THE TREE PROPERTY UP TO $\aleph_{\omega+1}$

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Abstract. Assuming ω supercompact cardinals we force to obtain a model where the tree property holds both at $\aleph_{\omega+1}$, and at $\aleph_n$ for all $2 \leq n < \omega$. A model with the former was obtained by Magidor–Shelah from a huge cardinal and ω supercompact cardinals above it, and recently by Sinapova from ω supercompact cardinals. A model with the latter was obtained by Cummings–Foreman from ω supercompact cardinals. Our model, where the two hold simultaneously, is another step toward the goal of obtaining the tree property on increasingly large intervals of successor cardinals.
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§1. Introduction. The tree property is a combinatorial principle that resembles large cardinal reflection properties, but may hold at successor cardinals. It states for a cardinal κ that every κ-tree, meaning every tree of height κ with levels of width $< \kappa$, has a branch of length κ. That it holds at $\kappa = \aleph_0$ is simply König's lemma. On the other hand it fails at $\aleph_1$ by a construction of Aronszajn. (Trees witnessing failure of the tree property are called Aronszajn trees.) The question of whether and to what extent it can hold at successor cardinals greater than $\aleph_1$ has been researched starting with work of Mitchell and Silver in Mitchell [5]. They show that the tree property can hold at $\aleph_2$, and is a remnant of a large cardinal property, specifically weak compactness, in the sense that given a weakly compact cardinal κ, a forcing extension defined by Mitchell turns κ into $\aleph_2$ while securing the tree property, and conversely, if $\aleph_2$ has the tree property in V, then it is weakly compact in an inner model.

One can use the same forcing techniques repeatedly to obtain the tree property simultaneously at many successor cardinals, provided there are gaps between them. It is substantially harder to obtain the tree property simultaneously at *consecutive* successor cardinals. Partly the reason is that the tree property at $\kappa = \tau^{++}$ has an effect on cardinal arithmetic already below τ^+; it implies that $2^\tau \geq \tau^{++}$. (This follows from the construction in Specker [9] showing that the tree property fails at δ^+ if $\delta^{<\delta} = \delta$.) Nonetheless, it is possible for the tree property to hold at consecutive successor cardinals. Abraham [1] produces a model where the tree property holds at both $\aleph_2$ and $\aleph_3$. Again it is a remnant of large cardinals, supercompactness and weak compactness for the cardinals that are turned into $\aleph_2$ and $\aleph_3$ respectively in Abraham's model. Since supercompactness is beyond

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the reach of current methods of inner model theory, it is not known whether it is necessary for Abraham's result. But some large cardinal, substantially beyond the weakly compact that was enough for the tree property at one cardinal, is needed by work of Magidor in [1] and later work of Foreman–Magidor. This need for substantially stronger large cardinals is a mathematical aspect of the added difficulty in obtaining the tree property at consecutive cardinals.

Moving further, Cummings–Foreman [2] produced a model where the tree property holds at $\aleph_n$ for all $2 \leq n < \omega$, starting from ω supercompact cardinals. For known lower bounds on the necessary large cardinals see Foreman–Magidor–Schindler [3]. A little earlier Magidor–Shelah [4] showed that the tree property can hold at $\aleph_{\omega+1}$. They used a huge cardinal and ω supercompact cardinals above it, but recent work of Sinapova [8] reduced the large cardinal assumption to ω supercompact cardinals.

Cummings–Foreman [2] asked whether it is consistent to have both these outcomes simultaneously, namely whether it is possible for the tree property to hold at all successor cardinals in the interval $[\aleph_2, \aleph_{\omega+1}]$.

Starting from ω supercompact cardinals, we prove in this paper that the answer is yes.

Whether one can go further is still open. It is not known whether the tree property can hold at all successor cardinals in the interval $[\aleph_2, \aleph_{\omega+2}]$, or even if it can hold simultaneously at $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$. By Specker's result above, the tree property at $\aleph_{\omega+2}$ implies that $2^{\aleph_\omega} \geq \aleph_{\omega+2}$, and it is not known if even this is consistent with the tree property at $\aleph_{\omega+1}$. This particular question has a long history, and we refer the reader to Neeman [6] and Sinapova [7] for positive answers at some singular strong limit cardinal κ and at $\aleph_{\omega^2}$ respectively.

