dense set of $\mathbb{Q}$ that belongs to M. $\mathbb{Q}$ is proper for M if every condition in M can be extended to a master condition for M.

Remark 3.1. If p is a strong mater condition for M, $\mathbb{Q} \in M$, and M is sufficiently elementary, then p is also an ordinary master condition for M. To see this note that for any dense set D of $\mathbb{Q}$ that belongs to M, by elementarity $D \cap M$ is dense in $\mathbb{Q} \cap M$.

Remark 3.2. Suppose $\mathbb{Q} \subseteq H(\theta)$ is strongly proper for $M \subseteq H(\theta)$. Let $\theta^* > \theta$ and suppose $M^* \prec H(\theta^*)$ is such that $M^* \cap H(\theta) = M$. Then $\mathbb{Q}$ is also strongly proper for M^*. This is immediate from the definitions, as only $M^* \cap \mathbb{Q} = M \cap \mathbb{Q}$ is relevant for determining strong properness.

The following claim gives standard consequences of properness. By the observation above, they are also consequences of strong properness.

Claim 3.3. *Suppose M is elementary in $H(\theta^*)$ and $\mathbb{Q} \in M$. Let G be generic for $\mathbb{Q}$ over V, and suppose that G includes a master condition for M. Then:*

1. *$M[G] \prec H(\theta^*)[G]$ and $M[G] \cap V = M$.*
2. *Let $\dot{f} \in M$ and suppose that $\dot{f}[G]$ is a function with ordinal domain. Let $\tau = \dot{f} \cap M$. Then $\tau[G] = \dot{f}[G] \restriction M$.*

Proof. The first part is well known. For the second, it is clear that $\tau[G] \subseteq \dot{f}[G]$, and (using the first part) that $\text{dom}(\tau[G]) \subseteq M$. For the reverse inclusion, if $\alpha \in M$ and $\dot{f}[G](\alpha) = b$, then there is some pair $\langle \sigma, q\rangle \in \dot{f}$ so that $\sigma[G] = \langle \alpha, b'\rangle$ and $q \in G$, and moreover for every such pair, $b' = b$. Using the first condition, such a pair $\langle \sigma, q\rangle$ can be found in M, so that it belongs to τ. □

Call a poset *proper for $\mathcal{S}$* if it is proper for every $M \in \mathcal{S}$, and similarly with strong properness.

Claim 3.4. *Suppose $\mathbb{Q} \in H(\theta^*)$ is proper for $\mathcal{S}^*$. Let $\delta < \theta^*$ be a cardinal and suppose that for each $\alpha < \delta$, the set $\{M \prec H(\theta^*) \mid \alpha \subseteq M, |M| < \delta, \text{ and } M \in \mathcal{S}^*\}$ is stationary. Then forcing with $\mathbb{Q}$ does not collapse δ.*

Proof. This is a standard application of Claim 3.3. Let $\dot{f}$ be a name for a function into δ, with domain $\alpha < \delta$. Using the stationarity assumed in the claim, find $M \in \mathcal{S}^*$ of size $< \delta$ with $\alpha \cup \{\dot{f}, \mathbb{Q}\} \subseteq M$. Let p be a master condition for M. By Claim 3.3, p forces the range of $\dot{f}$ to be contained in M. Since $M \in V$ has size $< \delta$, $M \not\supseteq \delta$, and hence p forces $\dot{f}$ to not be onto δ. □

Claim 3.5. *Suppose $\mathbb{Q} \subseteq H(\theta)$ is strongly proper for $\mathcal{S}$. Let $\delta \leq \theta$ be a cardinal and suppose that for each $\alpha < \delta$, the set $\{M \prec H(\theta) \mid \alpha \subseteq M, |M| < \delta$, and $M \in \mathcal{S}\}$ is stationary. Then forcing with $\mathbb{Q}$ does not collapse δ.*

Proof. Let $\theta^* > \theta$ be such that $\mathbb{Q} \in H(\theta^*)$. Let $\mathcal{S}^* = \{M^* \prec H(\theta^*) \mid M^* \cap H(\theta) \in \mathcal{S}\}$. Then the stationarity of $\mathcal{S}$ assumed in the claim implies similar stationarity for $\mathcal{S}^*$. Moreover $\mathbb{Q}$ is strongly proper for each $M^* \in \mathcal{S}^*$ by Remark 3.2, and hence proper for M^* by Remark 3.1. The current claim now follows from Claim 3.4. □

Lemma 3.6. *Let G be generic for $\mathbb{Q}$ over V, let α be an ordinal, and let $f = \dot{f}[G] \in V[G]$ be a function from α into the ordinals. Suppose that $\dot{f}, \mathbb{Q} \in M$, M is elementary in some $H(\theta^*)$, and G includes a strong master condition for M. If $f \restriction M$ belongs to V, then the entire function f must belong to V.*

Proof. Redefining the name $\dot{f}$ if necessary, we may assume that all elements of $\dot{f}$ are of the form $\langle \langle \xi, \mu \rangle\check{}, t \rangle$ where $\xi < \alpha$, $\mu \in \text{Ord}$, and $t \Vdash \dot{f}(\check{\xi}) = \check{\mu}$. Let $\tau = \dot{f} \cap M$. By Claim 3.3 and Remark 3.1, $\tau[G] = \dot{f}[G] \restriction M$. Suppose that $\tau[G]$ belongs to V. We prove that f belongs to V.

We are assuming that G includes a strong mater condition for M, and hence $G \cap M$ is generic for $\mathbb{Q} \cap M$ over V. Since $\tau \subseteq M$ and all elements of τ are of the form $\langle \langle \xi, \mu \rangle\check{}, t \rangle$, τ is a $\mathbb{Q} \cap M$-name. $\tau[G]$ (with τ viewed as a $\mathbb{Q}$ name) is equal to $\tau[G \cap M]$ (with τ viewed as a $\mathbb{Q} \cap M$ name). Since $\tau[G \cap M] = \tau[G]$ belongs to V, there is a condition $r \in G \cap M$ which forces (in $\mathbb{Q} \cap M$) a specific value for τ. In particular, for every $\xi \in \alpha \cap M$, there is $\mu \in \text{Ord} \cap M$ such that every $s \in \mathbb{Q} \cap M$ with $s \leq r$ extends to $t \in \mathbb{Q} \cap M$ so that $\langle \langle \xi, \mu \rangle\check{}, t \rangle \in \dot{f}$. By elementarity of M, it follows that for every $\xi \in \alpha$, there is $\mu \in \text{Ord}$ such that every $s \in \mathbb{Q}$ with $s \leq r$ extends to $t \in \mathbb{Q}$ so that $\langle \langle \xi, \mu \rangle\check{}, t \rangle \in \dot{f}$. This implies that r in fact completely forces, in $\mathbb{Q}$, all values of $\dot{f}$. So $f = \dot{f}[G] \in V$. □

Lemma 3.7. *Suppose $\mathbb{Q} \subseteq H(\theta)$ is strongly proper for $\mathcal{S}$, let δ be a regular cardinal, and suppose that $\{M \in \mathcal{S} \mid |M| < \delta\}$ is stationary in $H(\theta)$. Then forcing with $\mathbb{Q}$ does not add branches of length δ to trees in V. Precisely, suppose T is a tree in V, G is generic for $\mathbb{Q}$ over V, and $b \in V[G]$ is a branch through T of length δ. Then $b \in V$.*

Proof. Without loss of generality we may assume that nodes of T are ordinals, so that branches of length δ through T are functions from δ into ordinals. Suppose for contradiction that $b = \dot{b}[G]$ is a branch through T, of length δ, that belongs to $V[G]$ but not to V. Let $r \in G$ be a condition forcing this.

Let $\theta^* > \theta$ be large enough that $\mathbb{Q}, \dot{f} \in H(\theta^*)$. Using the stationarity assumed in the lemma, fix $M^* \prec H(\theta^*)$ so that $|M^*| < \delta$, r, $\dot{f}$, and $\mathbb{Q}$ belong to M^*, and $M^* \cap H(\theta) \in \mathcal{S}$. Let $p \leq r$ be a strong master condition for $M^* \cap H(\theta)$ (equivalently, since $\mathbb{Q} \subseteq H(\theta)$, a strong master condition for M^*). Replacing the generic G if needed, we may assume that $p \in G$.

Then since $b \notin V$ it follows by lemma 3.6 that $b \restriction M^* \notin V$. Let $\gamma = \sup(\delta \cap M^*)$. Since δ is regular and $|M^*| < \delta$, γ is smaller than δ. Since T is a *tree* that belongs to V, this implies that $b \restriction \gamma$ belongs to V (it is the function that enumerates all T-predecessors of the node $b(\gamma)$ according to their order in T). Since $b \restriction M^* = b \restriction \gamma \restriction M^*$ it follows that $b \restriction M^*$ belongs to V, contradiction. $\square$

The next claim, which deals with the product of proper and strongly proper posets, is an abstraction of Lemma 3 in Friedman [2].

Claim 3.8. *Let $\mathbb{A}, \mathbb{P} \in M \prec H(\theta^*)$. Suppose that $\mathbb{A}$ is strongly proper for M, and $\mathbb{P}$ is proper for M. Then:*

(1) *If a and p are strong and ordinary master conditions for M in $\mathbb{A}$ and $\mathbb{P}$ respectively, then $\langle a, p\rangle$ is a master condition for M in $\mathbb{A} \times \mathbb{P}$.*
(2) *$\mathbb{A} \times \mathbb{P}$ is proper for M.*
(3) *If G is generic for $\mathbb{A}$ over V with a strong master condition for M, then in $V[G]$, $\mathbb{P}$ is proper for $M[G]$.*

Proof. We prove the first part of the claim. The other two parts are immediate consequences of the first. Let $D \in M$ be a dense subset of $\mathbb{A} \times \mathbb{P}$, and let a and p be strong and ordinary master conditions for M in $\mathbb{A}$ and $\mathbb{P}$ respectively. It is enough to prove that $\langle a, p\rangle$ is compatible with an element of $D \cap M$. Let $Z = \{b \in \mathbb{A} \cap M \mid (\exists q \in M)\langle b, q\rangle \in D$ and q is compatible with $p\}$. The density of D, elementarity of M, and the fact that p is a master condition for M in $\mathbb{P}$, imply that Z is dense in $\mathbb{A} \cap M$. Since a is a strong master condition for M there is $b \in Z$ which is compatible with a. By definition of Z there is then $q \in M$ so that $\langle b, q\rangle \in D$ and q is compatible with p. $\square$

4. Sequence poset and strong properness

Let $\mathcal{S}$ and $\mathcal{T}$ be appropriate for κ, λ, and K. Let $\mathbb{P}$ be the sequence poset associated to κ, $\mathcal{S}$, $\mathcal{T}$, and K.

Claim 4.1. (1) *Let $s \in \mathbb{P}$ and let Q be a node in s. Then s is a strong mater condition for Q.*
(2) *$\mathbb{P}$ is strongly proper for $\mathcal{S} \cup \mathcal{T}$.*
(3) *If $W \in \mathcal{S} \cup \mathcal{T}$, then $\mathbb{P} \cap W$ is strongly proper for $(\mathcal{S} \cup \mathcal{T}) \cap W$. For any condition $s \in \mathbb{P} \cap W$ and any node $Q \in s$, s is a strong master condition for Q in $\mathbb{P} \cap W$.*

Proof. Consider first condition (1). Suppose for contradiction that s is not a strong master condition for Q. Extending s, we may fix a dense subset D of $\mathbb{P} \cap Q$ and assume that s forces the generic filter for $\mathbb{P}$ to avoid D. $\text{res}_Q(s)$ is a condition of $\mathbb{P}$ that belongs to Q. By density of D, there is $t \in D$ extending $\text{res}_Q(s)$. t belongs to Q as $D \subseteq Q$. By Corollary 2.31, s and t are directly compatible. Let r witness this. Then r is an extension of s that forces t into the generic object, contradicting the fact that s forces the generic object to avoid D.

By Corollary 2.32, every condition $u \in Q$ extends to a condition s with $Q \in s$, which by (1) is a strong master condition for $\mathbb{P}$. This establishes condition (2).

For condition (3), note that if s and t in the proof of condition (1) both belong to W, then so does r, since by elementarity W is closed under intersections. Similarly in the proof of condition (2), if u and Q belong to W then so does s. The same proofs can therefore be used to get the strong properness of $\mathbb{P} \cap W$. $\square$

Claim 4.2. *Let $\mathbb{P}^{\rm dec}$ be the decorated sequence poset associated to κ, λ, $\mathcal{S}$, and $\mathcal{T}$. Then $\mathbb{P}^{\rm dec}$ is strongly proper for $\mathcal{S} \cup \mathcal{T}$. Moreover, any condition $\langle t, g\rangle \in \mathbb{P}^{\rm dec}$ that belongs to a node $Q \in \mathcal{S} \cup \mathcal{T}$ extends to a condition $\langle s, f\rangle \in \mathbb{P}^{\rm dec}$ with $Q \in s$, and any such condition s is a strong master condition for Q.*

Proof. Similar to the proof of Claim 4.1, but using Claim 2.38 and Corollary 2.40 instead of Corollaries 2.31 and 2.32. □

Let W be a node of transitive type. $\{W\}$ is then a strong master condition for W, meaning that if G is generic for $\mathbb{P}$ over V with $\{W\} \in G$, then $\bar{G} = G \cap W$ is generic for $\bar{\mathbb{P}} = \mathbb{P} \cap W$ over V. This implies that the forcing to add G can be broken into two stages: first force with $\bar{\mathbb{P}}$ to add $\bar{G}$, then force with a factor poset to add G over $V[\bar{G}]$. It is easy to check that the factor poset is simply the restriction of $\mathbb{P}$ to conditions s so that $W \in s$ and $\text{res}_W(s) \in \bar{G}$. (This poset belongs to $V[\bar{G}]$.) We continue with a couple of claims establishing strong properness for the factor poset.

Claim 4.3. *Let W be a transitive node, let $\bar{\mathbb{P}} = \mathbb{P} \cap W$, and let $\bar{G}$ be generic for $\bar{\mathbb{P}}$ over V, Let $\hat{\mathcal{S}} = \{M[\bar{G}] \mid M \in \mathcal{S},\ W \in M,$ and $M \cap W \in \bigcup \bar{G}\}$. Then:*

1. *Every $M[\bar{G}] \in \hat{\mathcal{S}}$ is an elementary substructure of $H(\theta)[\bar{G}]$, and $M[\bar{G}] \cap V = M$.*
2. *In $V[\bar{G}]$, $\hat{\mathcal{S}}$ is stationary in $H(\theta)[\bar{G}]$.*

Proof. Let $M \in \mathcal{S}$ be such that $W \in M$ and $M \cap W \in \bigcup \bar{G}$. By Claim 4.1, any condition $\bar{s} \in \bar{\mathbb{P}}$ with $Q \in \bar{s}$ is a strong master condition for Q. Applying this with $Q = M \cap W$ and any condition $\bar{s} \in \bar{G}$ so that $M \cap W \in \bar{s}$, it follows by Remark 3.2 that $\bar{s}$ is also a strong master condition for M in $\bar{\mathbb{P}}$, and hence, by Remark 3.1 and Claim 3.3, $M[\bar{G}] \prec H(\theta)[\bar{G}]$ and $M[\bar{G}] \cap V = M$. This establishes condition (1).

Suppose for contradiction condition (2) fails, and let $\bar{s} \in \bar{G}$ force that $\dot{C}$ is a club disjoint from $\hat{\mathcal{S}}$. Let $\theta^* > \theta$ and let $M^* \prec H(\theta^*)$ be countable with $\mathbb{P}, W, \bar{s}, \dot{C} \in M^*$. Let $\bar{r} \in \bar{\mathbb{P}}$ be an extension of $\bar{s}$ with $M^* \cap W \in \bar{r}$. Such an extension exists in $\mathbb{P}$ by Corollary 2.32, and by closure of W under intersections it belongs to W, hence to $\bar{\mathbb{P}}$. It is easy to check that $\bar{r}$ forces $M[\bar{G}]$ into $\hat{\mathcal{S}}$ where $M = M^* \cap H(\theta)$. At the same time since $\dot{C} \in M^*$ it forces $M[\bar{G}]$ to belong to $\dot{C}[\bar{G}]$. This contradicts the fact that $\bar{r}$ extends $\bar{s}$. □

Claim 4.4. *Let W be a transitive node, let $\bar{\mathbb{P}} = \mathbb{P} \cap W$, let $\bar{G}$ be generic for $\bar{\mathbb{P}}$ over V, and let $\mathbb{Q}$ be the factor poset for adding a V generic for $\mathbb{P}$ extending $\bar{G}$, over $V[\bar{G}]$. Let $\hat{\mathcal{S}}$ be defined as in Claim 4.3. Then $\mathbb{Q}$ is strongly proper for $\hat{\mathcal{S}}$.*

Proof. Fix $M \in \mathcal{S}$ with $W \in M$ and $M \cap W \in \bigcup \bar{G}$. We have to show that the factor poset $\mathbb{Q}$ is strongly proper for $M[\bar{G}]$. We show this by proving that: (1) any condition $s \in \mathbb{Q}$ with $M \in s$ is a strong master condition for $M[\bar{G}]$ in $\mathbb{Q}$; and (2) any condition $t \in \mathbb{Q}$ that belongs to $M[\bar{G}]$ can be extended to such an s.

Consider (1) first. Suppose for contradiction that s is not a strong master condition for $M[\bar{G}]$. Extending s if necessary we may assume it forces that the generic avoids a specific set D which is dense in $\mathbb{Q} \cap M[\bar{G}]$. $\mathbb{Q} \cap M[\bar{G}]$ is equal to $\mathbb{Q} \cap M$ as $\mathbb{Q} \subseteq V$, and $M[\bar{G}] \cap V = M$ by Claim 4.3. So $D \subseteq M$, D is dense in $\mathbb{Q} \cap M$, and no extension r of s in $\mathbb{Q}$ extends an element of D.

Recall that the factor poset $\mathbb{Q}$ consists of conditions $u \in \mathbb{P}$ so that $W \in u$ and $\text{res}_W(u) \in \bar{G}$. The fact that s is such a condition and $W \in M$ implies that so is

$\text{res}_M(s)$. ($\text{res}_W(\text{res}_M(s))$ is weaker than $\text{res}_W(s)$ and therefore belongs to $\bar{G}$.) By density of D, there is some $t \in D$ extending $\text{res}_M(s)$. By Corollary 2.31 and since $D \subseteq M$, s and t are directly compatible in $\mathbb{P}$. To derive a contradiction we need to show they are compatible in the factor poset.

Let r witness that s and t are directly compatible in $\mathbb{P}$. By Claim 2.33, $\text{res}_W(r)$ is the closure of $\text{res}_W(s) \cup \text{res}_W(t)$ under intersections. From this, the fact that $\text{res}_W(s)$ and $\text{res}_W(t)$ both belong to $\bar{G}$, and the fact that $\bar{G}$ is a filter, it follows that $\text{res}_W(r)$ too belongs to $\bar{G}$. Hence r belongs to the factor poset, witnessing that s and t are compatible in this poset. This completes the proof of (1).

(2) is a consequence of the proof of (1). Let t be a condition in the factor poset that belongs to M. Let u be the condition $\{M \cap W, W, M\}$. Then u is a condition in the factor poset, $\text{res}_M(u) = \{W\} \subseteq t$, and the proof of (1) shows that u and t are compatible in the factor poset. Any condition s in the factor poset witnessing this is an extension of t with $M \in s$. □

Remark 4.5. We only proved strong properness of the factor poset for small nodes. Similar strong properness for transitive nodes is also true, and by simpler proofs.

5. Initial applications

We can now give several quick applications of the two-type model sequence posets defined in Section 2. We use the posets to obtain the tree property, to add clubs in subsets of $\theta \geq \omega_2$, to collapse to κ^+ without adding branches of length κ^+ to trees in V, and to obtain a model of PFA with an inner model that is correct about ω_2 but not about reals. The first two are the initial applications of the finite condition posets of countable models developed by Friedman [1] and Mitchell [4]. Our poset gives the same extensions, but the use of models of two types rather than only countable models makes the proof conceptually simpler. The last is a sketch of the argument of Friedman [2], but using the two-type model sequences.