Our proof that the tree property can hold at all successor cardinals in the interval $[\aleph_2, \aleph_{\omega+1}]$ builds on ideas and techniques from several of the papers mentioned above.

In Section 3 we obtain a fairly wide class of posets that, given supercompact cardinals κ_n, $2 \leq n < \omega$, collapse so that κ_n becomes $\aleph_n$ and the tree property holds at $\aleph_{\omega+1}$. One example of a poset in the class, assuming indestructibility of the supercompact cardinals, is simply the product $\mathrm{Col}(\omega, \mu) \times \mathrm{Col}(\mu^+, {<}\kappa_2) \times \prod_{2 \leq n < \omega} \mathrm{Col}(\kappa_n, {<}\kappa_{n+1})$ for some $\mu < \kappa_2$, whose successor becomes $\aleph_1$ in the extension. Note that the proof does not give the tree property in the extension for any particular μ; it only shows the existence of such a μ. This "retreat" to just showing the existence of μ was first used by Sinapova [8] and was a crucial part of her argument to obtain an extension with the tree property at $\aleph_{\omega+1}$, from ω supercompact cardinals. (Sinapova's argument involves a diagonal Prikry extension and other than this "retreat" it is completely different from ours.) More generally, we show that the tail-end of the poset above can be replaced by *any* poset that leaves the cardinals κ_n for $n > 2$ "generically supercompact", and that $\mathrm{Col}(\omega, \mu) \times \mathrm{Col}(\mu^+, {<}\kappa_2)$ can be replaced by any family of posets $\mathbb{L}(\mu)$, $\mu < \kappa_2$, that can, on a measure one set of substructures relative to a supercompactness measure on κ_2, be subsumed by Knaster posets. The precise formulation of this is given in Lemma 3.10.

In Section 4 we modify the Cummings–Foreman [2] poset for obtaining the tree property below $\aleph_\omega$, so that it (almost) fits the requirements of Lemma

3.10. In broad terms the modifications are necessary to bring the poset closer to a product, rather than an iteration, so that one can separate its tail-end from its initial segment below κ_2, and argue that the tail-end by itself preserves generic supercompactness for the cardinals κ_n, $n > 2$. We cannot quite bring the poset to this form, but we can get close in the sense that the poset we define is subsumed by a poset of this form (see Section 5), and the factor poset is μ closed. By a preservation theorem of Magidor–Shelah [4] this is enough to put the two constructions together. This final combination is done in Section 6.

§2. Preliminaries. We present in this section a few forcing claims that are used in later section. Most are folklore, with the exception of Claim 2.4 which is due to Unger [10]. Unger in a different paper [11] also proved a strengthening of Claim 2.3, that reduces the assumption on $\mathbb{P}$ to just the requirement that $\mathbb{P}\times\mathbb{P}$ is κ^+-c.c. More precisely he showed, and this implies the claim, that if $\mathbb{P}\times\mathbb{P}$ is τ-c.c. where τ is regular, then forcing with $\mathbb{P}$ does not add branches to trees of height τ in V. He used this to prove a generalization of the tree property in the model of Cummings–Foreman [2].

Definition 2.1. Let $K \subseteq V$ be a model of a sufficiently large fragment of ZFC. K has the $<\delta$ *covering property* (with respect to V) if for every $A \subseteq K$ in V with $|A| < \delta$, there is $B \in K$ so that $(|B| < \delta)^K$ and $B \supseteq A$.