5.1. **The tree property (after Mitchell** [4]**).** Let θ be a weakly compact cardinal. Let $K = H(\theta)$. Let $\mathcal{T}$ consist of all transitive $W \prec K$ of size $< \theta$ which are countably closed, and let $\mathcal{S}$ consist of all countable $M \prec K$. Both $\mathcal{S}$ and $\mathcal{T}$ are stationary. The fact that each $W \in \mathcal{T}$ is countably closed implies that $M \cap W \in W$ for all $M \in \mathcal{S}$. It is easy to check that if $W \in \mathcal{T}$, $M \in \mathcal{S}$, and $W \in M$, then $M \cap W \prec K$, and therefore $M \cap W \in \mathcal{S}$. (By elementarity of M, and since W can be wellordered inside K, there is a wellordering of W in M. This allows defining Skolem functions for W, inside M. Using the Skolem functions and the elementarity of M one can check that $M \cap W \prec W$, and this implies $M \cap W \prec K$.)

$\mathcal{S}$ and $\mathcal{T}$ are therefore appropriate for ω, ω_1, and $K = H(\theta)$. Let $\mathbb{P} = \mathbb{P}_{\omega,\mathcal{S},\mathcal{T},K}$. By Claim 4.1, $\mathbb{P}$ is strongly proper for every $Q \in \mathcal{S} \cup \mathcal{T}$, and indeed every condition s with $Q \in s$ is a strong master condition for Q. It follows by Claim 3.5 that forcing with $\mathbb{P}$ preserves ω_1 and θ, using the stationarity of $\mathcal{S}$ for the former, and the stationarity of $\mathcal{T}$ for the latter.

Let G be generic for $\mathbb{P}$ over V. Then $\bigcup G$ is a set of nodes. Using Claim 2.10 it is clear that $A = \{W \in \bigcup G \mid W \text{ is a transitive node}\}$ is increasing, both in $\in$ and in $\subseteq$. Moreover, by genericity and Corollary 2.32, A is unbounded in $H(\theta)$, meaning that for every $x \in H(\theta)$, there is $W \in A$ with $x \in W$. In particular this means that $\bigcup A = H(\theta)$.

For each $W \in A$, let $B_W = \{M \in \bigcup G \mid M$ is a small node between W and $W^*\}$, where W^* is the next element of A above W. Using Claim 2.10, it is clear that for every $W \in A$, B_W is increasing, both in $\in$ and in $\subseteq$. By genericity and Corollary 2.32, for every $x \in H(\theta)$ there is a small node M in $\bigcup G$ with $\{W, x\} \subseteq M$. Since all conditions in G are closed under intersection, and $W^* \in \bigcup G$, $M \cap W^*$ too belongs to $\bigcup G$, and must occur between W and W^* since $W \in M \cap W^*$. Letting x range over elements of W^* it follows that $\bigcup B_W = W^*$.

Since B_W is increasing, and all models in B_W are countable, the length of B_W is at most ω_1^V. Again using the fact that all models in B_W are countable, it follows that $W^* = \bigcup B_W$ has size at most $\omega_1^V = \omega_1^{V[G]}$ in $V[G]$.

Since this is true for each $W \in A$, and since every ordinal below θ belongs to $W \subseteq W^*$ for some $W \in A$, it follows that in $V[G]$, every ordinal between ω_1 and θ is collapsed to ω_1. From this and the preservation of θ, it follows that $\omega_2^{V[G]} = \theta$. We continue to prove that in $V[G]$, the tree property holds at θ.

Claim 5.1. *In $V[G]$, the tree property holds at $\theta = \omega_2$.*

Proof. Suppose not, and let $\dot{T} \in V$ be a name for a tree witnessing this. We may assume that elements of $T = \dot{T}[G]$ are pairs $\langle \xi, \mu \rangle \in \theta \times \omega_1$, and that level ξ of T consists exactly of $\{\xi\} \times \omega_1$. We may also assume that $\dot{T} \subseteq H(\theta)$. Finally, we may assume for definitiveness that it is forced outright in $\mathbb{P} = \mathbb{P}_{\omega, \mathcal{S}, \mathcal{T}, H(\theta)}$ that there are no cofinal branches through $\dot{T}$.

By standard arguments using the weak compactness of θ, there is an inaccessible cardinal $\kappa < \theta$, so that:

(1) $(H(\kappa); \dot{T} \cap H(\kappa)) \prec (H(\theta), \dot{T})$.
(2) It is forced in $\bar{\mathbb{P}} = \mathbb{P} \cap H(\kappa)$ that $\dot{T} \cap H(\kappa)$ has no branches of length κ.
(3) It is forced in $\bar{\mathbb{P}}$ that κ is regular.

Let $W = H(\kappa)$. W is then a node of transitive type, and changing G if necessary, we may assume $\{W\} \in G$. By strong properness of $\mathbb{P}$ then, $\bar{G} = G \cap W$ is generic over V for $\mathbb{P} \cap W$. By the conditions above, in $V[\bar{G}]$, κ is a regular cardinal, and $\bar{T} = (\dot{T} \cap W)[\bar{G}]$ is a tree on $\kappa \times \omega_1$, with no branches of length κ.

An application of Remark 3.1 and Claim 3.3 shows that $(H(\kappa); \bar{T}) \prec (H(\theta); T)$, and this implies that $\bar{T}$ is equal to $T \restriction \kappa$. In particular any node on level κ of T determines a branch of length κ in $\bar{T}$. Thus, in $V[G]$, there are branches of length κ through $\bar{T}$.

Let $\mathbb{Q} \in V[\bar{G}]$ be the factor poset to add G, forcing over $V[\bar{G}]$. Let $\hat{\mathcal{S}}$ be as in Claim 4.3. By the claim, $\hat{\mathcal{S}}$ is a stationary set of countable elementary substructures of $H(\theta)[\bar{G}]$ in $V[\bar{G}]$. By Claim 4.4, $\mathbb{Q}$ is strongly proper for $\hat{\mathcal{S}}$. It follows by Claim 3.7 that forcing with $\mathbb{Q}$ over $V[\bar{G}]$ does not add any branches of length κ to $\bar{T}$. But this is a contradiction, since $V[G]$ is an extension of $V[\bar{G}]$ by $\mathbb{Q}$, and $\bar{T}$, which has no branches of length κ in $V[\bar{G}]$, has such branches in $V[G]$. □

By similar proofs, but with $\kappa > \omega$ (and $\lambda = \kappa^+$) one can of course obtain the tree property at greater cardinals. The sequence poset can also be used to obtain the tree property at two successive cardinals. For example, suppose θ is supercompact and $\theta^* > \theta$ is weakly compact. Force with the poset $\mathbb{P}$ above to add G. Then follow this by forcing, over $V[G]$, with the sequence poset associated to ω_1, $\theta = \omega_2$, $\mathcal{S}$ consisting of models of the form $M[G]$ where $\theta \in M \prec H(\theta^*)$, M belongs to V, has size $< \theta$, and has ordinal intersection with θ, and $\mathcal{T}$ consisting of models $W[G]$

where $W \prec H(\theta^*)$ is transitive and $\theta \in W$. This will collapse all ordinals between θ and θ^* to θ, resulting in a model where $\theta^* = \omega_3$, and where the tree property holds at both θ and θ^*.

However the proof in this case is substantially more involved, for a couple of reasons. The second stage sequence poset forcing uses small nodes which are not countably closed in $V[G]$. While θ and θ^* are preserved by this forcing using strong properness, preservation of ω_1 requires a special argument. And while the proof of the tree property at θ^* is identical to the proof given above using strong properness of the factor poset, the proof at θ is more involved, since factors of the second stage poset to its small nodes are not strongly proper after forcing with the first stage.

5.2. **Adding clubs with finite conditions (after Friedman** [1] **and Mitchell** [4]**).** Let $\theta \geq \omega_2$ be a regular cardinal. Suppose there is a *thin stationary subset* $\mathcal{R}$ of $\mathcal{P}_{<\omega_1}(\theta)$. Precisely this means that $\mathcal{R}$ is a set of countable subsets of θ, $\mathcal{R}$ is stationary, and the set $\{\delta < \theta \mid \{x \cap \delta \mid x \in \mathcal{R}\}$ has size $\delta\}$ is unbounded in θ. Equivalently this set is club in θ relative to ordinals of uncountable cofinality. (For more on the existence and nonexistence of thin stationary sets see Friedman–Krueger [3].)

Let $f \colon \theta \to \mathcal{P}_{<\omega_1}(\theta)$ be a function so that on a club Z of $\delta < \theta$ of uncountable cofinality, $\{x \cap \delta \mid x \in \mathcal{R}\} \subseteq f''\delta$. Such a function can be constructed using the assumption that $\mathcal{R}$ is thin.

Let $U \subseteq \theta$ be stationary, and *fat*, meaning that if B is club in ω_2, then $U \cap B$ contains a club of order type $\omega_1 + 1$. We show how to add a club subset of U, without collapsing ω_1, θ, or any cardinals above θ, using a poset with finite conditions.

By passing to a stationary subset of U we may assume that if $\alpha \in U \wedge \mathrm{cof}(\alpha) \geq \omega_1$ then $U \cap \alpha$ contains a club in α. (If the set of $\alpha \in U$ of cofinality $\geq \omega_1$ so that $U \cap \alpha$ contains a club in α is not stationary, let B be a club disjoint from this set, and derive a contradiction applying fatness of U to B.) Let c be a function that assigns to each $\alpha \in U$ of uncountable cofinality a club in $U \cap \alpha$.

Let $K = L_\theta[f, Z, U, c]$ (constructing relative to Z, U, and the graphs of f and c). Let $\mathcal{T}$ consist of transitive W which belong to K and are elementary in $(K; f, Z, U)$, with $\sup(W \cap \mathrm{Ord}) \in U$ and $\mathrm{cof}(\sup(W \cap \mathrm{Ord}))$ uncountable. Let $\mathcal{S}$ consist of countable M which belong to K and are elementary in $(K; f, Z, U)$, with $\sup(M \cap \mathrm{Ord}) \in U$, and $M \cap W \in W$ for every $W \in \mathcal{T}$ which belongs to M. An argument similar to that in Subsection 5.1 shows that $M \cap W$ is elementary in $(K; f, Z, U)$ whenever $M \in \mathcal{S}$, $W \in \mathcal{T}$, and $W \in M$. Moreover $\sup(M \cap W \cap \mathrm{Ord}) \in U$ because U is fat. ($\sup(W \cap \mathrm{Ord})$ is a point of uncountable cofinality in U. A club witnessing fatness of U at $\sup(W \cap \mathrm{Ord})$ belongs to M by elementarity and inclusion of the function c in the structure K. This implies that $\sup(M \cap W \cap \mathrm{Ord})$ belongs to this club, and hence $\sup(M \cap W \cap \mathrm{Ord}) \in U$.) Hence $M \cap W \in \mathcal{S}$. It follows from this, using also the requirement $M \cap W \in W$ in the definition of $\mathcal{S}$, that $\mathcal{T}$ and $\mathcal{S}$ are appropriate for ω, ω_1, and K.