Claim 2.2. *Suppose $\delta < \kappa$ are regular cardinals, K is a model of some large enough fragment of ZFC, K has the $<\kappa$ covering property in V, and $(\forall\gamma < \kappa)(\gamma^{<\delta} < \kappa)^K$. Let $\mathbb{P}$ be a forcing notion in K, whose conditions are all functions with domain of size $< \delta$ in K. Then any family of size κ in V of conditions in $\mathbb{P}$, can be refined to a family of the same size whose domains form a Δ system.*

Proof. It is enough to show that for any A of size $< \kappa$ in V, the set $\{x \cap A \mid x \in K \text{ and } (|x| < \delta)^K\}$ has size $< \kappa$. Standard arguments then yield a Δ-system lemma for families of size κ in V, consisting of sets of size $< \delta$ in K.

Using the covering property we may assume that $A \in K$ and $(|A| < \kappa)^K$. Then since K is closed under intersections (a consequence of some fragment of ZFC in K), $\{x \cap A \mid x \in K \text{ and } (|x| < \delta)^K\}$ is equal to $\mathcal{P}_{<\delta}(A)^K$. Since $(|A| < \kappa)^K$, by the claim assumptions $\mathcal{P}_{<\delta}(A)^K$ has size $< \kappa$ in K, and therefore also in V. $\dashv$

Claim 2.3. *Let T be a tree of height of cofinality at least κ^+, and levels of width less than λ, for some $\lambda \geq \kappa^+$. Let $\mathbb{P}$ be κ^+-c.c. Suppose there is some κ^+-c.c. forcing notion $\mathbb{P}^\lambda$ which adds λ filters, all mutually generic for $\mathbb{P}$. (This holds for example if $\mathbb{P}$ is isomorphic to some λ product of itself.) Then forcing with $\mathbb{P}$ does not add any new cofinal branches through T.*

Proof. Without loss of generality, elements of T are sequences of ordinals ordered by extension. Let $\dot{b}$ be a $\mathbb{P}$ name for a cofinal branch through T, viewed as a sequence of ordinals of length κ^+. Let $G = \langle G_\xi \mid \xi < \lambda\rangle$ be generic for $\mathbb{P}^\lambda$. Let R be a large initial segment of V and let $M \prec R$ with $\kappa \cup \{T, \kappa, \kappa^+, \mathbb{P}, \mathbb{P}^\lambda\} \subseteq M$ and $|M| = \kappa$. Let $\alpha = \sup(M \cap \text{height}(T))$. Note that $\alpha < \text{height}(T)$ since height(T) has cofinality at least κ^+. For each ξ let $\delta_\xi = \dot{b}[G_\xi](\alpha)$. Since $\mathbb{P}^\lambda$ is κ^+-c.c., it does not collapse λ. Since level α of T has width less than λ in V it

follows that there are $\xi \neq \zeta$ so that $\delta_\xi = \delta_\zeta$. Hence $\dot{b}[G_\xi]\restriction\alpha = \dot{b}[G_\zeta]\restriction\alpha$. This implies that $M[G_\xi \times G_\zeta] \models \dot{b}[G_\xi] = \dot{b}[G_\zeta]$. (We are using the fact that $\mathbb{P} \times \mathbb{P}$ is κ^+-c.c., and therefore $M[G_\xi \times G_\zeta] \cap V = M$ so that $M[G_\xi \times G_\zeta] \cap \text{height}(T) \subseteq \alpha$.) By elementarity of $M[G_\xi \times G_\zeta]$ in $R[G_\xi \times G_\zeta]$ it follows that $\dot{b}[G_\xi] = \dot{b}[G_\zeta]$, and since the two filters are mutually generic, $\dot{b}[G_\xi] = \dot{b}[G_\zeta]$ must belong to V. $\dashv$

Claim 2.4 (Unger [10]). *Let $\tau < \kappa$. Let T be a κ^+ tree, i.e., a tree of height κ^+ with levels of width κ. Let $W \subseteq V$ and suppose that V is a κ-c.c. forcing extension of W. Let $\mathbb{P} \in W$ be $< \kappa$ closed in W. Suppose that $2^\tau > \kappa$ in W. Then forcing with $\mathbb{P}$ (over V) does not add cofinal branches to T.*

Proof. Let A be generic for $\mathbb{A}$ over W, where $\mathbb{A}$ is κ-c.c. in W and $V = W[A]$. Let $\dot{T} \in W$ be an $\mathbb{A}$ name for T, and suppose without loss of generality that $\dot{T}$ is forced to be a tree, meaning in particular that if $a \in \mathbb{A}$ forces that $\langle \alpha, \xi_1 \rangle$ and $\langle \alpha, \xi_2 \rangle$ are both predecessors of $\langle \alpha', \xi' \rangle$ in $\dot{T}$, then $\xi_1 = \xi_2$.

Let $\dot{b} \in W$ be an $\mathbb{A} \times \mathbb{P}$ name for a cofinal branch through T. Suppose for contradiction that $\dot{b}$ is forced to not belong to $V = W[A]$. It is then forced in $\mathbb{A} \times \mathbb{P} \times \mathbb{P}$ that, letting $A \times G_1 \times G_2$ be generic, $\dot{b}[A \times G_1] \neq \dot{b}[A \times G_2]$. Thus, for any conditions $p_1, p_2 \in \mathbb{P}$, and for any condition $a \in A$, there is $\alpha < \kappa^+$, $a' \leq a$, $p_1' \leq p_1$, and $p_2' \leq p_2$, so that $\langle a', p_1' \rangle$ and $\langle a', p_2' \rangle$ force distinct values for $\dot{b}(\alpha)$. By repeated applications of this inside W, using the closure of $\mathbb{P}$ and the κ-chain condition for $\mathbb{A}$, it follows that there are $p_1^* \leq p_1$, $p_2^* \leq p_2$, and a set $\{\langle a_\xi, \alpha_\xi \rangle \mid \xi < \gamma\}$ of size $< \kappa$, so that $\langle a_\xi, p_1^* \rangle$ and $\langle a_\xi, p_2^* \rangle$ force distinct values for $\dot{b}(\alpha_\xi)$, and $\{a_\xi \mid \xi < \gamma\}$ is a maximal antichain in $\mathbb{A}$. As $\dot{T}$ is forced to be a tree, letting $\alpha = \sup\{\alpha_\xi \mid \xi < \gamma\} < \kappa^+$, it then follows that there is no a and no $p_1^{**} \leq p_1^*$, $p_2^{**} \leq p_2^*$, so that $\langle a, p_1^{**} \rangle$ and $\langle a, p_2^{**} \rangle$ force the same value for $\dot{b}(\alpha)$. We say in such a case that p_1^* and p_2^* *enforce complete separation* at α. Note that if p_1^* and p_2^* enforce complete separation at α, then they also enforce complete separation at every $\alpha' \geq \alpha$. This again uses the fact that $\dot{T}$ is forced to be a tree. Note also that if p_1^* and p_2^* enforce complete separation at α, then so do all extensions of p_1^* and p_2^*.

Let $\delta \leq \tau$ be least so that $2^\delta > \kappa$ in W. Working inside W, using the closure of $\mathbb{P}$ and the conclusion of the previous paragraph, construct an extension preserving embedding π from $2^{\leq \delta}$ into $\mathbb{P}$ with the property that for any $s \in 2^{<\delta}$, there is an ordinal α_s so that $\pi(s^\frown 0)$ and $\pi(s^\frown 1)$ enforce complete separation at α_s. Let $\alpha = \sup\{\alpha_s \mid s \in 2^{<\delta}\}$. By minimality of δ, $\alpha < \kappa^+$. By construction, for every distinct $s, t \in 2^\delta$, there is $\bar{\alpha} < \alpha$, $p_1 \geq \pi(s)$, and $p_2 \geq \pi(t)$, so that p_1 and p_2 enforce complete separation at $\bar{\alpha}$. Hence $\pi(s)$ and $\pi(t)$ enforce complete separation at α.