Claim 5.2. *$\mathcal{T}$ and $\mathcal{S}$ are both stationary.*

Proof. The stationarity of $\mathcal{T}$ is immediate from the stationarity of U. We prove that $\mathcal{S}$ is stationary. Let h be a function from $K^{<\omega}$ into K. We have to prove that there are countable M which belong to $\mathcal{S}$ and are closed under h.

Let $W \in \mathcal{T}$ be closed under h. Such W exists since $\mathcal{T}$ is stationary. Let $Q \subseteq K$ be countable, elementary in $(K; f, Z, U)$, and closed under h, with $W \in Q$ and

$Q \cap \text{Ord} \in \mathcal{R}$. Such Q exists since $\mathcal{R}$ is stationary. Set $M = Q \cap W$. Then M is closed under h and elementary in $(K; f, Z, U)$. The fact that U is fat and $\sup(W \cap \text{Ord}) \in U$ implies that $\sup(M \cap \text{Ord}) = \sup(Q \cap W \cap \text{Ord}) \in U$. Since $Q \cap \text{Ord}$ belongs to $\mathcal{R}$, $M \cap \text{Ord} = Q \cap \sup(W \cap \text{Ord})$ belongs to $f'' \sup(W \cap \text{Ord}) \subseteq W$. It follows from this that $M \in W$ and in particular $M \in K$. Moreover for any $\bar{W} \in \mathcal{T}$ which belongs to M, a similar argument shows that $M \cap \bar{W} = Q \cap \bar{W}$ belongs to $\bar{W}$, and this establishes that $M \in \mathcal{S}$. □

Remark 5.3. The requirement that $\mathcal{R}$ is thin can be weakened slightly, to require that the sequence of sets $\{x \cap \delta \mid x \in \mathcal{R} \text{ and } \delta \in x\}$ is approachable on a stationary subset of U of points of uncountable cofinality. Precisely this means that there is an enumeration f of sets so that for stationarily many $\delta \in U$ of uncountable cofinality, $\{x \cap \delta \mid x \in \mathcal{R} \text{ and } \delta \in x\} \subseteq f''\delta$. (If we removed the part "$\delta \in x$", this would be equivalent to thinness.) The proofs above go through essentially unmodified with this condition, restricting $\mathcal{T}$ to W so that $\delta = \sup(W \cap \text{Ord})$ is in the stationary set witnessing approachability. The condition cannot be weakened further since the existence of such $\mathcal{R}$ follows from the existence of stationary $\mathcal{S}$ and $\mathcal{T}$ which are appropriate for K of size θ. To see this, let f enumerate K. Then the set $\{M \cap W \cap \text{Ord} \mid M \in \mathcal{S},\ W \in \mathcal{T},\ M \text{ and } W \text{ are elementary with respect to } f, \text{ and } W \in M\}$ is stationary in $\mathcal{P}_{<\omega_1}(\theta)$, and approachable on the stationary set $\{\sup(W \cap \text{Ord}) \mid W \in \mathcal{T} \text{ and } W \text{ is elementary with respect to } f\}$.

Remark 5.4. If $\delta < \theta \to \delta^\omega < \theta$ then a thin stationary set on $\mathcal{P}_{<\omega_1}(\theta)$ exists trivially. Indeed $\mathcal{P}_{<\omega}(\theta)$ itself is such a set.

Let $\mathbb{P}^{\text{dec}}$ be the *decorated* sequence poset associated to ω, $\mathcal{S}$, $\mathcal{T}$, and K. By Claim 4.2, $\mathbb{P}^{\text{dec}}$ is strongly proper for $\mathcal{S} \cup \mathcal{T}$.

Claim 5.5. *Forcing with $\mathbb{P}^{\text{dec}}$ does not collapse ω_1 and θ, and does not collapse cardinals or change cofinalities above θ.*

Proof. The first two parts are immediate from strong properness and the stationarity of $\mathcal{S}$ and $\mathcal{T}$ using Remark 3.2, Remark 3.1, and Claim 3.3. Let us only note that if $Q \prec H(\theta)$ and $\mathbb{P}^{\text{dec}}$ is strongly proper for $Q \cap K$, then it is also strongly proper for Q, as strong properness for Q depends only on $Q \cap \mathbb{P}^{\text{dec}}$, and $\mathbb{P}^{\text{dec}} \subseteq K$.

The third part is immediate from the fact that $|\mathbb{P}^{\text{dec}}| = |K| = \theta$. □

Claim 5.6. *Let G be generic for $\mathbb{P}^{\text{dec}}$ over V. Let $C = \{\sup(N \cap \text{Ord}) \mid N \in \bigcup G\}$. Then C is a club subset of U.*

Proof. That $C \subseteq U$ is immediate from the requirements that $\sup(W \cap \text{Ord}) \in U$ and $\sup(M \cap \text{Ord}) \in U$ in the definitions of $\mathcal{S}$ and $\mathcal{T}$. Using Corollary 2.40, the fact that $\mathcal{S} \cup \mathcal{T}$ is unbounded in K implies that for every $\alpha < \theta$, the set of conditions forcing an ordinal above α into C is dense. It follows that C is unbounded in θ. We prove that C is closed.

Let $\alpha < \theta$, and let $\langle s, f \rangle \in \mathbb{P}^{\text{dec}}$ force that $\alpha \notin C$. It is enough to show that $\langle s, f \rangle$ can be extended to a condition forcing that α is not a limit point of C. Let Q be the first node in s so that $\sup(Q \cap \text{Ord}) \geq \alpha$. (Extending s if necessary we may assume there is such a node, using the unboundedness of C.) Let M be the largest node of s below Q. Since $\langle s, f \rangle$ forces α outside C, $\sup(Q \cap \text{Ord}) > \alpha$. Hence there is $\xi \geq \alpha$ which belongs to Q. Let f' be the function on s that differs from f only on M, with $f'(M) = f(M) \cup \{\xi\}$. Then $\langle s, f' \rangle \in \mathbb{P}^{\text{dec}}$ and $\langle s, f' \rangle \leq \langle s, f \rangle$. Moreover, in every extension $\langle t, g\rangle$ of $\langle s, f'\rangle$ the successor of M is a node M^* with $\xi \in M^*$, hence $\sup(M^* \cap \mathrm{Ord}) > \alpha$. It follows that $\langle s, f'\rangle$ forces that C has no elements between $\sup(M \cap \mathrm{Ord})$ and α, and in particular α is not a limit point of C. □

Claims 5.5 and 5.6 show that forcing with $\mathbb{P}^{\mathrm{dec}}$, which is a poset of finite conditions, adds a club through U, without collapsing ω_1, θ, or any cardinals above θ. (θ itself, if greater than ω_2, becomes ω_2 of the extension, by arguments similar to the arguments on collapsing in Subsection 5.1.)

5.3. **Collapsing without adding branches.** Let κ be a cardinal and suppose that $\kappa^{<\kappa} = \kappa$. Let $\theta > \kappa^+$ be regular, with $|H(\theta)| = \theta$, and such that $\delta < \theta \to \delta^\kappa < \theta$.

Set $\mathcal{T}$ to consist of all transitive $W \prec H(\theta)$ so that $|W| < \theta$ and W is κ closed. Set $\mathcal{S}$ to consist of all $M \prec H(\theta)$ of size κ which are closed under sequences of length $< \kappa$. Then $\mathcal{S}$ and $\mathcal{T}$ are appropriate for κ, κ^+, and θ. Our assumptions on κ and θ imply that $\mathcal{S}$ and $\mathcal{T}$ are both stationary.

Let $\mathbb{S}_{\kappa,\theta}$ be the poset $\mathbb{P}_{\kappa,\mathcal{S},\mathcal{T},H(\theta)}$. By arguments similar to the collapsing arguments in Subsection 5.1, forcing with $\mathbb{S}_{\kappa,\theta}$ collapses all cardinals between κ^+ and θ to κ^+. By Claim 2.8, $\mathbb{S}_{\kappa,\theta}$ is $<\kappa$ closed. It follows that forcing with $\mathbb{S}_{\kappa,\theta}$ does not collapse κ or any smaller cardinals, and indeed does not add sequences of ordinals of length $< \kappa$. By Claim 4.1, $\mathbb{S}_{\kappa,\theta}$ is strongly proper for $\mathcal{S} \cup \mathcal{T}$. Since $\mathcal{S}$ and $\mathcal{T}$ are both stationary, it follows using Remark 3.2, Remark 3.1, and Claim 3.3 that forcing with $\mathbb{S}_{\kappa,\theta}$ does not collapse κ^+ and θ. Since $\mathbb{S}_{\kappa,\theta}$ has size $|H(\theta)| = \theta$, it does not collapse cardinals or change cofinalities above θ.

Altogether then, $\mathbb{S}_{\kappa,\theta}$ has the same collapsing effects as $\mathrm{Col}(\kappa^+, {<}\theta)$.

The collapse poset $\mathrm{Col}(\kappa^+, {<}\theta)$ can be split into forcing first with an initial segment $\mathrm{Col}(\kappa^+, \gamma)$ and then with a tail-end $\mathrm{Col}(\kappa^+, [\gamma, \theta))$. A similar splitting is possible with $\mathbb{S}_{\kappa,\theta}$. For any $W \in \mathcal{T}$, forcing with $\mathbb{S}_{\kappa,\theta}$ below the condition $\{W\}$ is, by strong properness, the same as forcing first with $\mathbb{S}_{\kappa,\theta} \cap W$, and then with the factor poset $\mathbb{S}_{\kappa,\theta}/(\mathbb{S}_{\kappa,\theta} \cap W)$. (Recall that the factor poset consists of all conditions in $s \in \mathbb{S}_{\kappa,\theta}$ so that $W \in s$ and $\mathrm{res}_W(s)$ belongs to the generic added by $\mathbb{S}_{\kappa,\theta} \cap W$.)

However, in contrast with $\mathrm{Col}(\kappa^+, {<}\theta)$, $\mathbb{S}_{\kappa,\theta}$ does *not* add branches of length κ^+ or greater through trees in V. More precisely, if T is a tree in V, G is generic for $\mathbb{S}_{\kappa,\theta}$ over V, and f is a branch of T that belongs to $V[G]$ and has length $\geq \kappa^+$, then $f \in V$. This follows from strong properness for $\mathcal{S}$ and the stationarity of $\mathcal{S}$, by Lemma 3.7. Furthermore, the factor posets $\mathbb{S}_{\kappa,\theta}/(\mathbb{S}_{\kappa,\theta} \cap W)$ too do not add branches of length $\geq \kappa^+$, to trees in the extension of V by $\mathbb{S}_{\kappa,\theta} \cap W$. This again follows by Lemma 3.7, this time using strong properness of the factor poset, given by Claims 4.3 and 4.4.

(Note that the ordinary collapse poset, $\mathrm{Col}(\kappa^+, {<}\theta)$, *does* add new branches of length κ^+ through trees in V. Indeed every segment $\mathrm{Col}(\kappa^+, \gamma)$ of the poset adds such a branch, for example through the tree of functions into γ with domain $< \kappa^+$, ordered by extension.)

This property of $\mathbb{S}_{\kappa,\theta}$ makes it a useful substitute for the ordinary collapse in arguments that deal with the tree property.

One example involves models of the tree property at successors of singular cardinals together with failure of the Singular Cardinals Hypothesis. Neeman [6] produced such models for large singular cardinals. Sinapova [8] then produced such models for $\aleph_{\omega^2}$. Passing from Neeman's construction to Sinapova's requires several collapses, that ultimately will turn the large cardinal at the starting point to $\aleph_{\omega^2}$. The initial impediment for combining such collapses with Neeman's construction is in adapting Lemma 3.2 of [6], which absorbs a branch through a tree from a generic extension of a model to the model itself. Later impediments are of a similar nature.

Sinapova overcame these impediments in [8] using a very clever argument on narrow systems, and systems of branches through them. If one were to replace the ordinary collapses used for this by the model sequence collapses defined above, then less clever arguments would suffice. (The parallel of Lemma 3.2 in [6] for settings with the collapse is a consequence of the "no new branches" property for factors of the model sequence collapses. Similar arguments, and other arguments using closure of the model sequence collapses, can handle later issues in the adaptation.)

5.4. **A particular model of PFA (after Friedman [2]).** Friedman [2] obtains a model of PFA with an inner model that is correct about ω_2, but does not contain all reals (and similarly of BPFA, from smaller large cardinal assumptions). The proof uses the side conditions of Friedman [1]. We briefly sketch an adaptation of the proof to use the two-type model sequences.

(One can also obtain such a model from the forcing construction in Section 6. With the Laver function F changed to a homogenized version F', for example letting $F'(\alpha)$ name the sum of all proper posets in $V_{f(\alpha)}[G \cap H(\alpha)]$, the inner model generated by the side conditions in G of Section 6 does not contain all reals.)

Let θ be a supercompact cardinal. Let $K = H(\theta)$. Let $\mathcal{S}$ consist of all countable elementary substructures of K. Let $Z = \{\alpha < \theta \mid H(\alpha) \prec H(\theta)$ and $H(\alpha)$ is countably closed$\}$. Let $\mathcal{T} = \{H(\alpha) \mid \alpha \in Z\}$. $\mathcal{S}$ and $\mathcal{T}$ are appropriate for ω, ω_1, and K. Let $\mathbb{P} = \mathbb{P}_{\omega,\mathcal{S},\mathcal{T},K}$. Let G be generic for $\mathbb{P}$ over V. As in previous subsections, θ is turned to ω_2 in the extension $V[G]$, and apart from this no cardinals are collapsed.

We intend to work with the set $\{\alpha \mid H(\alpha) \in \bigcup G\}$. As in previous subsections the set is club in θ relative to Z, and this is all we need for the argument on PFA. But in fact, in this case, the set is outright equal to Z: (This holds in general when $\mathcal{T}$ consists of exactly the countably closed $H(\alpha)$ which are elementary in some expansion of $H(\theta)$.)

Claim 5.7. *For each $\alpha \in Z$, $H(\alpha)$ is a node in $\bigcup G$.*

Proof. It is enough to check that every condition s can be extended to t with $H(\alpha) \in t$. Fix s. If $s \subseteq H(\alpha)$ then $s \cup \{H(\alpha)\}$ is a condition. Suppose then that $s \not\subseteq H(\alpha)$, and let M be the first node of s outside $H(\alpha)$. If $M = H(\alpha)$, take $t = s$. If $\alpha \in M$, then $\text{res}_M(s) \cup \{H(\alpha)\}$ is a condition that belongs to M and extends $\text{res}_M(s)$. By Corollary 2.31 it is compatible with s, giving an extension of s which includes $H(\alpha)$.

Suppose finally that M is above $H(\alpha)$ and $\alpha \notin M$. In particular M is of countable type. Let α^* be the first ordinal in M above α. Such an ordinal exists since $M \not\subseteq H(\alpha)$. The minimality of α^* implies that $M \cap H(\alpha^*) = M \cap H(\alpha)$. From this, the elementarity of M, and the elementarity of $H(\alpha)$, it follows that $H(\alpha^*) \prec K$. (Suppose not. Then there are $a_1, \dots, a_n \in H(\alpha^*)$ and a formula φ so that in K there exists some y so that $K \models \varphi(a_1, \dots, a_n, y)$, but there is no such y in $H(\alpha^*)$. By elementarity of M, $a_1, \dots, a_n$ can be found in M. They then belong to $H(\alpha)$. By elementarity of $H(\alpha)$, y can be found in $H(\alpha)$, hence also in $H(\alpha^*)$.) It also follows that α^* has uncountable cofinality. So $H(\alpha^*) \in \mathcal{T}$. The argument of the previous paragraph shows that there is an extension s^* of s with $H(\alpha^*) \in s^*$. The first node of s^* above α has smaller von Neumann rank than the first node of s above α, and by induction it follows that there is an extension of s^*, hence also of s, that contains $H(\alpha)$. □

We now briefly sketch Friedman's argument to force over $V[G]$ to obtain PFA, without collapsing any cardinals. The forcing will add reals, and so in the resulting model of PFA, $V[G]$ will be a submodel that is correct about ω_2 and does not include all reals.

Let $F \in V$ be a Laver function for the supercompact cardinal θ. Let $\theta^* > \theta$ and let $\mathcal{S}^*$ be the set of countable $M \prec H(\theta^*)$ with $\theta, F \in M$. For each $\alpha \in Z \cup \{\theta\}$, let $\mathcal{S}^*_\alpha = \{M[G \cap H(\alpha)] \mid M \in \mathcal{S}^*, \alpha \in M, \text{ and } M \cap H(\alpha) \in G\}$. Let $\mathbb{Q}_\alpha$ be the factor poset for adding G over $V[G \cap H(\alpha)]$. By Claim 4.4, $\mathbb{Q}_\alpha$ is strongly proper for $\mathcal{S}^*_\alpha$.

A *diagonal* iteration over $V[G]$ is an iteration $\mathbb{R}$ of length θ with the additional restriction that $r \restriction \alpha \in V[G \cap H(\alpha)]$ for every $r \in \mathbb{R}$ and every $\alpha \in Z$. (All such α have uncountable cofinality.) The notion is due to Friedman [2].

Let $\bar{Z} \subseteq Z$ be the set of inaccessible α closed under F. Working in $V[G]$, let $\mathbb{R}$ be the length θ countable support diagonal iteration of the posets given by $F(\alpha)$, for $\alpha \in \bar{Z}$ so that, in $\mathbb{R}\restriction\alpha$ over $V[G\restriction\alpha]$, it is forced that $F(\alpha)$ is proper with respect to $\mathcal{S}^*_\alpha$.

The use of diagonal iteration in the definition of $\mathbb{R}$, together with the fact that the individual posets being iterated all have size $< \theta$, implies that $\mathbb{R}$ is θ-c.c. over $V[G]$, and therefore no cardinals $\geq \omega_2^{V[G]} = \theta$ are collapsed by $\mathbb{R}$.

Recall that $\mathcal{S}^*_\alpha$ consists of structures of the form $M[G \cap H(\alpha)]$. By definition $\mathbb{R}$ only uses $F(\alpha)$ if it is forced to be proper for these structures. The next claim shows that $F(\alpha)$ is then forced to be proper also for the extended structures $M[G]$.

Claim 5.8. *Let α be as above, and let $M \prec H(\theta^*)$ be countable with $\alpha, F \in M$. Then it is forced, in $\mathbb{R}\restriction\alpha$ over $V[G]$, that $F(\alpha)$ is proper for $M[G]$.*

Proof. This follows by condition (3) in Claim 3.8, using the fact that $\mathbb{Q}_\alpha$, which leads from $V[G \cap H(\alpha)]$ to $V[G]$, is strongly proper. □

The poset $\mathbb{R}$ is, by the last claim, a countable support diagonal iteration of proper posets in $V[G]$, with properness restricted to elementary substructures in $\mathcal{S}^*_\theta$. Were it not for the use of a *diagonal* iteration, preservation of properness under countable support iterations would directly imply that $\mathbb{R}$ is itself proper for these substructures. With the use of a diagonal iteration, at each limit stage γ, the inductive step of the preservation argument only yields that $\mathbb{R}\restriction\gamma$ is proper for structures $M[G \cap H(\alpha^*)]$ in $V[G \cap H(\alpha^*)]$, where α^* is the first element of Z that is $\geq \gamma$. Fortunately, by condition (3) of Claim 3.8 this implies properness of $\mathbb{R}\restriction\gamma$ for structures $M[G]$ in $V[G]$, allowing the inductive preservation argument to proceed, and showing ultimately that $\mathbb{R}$ is proper in $V[G]$ for structures in $\mathcal{S}^*_\theta$. In particular then $\mathbb{R}$ does not collapse ω_1 over $V[G]$.

Let I be generic for $\mathbb{R}$ over $V[G]$. We have seen above that $V[G][I]$ and $V[G]$ have the same cardinals. It is clear that in $V[G][I]$ there are more reals. Standard arguments, directly from the definition of $\mathbb{R}$, using the supercompactness of θ and the fact that F is a Laver function, show that PFA holds in $V[G][I]$.