Continuing to work inside W, find for each $s \in 2^\delta$, some $a_s \in \mathbb{A}$ and $q_s \leq \pi(s)$ so that $\langle a_s, q_s \rangle$ forces a value for $\dot{b}(\alpha)$. Since T is forced to be a κ^+ tree, the values forced for $\dot{b}(\alpha)$ belong to κ. Since $2^\delta > \kappa$ in W, and since $\mathbb{A}$ is κ-c.c. in W, there must be $s \neq t$ both in 2^δ, so that a_s and a_t are compatible, and so that $\langle a_s, q_s \rangle$ and $\langle a_t, q_t \rangle$ force the same value for $\dot{b}(\alpha)$. Letting $a \in \mathbb{A}$ extend a_s and a_t, it follows that $\langle a, q_s \rangle$ and $\langle a, q_t \rangle$ force the same value for $\dot{b}(\alpha)$, but this contradicts the fact that $\pi(s)$ and $\pi(t)$ enforce complete separation at α. $\dashv$

CLAIM 2.5. *Let $\mathbb{P}$ be $<\kappa$ closed in W, where V is a κ-c.c. forcing extension of W. Then forcing with $\mathbb{P}$ over V does not add any sequences of ordinals of length $< \kappa$.*

PROOF. Let $\mathbb{A}$ be a κ-c.c. poset in W so that V is an extension of W by $\mathbb{A}$. Let $A \times P$ be generic for $\mathbb{A} \times \mathbb{P}$ over W. Then by closure of $\mathbb{P}$, $\mathbb{A}$ remains κ-c.c. in $W[P]$. Hence any $\mathbb{A}$ name in $W[P]$ for a sequence of ordinals of length $< \kappa$, is equivalent to a name of size $< \kappa$, which by closure of $\mathbb{P}$ belongs to W. So all sequences of ordinals of length $< \kappa$ in $W[A][P] = W[P][A]$ belong to $W[A]$. $\dashv$

§3. The tree property at $\aleph_{\omega+1}$. Magidor–Shelah [4] were the first to obtain the tree property at $\aleph_{\omega+1}$. They used a huge cardinal with ω supercompact cardinals above it. Sinapova [8] found an argument that requires only ω supercompact cardinals. Her model is obtained by a diagonal Prikry extension that turns the lowest of the supercompact cardinals into $\aleph_\omega$. One of the crucial novelties in her argument is that the poset itself selects which cardinal is turned into $\aleph_1$. We show here that with a similar selection mechanism, and assuming indestructibility of the supercompact cardinals, the product of ordinary collapse posets between and below ω supercompact cardinals leads to a model where the tree property holds at $\aleph_{\omega+1}$. This is Corollary 3.9. Moreover the same is true for other posets, so long as they leave enough "generic supercompactness" at κ_n for $n > 2$, and so long as their component below κ_2 has many hulls that are subsumed by Knaster posets. The exact formulation of this is given by Lemma 3.10. We will use several tools from a different paper by Sinapova, [7], and from Magidor–Shelah [4].

Let ν be a strong limit cardinal of cofinality ω. Whenever we talk about ν^+ trees, we view them as relations on $\nu^+ \times \nu$, with level α of the tree consisting of pairs in $\{\alpha\} \times \nu$.

DEFINITION 3.1 (Magidor–Shelah [4]). A *system* on $D \times \tau$ is a collection of transitive, reflexive relations R_i ($i \in I$) on $D \times \tau$, so that:

1. If $\langle \alpha, \xi \rangle \; R_i \; \langle \beta, \zeta \rangle$ and $\langle \alpha, \xi \rangle \neq \langle \beta, \zeta \rangle$ then $\alpha < \beta$.
2. If $\langle \alpha_0, \xi_0 \rangle$ and $\langle \alpha_1, \xi_1 \rangle$ are both below $\langle \beta, \zeta \rangle$ in R_i, then $\langle \alpha_0, \xi_0 \rangle$ and $\langle \alpha_1, \xi_1 \rangle$ are comparable in R_i. (By condition (1) this implies that $\langle \alpha_0, \xi_0 \rangle \; R_i \; \langle \alpha_1, \xi_1 \rangle$ if $\alpha_0 < \alpha_1$, $\langle \alpha_1, \xi_1 \rangle \; R_i \; \langle \alpha_0, \xi_0 \rangle$ if $\alpha_1 < \alpha_0$, and $\xi_0 = \xi_1$ if $\alpha_0 = \alpha_1$.)
3. For every $\alpha < \beta$ both in D, there is $i \in I$, and $\xi, \zeta \in \tau$, so that $\langle \alpha, \xi \rangle \; R_i \; \langle \beta, \zeta \rangle$.