6. The consistency of PFA using finite supports

Our main application of the sequence models poset is a new proof of the consistency of the Proper Forcing Axiom, that does *not* use preservation of properness under countable support iterations. Instead, we will work with a finite support iteration, and use side conditions from the sequence poset to enforce properness.

Let θ be a supercompact cardinal. Let $F\colon \theta \to H(\theta)$ be a Laver function. Set $K = H(\theta)$. Let Z be the set of $\alpha < \theta$ so that $(H(\alpha); F\restriction\alpha)$ is elementary in $(H(\theta); F)$. For each $\alpha \in Z$, let $f(\alpha)$ be the least cardinal so that $F(\alpha) \in H(f(\alpha))$. Note that $f(\alpha)$ is smaller than the next element of Z above α. Set $\mathcal{T}$ to be the set of models $W = H(\alpha)$ for $\alpha \in Z$ so that $H(\alpha)$ is countably closed (equivalently, α has uncountable cofinality). Set $\mathcal{S}$ to be the set of countable models which are elementary in $(H(\theta); F)$. $\mathcal{S}$ and $\mathcal{T}$ are then stationary and appropriate for ω, ω_1, and $K = H(\theta)$.

We describe a poset $\mathbb{A}$ that forces PFA. Conditions in the poset have two components. One corresponds to a finite support iteration of proper posets given by the Laver function F. This is similar to the standard consistency proof for PFA, except that finite supports are used instead of countable. The other component consists simply of conditions in the model sequence poset $\mathbb{P}_{\omega,\mathcal{S},\mathcal{T},H(\theta)}$. We will connect the two components by restricting conditions in the first to master conditions for models in the second.

Definition 6.1. Conditions in the poset $\mathbb{A}$ are pairs $\langle s, p\rangle$ so that:

1. s is a condition in $\mathbb{P}_{\omega,\mathcal{S},\mathcal{T},H(\theta)}$. In other words it is a finite, $\in$-increasing sequence of models from $\mathcal{S} \cup \mathcal{T}$, closed under intersections.
2. p is a partial function on θ, with domain contained in the (finite) set $\{\alpha < \theta \mid H(\alpha) \in s$ and $\Vdash_{\mathbb{A}\cap H(\alpha)}$ "$F(\alpha)$ is a proper poset"$\}$.
3. For $\alpha \in \mathrm{dom}(p)$, $p(\alpha) \in H(f(\alpha))$.
4. $\Vdash_{\mathbb{A}\cap H(\alpha)} p(\alpha) \in F(\alpha)$.
5. For each small node $M \in s$ so that $\alpha \in M$, $\langle s \cap H(\alpha), p\restriction\alpha\rangle \Vdash_{\mathbb{A}\cap H(\alpha)}$ "$p(\alpha)$ is a master condition for M".

The ordering on $\mathbb{A}$ is the following: $\langle s^*, p^*\rangle \leq \langle s, p\rangle$ iff $s^* \leq s$ and for every $\alpha \in \mathrm{dom}(p)$, $\langle s^* \cap H(\alpha), p^*\restriction\alpha\rangle \Vdash_{\mathbb{A}\cap H(\alpha)} p^*(\alpha) \leq_{F(\alpha)} p(\alpha)$.

The definition is an induction on $\alpha \in Z \cup \{\theta\}$, as knowledge of $\mathbb{A}\cap H(\alpha)$ is needed to evaluate the conditions on $p(\alpha)$. Conditions (2)–(4) are the standard conditions in an iteration of proper posets given by the Laver function F. Condition (5) connects this iteration with the side conditions given by $\mathbb{P}_{\omega,\mathcal{S},\mathcal{T},H(\theta)}$.

Remark 6.2. If $\alpha \in Z$, then $\mathbb{A} \cap H(\alpha)$ is definable in $(H(\theta); F)$ from α. This is because $F\restriction\alpha$ is definable, and so are $\mathcal{S}\cap H(\alpha)$ and $\mathcal{T}\cap H(\alpha)$. (The parts of $\mathcal{S}$ and $\mathcal{T}$ below α can be defined using elementarity in $(H(\alpha); F\restriction\alpha)$ instead of elementarity in $(H(\theta); F)$, and this can be done inside $(H(\theta); F)$ with α as parameter.) In particular it follows that $\mathbb{A} \cap H(\alpha) \in M$ for every $M \in \mathcal{S}$ with $\alpha \in M$.

Remark 6.3. Condition (5) involves some abuse of notation, since M is not an elementary substructure of the extension of $H(\theta)$ by $\mathbb{A} \cap H(\alpha)$. What we mean precisely is that $\langle s \cap H(\alpha), p\restriction\alpha\rangle \Vdash_{\mathbb{A}\cap H(\alpha)}$ "$p(\alpha)$ is a master condition for $M[\dot{G}_\alpha]$" where $\dot{G}_\alpha$ names the $\mathbb{A} \cap H(\alpha)$ generic.

Remark 6.4. Condition (5) holds for α and M iff it holds for α and $M \cap H(\gamma)$, whenever $\gamma \in Z \cup \{\theta\}$ is larger than α. The reason is that $F(\alpha) \in H(f(\alpha)) \subseteq H(\gamma)$, and so being a master condition for M in the interpretation of $F(\alpha)$ is equivalent to being a master condition for $M \cap H(\gamma)$.

Claim 6.5. *Condition (5) in Definition 6.1 is equivalent to the same condition with the restriction to M so that $\alpha \in M$ replaced by restriction to M which occur above $H(\alpha)$ in s and so that there are no transitive nodes in s between $H(\alpha)$ and M.*

Proof. It is clear that the original condition implies the version in the claim, since by Claim 2.10, $\alpha \in M$ for every M which occurs above $H(\alpha)$ in s with no transitive nodes between $H(\alpha)$ and M. For the converse, let W^* be the first transitive node of s above $H(\alpha)$. By Remark 6.4, being a master condition for M in $F(\alpha)$ is equivalent to being a master condition for $M \cap W^*$. From this, the closure of s under intersection with W^*, and the fact that if $\alpha \in M$ then $\alpha \in M \cap W^*$ and hence $M \cap W^*$ occurs above $H(\alpha)$, it follows that being a master condition for all $M \in s$ with $\alpha \in M$ is a consequence of being a master condition for all $M \in s$ between $H(\alpha)$ and W^*. □

For $\beta \in Z \cup \{\theta\}$, let $\mathbb{A}_\beta$ denote the poset given by Definition 6.1 with the added restriction that $\mathrm{dom}(p) \subseteq \beta$. For $\beta \in Z$ this poset is related to $\mathbb{A} \cap H(\beta)$, but the two are not the same, since the latter restricts the side conditions to belong to $H(\beta)$, while the former does not. $\mathbb{A}_\theta$ is equal to $\mathbb{A}$.

Claim 6.6. *Let $\alpha < \beta$ belong to $Z \cup \{\theta\}$. Let $\langle s, p\rangle \in \mathbb{A}_\beta$ with $W = H(\alpha) \in s$. Let $\langle t, q\rangle \in \mathbb{A} \cap H(\alpha)$ extend $\langle s \cap H(\alpha), p\restriction\alpha\rangle$. Then $\langle s, p\rangle$ and $\langle t, q\rangle$ are compatible in $\mathbb{A}_\beta$. Moreover this is witnessed by the condition $\langle u, h\rangle$ with $u = t \cup s$, $h\restriction\alpha = q$, and $h\restriction[\alpha, \beta) = p\restriction[\alpha, \beta)$.*

Proof. $s \cap H(\alpha)$ is equal to $\mathrm{res}_W(s)$, so by Corollary 2.31, t and s are directly compatible in $\mathbb{P}_{\omega,\mathcal{S},\mathcal{T},H(\theta)}$. Let u witness this. u is the closure of $s \cup t$ under intersections, and since t extends an initial segment of s, u is simply equal to $s \cup t$.

Let $h = q \cup p\restriction[\alpha, \beta)$. It is easy, with some uses of Claim 6.5, to check that $\langle u, h\rangle$ is a condition in $\mathbb{A}_\beta$ and that it extends both $\langle t, q\rangle$ and $\langle s, p\rangle$. □

Lemma 6.7. *Let β belong to $Z \cup \{\theta\}$.*

(1) *Let $\langle s, p\rangle \in \mathbb{A}_\beta$ and let $W = H(\alpha)$ be a transitive node in s. Then $\langle s, p\rangle$ is a strong master condition for W in $\mathbb{A}_\beta$.*
(2) *Let $\langle s, p\rangle \in \mathbb{A}_\beta$, let $W \in \mathcal{T}$, and suppose $\langle s, p\rangle \in W$. Then $\langle s \cup \{W\}, p\rangle$ is a condition in $\mathbb{A}_\beta$. (It trivially extends $\langle s, p\rangle$.)*
(3) *$\mathbb{A}_\beta$ is strongly proper for $\mathcal{T}$.*

Proof. Condition (2) is immediate from the definitions, using Corollary 2.32 which implies that $s \cup \{W\} \in \mathbb{P}_{\omega,\mathcal{S},\mathcal{T},H(\theta)}$. Condition (3) is immediate from (1) and (2). Condition (1) in case $\alpha < \beta$ follows from Claim 6.6: Let D be dense in $\mathbb{A}_\beta \cap H(\alpha)$. It is enough to prove that every $\langle s, p\rangle \in \mathbb{A}_\beta$ with $H(\alpha) \in s$ is compatible with a condition in D. Fix $\langle s, p\rangle$. By density of D in $\mathbb{A}_\beta \cap H(\alpha)$, there is $\langle t, q\rangle \in D$ which extends $\langle s \cap H(\alpha), p\restriction\alpha\rangle$. By Claim 6.6, $\langle s, p\rangle$ and $\langle t, q\rangle$ are compatible, and the condition $\langle u, h\rangle$ given by the claim to witness this belongs to $\mathbb{A}_\beta$. In case $\alpha \geq \beta$, condition (1) is proved similarly with a direct use of Corollary 2.31 instead of Claim 6.6. □

Claim 6.8. *Let $\mathbb{Q}$ be proper. Let κ be large enough that $\mathbb{Q} \in H(\delta)$ for some $\delta < \kappa$. Let $M_0 \in M_1 \in \dots M_{l-1}$ be an increasing sequence of countable elementary* *substructures of $H(\kappa)$ with $\mathbb{Q} \in M_i$. Suppose that $k < l$ and $q \in M_k$ is a master condition for $M_0, \dots, M_{k-1}$. Then there is $q^* \leq q$ which is a master condition for $M_0, \dots, M_{l-1}$.*

Proof. By standard arguments using the fact that κ is larger than the least δ with $\mathbb{Q} \in H(\delta)$, for every countable $M \prec H(\kappa)$, and every $r \in M$, there is $r^* \leq r$ which is a master condition for M. The claim follows by successive applications of this fact, setting $r_k = q \in M_k$, obtaining for each $i \geq k$ a master condition $r_{i+1} \leq r_i$ for M_i in M_{i+1}, and taking $q^* = r_l$. r_{i+1} can be found in M_{i+1} by elementarity. □

Claim 6.9. *Let $\langle s, p\rangle \in \mathbb{A}$. Suppose $H(\alpha) \in s$, and $\alpha \notin \mathrm{dom}(p)$. Let M be a small node of s and let $\langle t, q\rangle \in \mathbb{A} \cap M$. Suppose that $\alpha \in \mathrm{dom}(q)$ and $\langle s, p\rangle \leq \langle t, q{\restriction}\theta - \{\alpha\}\rangle$. Suppose further that $\mathrm{res}_M(s) - H(\alpha) \subseteq t$. Then there is p' extending p with $\mathrm{dom}(p') = \mathrm{dom}(p) \cup \{\alpha\}$ and so that $\langle s, p'\rangle$ is a condition in $\mathbb{A}$ extending $\langle t, q\rangle$.*

Proof. Let W be the first transitive node of s above $H(\alpha)$ if there is one, and $H(\theta)$ otherwise. Let $N_0 \in \cdots \in N_{l-1}$ list the small nodes of s above $H(\alpha)$ and below W. Since $q \in M$ and $\alpha \in \mathrm{dom}(q)$, $\alpha \in M$. This implies that $M \cap W$ appears on the list $N_0, \dots, N_{l-1}$. Fix k so that $M \cap W = N_k$. Since $\langle t, q\rangle \in \mathbb{A}$ and $\alpha \in \mathrm{dom}(q)$, $\Vdash_{\mathbb{A} \cap H(\alpha)}$ "$F(\alpha)$ is a proper poset and $q(\alpha) \in F(\alpha)$". Moreover for every small node $N \in t$ with $\alpha \in N$, it is forced by $\langle t \cap H(\alpha), q{\restriction}\alpha\rangle$, and hence also by $\langle s \cap H(\alpha), p{\restriction}\alpha\rangle$, that $q(\alpha)$ is a master condition for N. In particular, since $t \supseteq \mathrm{res}_M(s) - H(\alpha) \supseteq \{N_0, \dots, N_{k-1}\}$, this holds for $N_0, \dots, N_{k-1}$.

By Claim 6.8 there is a name $\dot{q}^*$ which is forced by $\langle s \cap H(\alpha), p{\restriction}\alpha\rangle$ to be a master condition for all models $N_0, \dots, N_{l-1}$, and to extend $q(\alpha)$. $\dot{q}^*$ can be picked in $H(f(\alpha))$.

Set $p' = p \cup \{\alpha \mapsto \dot{q}^*\}$. It is easy to check that $\langle s, p'\rangle$ is a condition in $\mathbb{A}$, and that it extends $\langle t, q\rangle$. □

Claim 6.10. *Let $\langle s, p\rangle, \langle t, q\rangle \in \mathbb{A}$. Let M be a small node of s and suppose that $\langle t, q\rangle \in M$. Suppose that for some $\delta < \theta$, $\langle s, p\rangle$ extends $\langle t, q{\restriction}\delta\rangle$ and $\mathrm{dom}(q) - \delta$ is disjoint from $\mathrm{dom}(p)$. Suppose further that $\mathrm{res}_M(s) - H(\delta) \subseteq t$. Then there is p' extending p so that $\mathrm{dom}(p') = \mathrm{dom}(p) \cup (\mathrm{dom}(q) - \delta)$ and so that $\langle s, p'\rangle$ is a condition in $\mathbb{A}$ extending $\langle t, q\rangle$.*

Proof. Immediate by successive applications of Claim 6.9, going over all $\alpha \geq \delta$ in $\mathrm{dom}(q)$ in increasing order. □

Lemma 6.11. *Let $\beta \in Z \cup \{\theta\}$. Let $\langle s, p\rangle$ be a condition in $\mathbb{A}_\beta$. Let $\theta^* > \theta$ and let $M^* \prec H(\theta^*)$ be countable with $F, \beta \in M^*$. Let $M = M^* \cap H(\theta)$ and suppose that $M \in s$. Then:*

(1) *for every $D \in M^*$ which is dense in $\mathbb{A}_\beta$, there is $\langle t, q\rangle \in D \cap M^*$ which is compatible with $\langle s, p\rangle$. Moreover there is $\langle s^*, p^*\rangle \in \mathbb{A}_\beta$ extending both $\langle s, p\rangle$ and $\langle t, q\rangle$, so that $\mathrm{res}_M(s^*) - H(\beta) \subseteq t$, and all small nodes of s^* above β and outside M are either nodes of s or of the form $N' \cap W$ where N' is a small node of s.*
(2) *$\langle s, p\rangle$ is a master condition for M^* in $\mathbb{A}_\beta$.*

Proof. Condition (2) is immediate from condition (1). We prove condition (1) by induction on β. If β is the first element of Z then $\mathbb{A}_\beta$ is isomorphic to $\mathbb{P}_{\omega,\mathcal{S},\mathcal{T},H(\theta)}$, and the condition holds by Corollary 2.31. We handle the limit and successor cases below.

Suppose first that β is a limit point of $Z \cup \{\theta\}$. Let $\bar{\beta} = \sup(\beta \cap M^*)$. This may be β itself if $\mathrm{cof}(\beta) = \omega$. Let $\delta < \bar{\beta}$ in $Z \cap M^*$ be large enough that $\mathrm{dom}(p) \cap \bar{\beta} \subseteq \delta$. Such δ exists since $\mathrm{dom}(p)$ is finite, while $\bar{\beta}$ is a limit point of $Z \cap M$.

Let E be the set of conditions $\langle t, \bar{q} \rangle \in \mathbb{A}_\delta$ which extend to conditions $\langle t, q \rangle \in D$ with $q \restriction \delta = \bar{q}$. E is dense in $\mathbb{A}_\delta$, and belongs to M^*. By induction there is $\langle t, \bar{q} \rangle \in E \cap M^*$ which is compatible with $\langle s, p \restriction \delta \rangle$. Moreover there is $\langle s^*, p_1 \rangle \in \mathbb{A}_\delta$ which extends both $\langle t, \bar{q} \rangle$ and $\langle s, p \restriction \delta \rangle = \langle s, p \restriction \bar{\beta} \rangle$, with $\mathrm{res}_M(s^*) - H(\delta) \subseteq t$, and so that all small nodes of s^* above δ and outside M are either nodes of s or of the form $N' \cap W$ where N' is a small node of s.

Let $\langle t, q \rangle \in D$ witness that $\langle t, \bar{q} \rangle \in E$. Using elementarity of M^*, pick q inside M^*. By Claim 6.10, there is p_2 extending p_1 so that $\mathrm{dom}(p_2) = \mathrm{dom}(p_1) \cup (\mathrm{dom}(q) - \delta)$, and so that $\langle s^*, p_2 \rangle$ extends $\langle t, q \rangle$.

Set $p^* = p_2 \cup p \restriction [\bar{\beta}, \beta)$. It is enough now to prove that $\langle s^*, p^* \rangle$ is a condition in $\mathbb{A}_\beta$. (By construction it extends both $\langle t, q \rangle$ and $\langle s, p \rangle$.) Fix $\alpha \in \mathrm{dom}(p) \restriction [\bar{\beta}, \beta)$, and fix a small node $N \in s^*$ with $\alpha \in N$. We must check that $p^*(\alpha) = p(\alpha)$ is forced by $\langle s^* \cap H(\alpha), p^* \restriction \alpha \rangle$ to be a master condition for N. It is enough to check that this is forced by $\langle s \cap H(\alpha), p \restriction \alpha \rangle$.

If N belongs to M then $N \subseteq M$, and since $\alpha \in [\bar{\beta}, \beta)$ this contradicts the fact that $\alpha \in N$. So N must be outside M. It follows by properties of s^* above that N is either a node of s or an intersection $N' \cap W$ where N' is a small node in s. If N is a node of s then $\langle s \cap H(\alpha), p \restriction \alpha \rangle$ forces $p(\alpha)$ to be a master condition for N, because $\langle s, p \rangle$ is a condition in $\mathbb{A}_\beta$. The same is true in case N has the form $N' \cap W$ for $N' \in s$, using Remark 6.4. This completes the proof of the limit case of the lemma.

Suppose next that β is a successor point of Z. Let α be the predecessor of β in Z. By elementarity of M^*, $\alpha \in M^*$. For expository simplicity, fix G which is generic for $\mathbb{A}_\alpha$ over V, with $\langle s, p \restriction \alpha \rangle \in G$. By induction $\langle s, p \restriction \alpha \rangle$ is a master condition for M^* in $\mathbb{A}_\alpha$, so $M^*[G] \prec H(\theta^*)[G]$ and $M^*[G] \cap V = M^*$.

Suppose that $H(\alpha)$ is a node in s. (We will handle the case that $H(\alpha) \notin s$ later.) By Lemma 6.7, $G \cap H(\alpha)$ is generic for $\mathbb{A} \cap H(\alpha)$ over V.

If it is not forced in $\mathbb{A} \cap H(\alpha)$ that $F(\alpha)$ is a proper poset, then $\mathbb{A}_\beta$ is equal to $\mathbb{A}_\alpha$ and the lemma at β follows immediately by induction. Suppose then that $F(\alpha)$ is forced to be a proper poset, and let $\mathbb{Q} = F(\alpha)[G \cap H(\alpha)]$. $\mathbb{Q}$ belongs to $M^*[G \cap H(\alpha)] \subseteq M^*[G]$.

Let W be the first transitive node of s above $H(\alpha)$ if there is one, and $H(\theta)$ otherwise. Let $N_0 \in N_1 \in \dots N_{l-1}$ list the small nodes of s between $H(\alpha)$ and W, in increasing order. Let $k < l$ be such that $N_k = M \cap W$. Note that the nodes in $\{N_0, \dots, N_{l-1}\}$ that belong to $\mathrm{res}_M(s)$ are exactly the nodes $N_0, \dots, N_{k-1}$.

Fix $D \in M^*$ which is dense in $\mathbb{A}_\beta$. Let E be the set of $u \in \mathbb{Q}$ so that one of the following two conditions holds:

(i) No extension of u is a master condition for the models $N_i[G \cap H(\alpha)]$ for all $i < k$.

(ii) There exists $\langle t, q \rangle \in D$ with $\langle t \cap H(\alpha), q \restriction \alpha \rangle \in G$, $t \leq \mathrm{res}_M(s)$, and $q(\alpha)[G \cap H(\alpha)] = u$.

In the case of condition (ii), by Definition 6.1 and the fact that $t \leq \mathrm{res}_M(s)$, u is a master condition for $N_i[G \cap H(\alpha)]$ for all $i < k$.

All the parameters in the definition of E belong to $M^*[G \cap H(\alpha)]$, and therefore by elementarity so does E. The density of D in $\mathbb{A}_\beta$ implies that E is dense in $\mathbb{Q}$: Let $\bar{u} = \dot{\bar{u}}[G \cap H(\alpha)]$ be any condition in $\mathbb{Q}$ and suppose for contradiction it has no extension in E. Let $\langle a, h\rangle \in G \cap H(\alpha)$ force this. Extending $\bar{u}$ using failure of (i) we may assume that it is a master condition for $N_i[G \cap H(\alpha)]$ for all $i < k$. We may assume $\langle a, h\rangle$ forces this, and extending $\langle a, h\rangle$ further if needed we may assume also that $\mathrm{res}_M(s) \cap H(\alpha) \subseteq a$. Let $a^* = a \cup \mathrm{res}_M(s)$ (which below W is exactly $a \cup \{H(\alpha), N_0, \ldots, N_{k-1}\}$). Let $h^* = h \cup \{\alpha \mapsto \dot{\bar{u}}\}$. Then $\langle a^*, h^*\rangle$ is a condition in $\mathbb{A}_\beta$, and any $\langle t, q\rangle \in D$ which extends it provides a contradiction to the fact that $\dot{\bar{u}}$ is forced to have no extensions in E.

We proceed by using the density of E, and properness of $\mathbb{Q}$, to prove condition (1) of the lemma.

If $\alpha \in \mathrm{dom}(p)$, then by Definition 6.1, $p(\alpha)[G \cap H(\alpha)]$ is a master condition for $M^*[G \cap H(\alpha)]$ in $\mathbb{Q}$. It follows using the density of E that there is $u \in E \cap M^*[G \cap H(\alpha)]$, and u^* which extends both $p(\alpha)[G \cap H(\alpha)]$ and u. Since u^* extends $p(\alpha)[G \cap H(\alpha)]$, it is a master condition for $N_i[G \cap H(\alpha)]$ for all $i < l$, and equivalently for all small nodes N of s with $\alpha \in N$.

If $\alpha \notin \mathrm{dom}(p)$, then u and u^* with the same properties can be obtained as follows: First let $v \in \mathbb{Q}$ be a master condition for $N_i[G \cap H(\alpha)]$ for all $i < l$. This is possible using Claim 6.8. Then obtain u and u^* as in the previous paragraph, starting from v instead of $p(\alpha)[G \cap H(\alpha)]$.

Since $u^* \leq u$ is a master condition for each $N_i[G \cap H(\alpha)]$, the membership of u in E must hold through condition (ii) in the definition of E. Let $\langle t, q\rangle$ witness the condition. Using the elementarity of $M^*[G \cap H(\alpha)]$, pick $\langle t, q\rangle$ in this model. Since $M^*[G \cap H(\alpha)] \cap V = M^*$, $\langle t, q\rangle$ belongs to M^*.

$\langle t, q\rangle$ is a condition in $\mathbb{A}_\beta$, and in particular $q(\alpha)$ is forced by $\langle t \cap H(\alpha), q\restriction\alpha\rangle$ to be a master condition for all small nodes $N \in t$ with $\alpha \in N$. Since $q(\alpha)[G \cap H(\alpha)] = u$ and $\langle t \cap H(\alpha), q\restriction\alpha\rangle \in G$, u is a master condition for these nodes.

By Corollary 2.31, t and s are directly compatible. Let r witness this. By the same corollary, $\mathrm{res}_M(r) = t$ and hence the small nodes of r inside M are nodes of t. Again by the corollary, the small nodes of r outside M are either nodes of s or intersections of small nodes of s with transitive nodes of t. Since u is a master condition for small $N \in t$ with $\alpha \in N$, and $u^* \leq u$ is a master condition for small $N \in s$ with $\alpha \in N$, it follows using Remark 6.4 that u^* is a master condition for all $N \in r$ with $\alpha \in N$.

Let $\dot{u}, \dot{u}^* \in M^*$ name u and u^* respectively. Let $\langle a, h\rangle \in G \cap H(\alpha)$ be strong enough to force all the properties of u, u^*, and $\langle t, q\rangle$ proved above. Extending $\langle a, h\rangle$ we may assume it is stronger than both $\langle s \cap H(\alpha), p\restriction\alpha\rangle$ and $\langle t \cap H(\alpha), q\restriction\alpha\rangle$.

By Claim 2.33, $r \cap H(\alpha) = \mathrm{res}_{H(\alpha)}(r)$ is the closure of $\mathrm{res}_{H(\alpha)}(s) \cup \mathrm{res}_{H(\alpha)}(t) = (s \cup t) \cap H(\alpha)$. Since $\langle a, h\rangle$ extends both $\langle s \cap H(\alpha), p\restriction\alpha\rangle$ and $\langle t \cap H(\alpha), q\restriction\alpha\rangle$, and since a is closed under intersections, $r \cap H(\alpha) \subseteq a$. By Corollary 2.31, a and r are then directly compatible, and this is witnessed by $a \cup r$.

Let $s^* = a \cup r$. Let $p^* = h \cup \{\alpha \mapsto \dot{u}^*\}$. It is easy now to check that $\langle s^*, p^*\rangle$ is a condition in $\mathbb{A}_\beta$, and extends both $\langle s, p\rangle$ and $\langle t, q\rangle$. This, together with properties of s^* proved above, completes the proof of condition (1) in case that β is a successor, α is the predecessor of β in Z, and $H(\alpha) \in s$.

Suppose finally that β is a successor point of Z, α is the predecessor of β in Z, and $H(\alpha) \notin s$. If there is no extension s' of s with $H(\alpha) \in s'$, then α cannot belong to the domain of p' for any extension $\langle s', p'\rangle$ of $\langle s, p\rangle$, and so below $\langle s, p\rangle$ the poset $\mathbb{A}_\beta$ is the same as the poset $\mathbb{A}_\alpha$. Condition (1) of the lemma then holds trivially by induction. If there is an extension s' of s with $H(\alpha) \in s$, then there is such an extension where the only added nodes are transitive nodes and intersections of small nodes of s with transitive nodes. By the arguments above the lemma holds for such an extension, and this implies that it holds for s. □

Corollary 6.12. *Forcing with $\mathbb{A}$ preserves ω_1 and θ as cardinals. All cardinals between ω_1 and θ are collapsed to ω_1.*

Proof. Preservation of θ is immediate from the strong properness given by Lemma 6.7 and stationarity of $\mathcal{T}$. Preservation of ω_1 is immediate from the properness given by Lemma 6.11 for $\mathbb{A} = \mathbb{A}_\theta$ and stationarity of $\mathcal{S}$. The arguments in Subsection 5.1 applied to the sequence poset component of $\mathbb{A}$ show that all cardinals between ω_1 and θ are collapsed to ω_1. □

Lemma 6.13. *The extension of V by $\mathbb{A}$ satisfies the proper forcing axiom.*

Proof. This follows from the inclusion of a Laver iteration of proper posets in $\mathbb{A}$. The argument is standard but uses the strong properness given by Lemma 6.7 rather than the automatic strong properness that holds for iterations. We give a brief sketch.

Suppose the lemma fails and let $\langle a, h\rangle \in \mathbb{A}$ force that $\dot{\mathbb{Q}}$ and $\dot{D}_\xi$, $\xi < \omega_1$, provide a counterexample. Pick $\dot{\mathbb{Q}}$ so that it is outright forced to be proper. Let γ be large enough that $\dot{\mathbb{Q}}$ and $\dot{D}_\xi$ belong to $H(\gamma)$. Since F is a Laver function and θ is supercompact, there is $\bar\theta < \theta$, $\bar\gamma < \theta$, $\dot{\mathbb{P}} \in H(\bar\gamma)$, and $\dot{E}_\xi \in H(\bar\gamma)$ for $\xi < \omega_1$, so that $(H(\bar\gamma); F\restriction\bar\theta, \dot{\mathbb{P}}, \dot{E}_\xi)$ embeds elementarily into $(H(\gamma); F, \dot{\mathbb{Q}}, \dot{D}_\xi)$ via an embedding, π say, with critical point $\bar\theta$, and so that $F(\bar\theta) = \dot{\mathbb{P}}$.

$\bar\theta$ may be picked large enough that $\langle a, h\rangle \in \mathbb{A} \cap H(\bar\theta)$. Then by Lemma 6.7, $\langle a \cup \{H(\bar\theta)\}, h\rangle$ is a condition in $\mathbb{A}$, and a strong master condition for $H(\bar\theta)$.

Let G be generic for $\mathbb{A}$ with $\langle a\cup\{H(\bar\theta)\}, h\rangle \in G$. By strong properness, $G\cap H(\bar\theta)$ is generic for $\mathbb{A}\cap H(\bar\theta)$ over V. π extends trivially to an embedding of $H(\bar\gamma)[G\cap H(\bar\theta)]$ into $H(\gamma)[G]$.

It is easy to check, from the definition of $\mathbb{A}$, genericity of G, and the fact that $F(\bar\theta) = \dot{\mathbb{P}}$, that $\{p(\bar\theta)[G \cap H(\bar\theta)] \mid \langle s, p\rangle \in G \text{ and } \bar\theta \in \mathrm{dom}(p)\}$ is generic for $\dot{\mathbb{P}}[G \cap H(\bar\theta)]$ over $V[G \cap H(\bar\theta)]$. The image of this set under the extended embedding π is a filter on $\mathbb{Q} = \dot{\mathbb{Q}}[G]$ that meets each of the sets $D_\xi = \dot{D}_\xi[G]$. □

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