If T is a ν^+ tree, then the singleton relation T is a system on $\nu^+ \times \nu$. Any restriction $T \restriction (D \times \tau)$ of T satisfies conditions (1) and (2) in Definition 3.1, but condition (3) may in general fail.

DEFINITION 3.2 (Sinapova [7]). Let $\{R_i\}_{i \in I}$ be a system on $D \times \tau$. A *system of branches* through $\{R_i\}_{i \in I}$ is a collection $\{b_j\}_{j \in J}$ so that:

1. Each b_j is a branch through R_i for some $i = i_j \in I$. This means that b_j is a partial function from D into τ, and for any $\beta \in \mathrm{dom}(b_j)$ and any $\alpha < \beta$ in D, $\alpha \in \mathrm{dom}(b_j)$ iff $(\exists \xi) \langle \alpha, \xi \rangle \; R_i \; \langle \beta, b_j(\beta) \rangle$, and $b_j(\alpha)$ is equal to the unique ξ witnessing this. (ξ is unique by condition (2) of Definition 3.1.)
2. For every $\alpha \in D$, there is j so that $\alpha \in \mathrm{dom}(b_j)$.

We do not require the branches b_j to be cofinal (meaning that $\mathrm{dom}(b_j)$ is cofinal in D). But if $|J|$ is smaller than the cofinality of D, then by condition (2), at least one of the branches has to be cofinal.

LEMMA 3.3 (Sinapova [7]). *Let $\{R_i\}_{i\in I}$ be a system on $D \times \tau$, with D cofinal in ν^+, and $\max\{|I|, \tau\} < \nu$. Suppose that there is $W \subseteq V$, a poset $\mathbb{P} \in W$, and a regular cardinal $\kappa < \nu$ above $\max\{|I|, \tau\}^+$, so that:*

1. *The empty condition in $\mathbb{P}$ forces that there exists a system $\{b_j\}_{j\in J}$ of branches through $\{R_i\}_{i\in I}$, with $|J|^+ < \kappa$.*
2. *$\mathbb{P}$ is $<\kappa$ closed in W, and V is a forcing extension of W by a κ-c.c. poset.*

Then there exists j so that b_j is cofinal and belongs to V. In particular there is $i \in I$ so that in V, R_i has a cofinal branch.

PROOF. Let $\mathbb{A}$ be a κ-c.c. poset so that V is an extension of W by $\mathbb{A}$, and let E be generic for $\mathbb{A}$ over W with $V = W[E]$.

Let $\dot{b}_j \in V = W[E]$ name b_j in the poset $\mathbb{P}$. Suppose for contradiction that no cofinal b_j belongs to V. Without loss of generality we may assume that the empty condition in $\mathbb{P}$ forces $\dot{b}_j \notin V$ if $\dot{b}_j$ is cofinal.

Let $\lambda = \max\{|I|, |J|, \tau\}^+$. By assumption, $\lambda < \kappa$. Let $\mathbb{P}^\lambda$ be the full support λth power of $\mathbb{P}$, defined in W. Let $\langle G_\xi \mid \xi < \lambda\rangle$ be generic for $\mathbb{P}^\lambda$ over $V = W[E]$.

$\mathbb{P}^\lambda$ is $<\kappa$ closed in W, and V is a κ-c.c. extension of W. It follows by Claim 2.5 that forcing with $\mathbb{P}^\lambda$ over V does not add sequences of ordinals of length $< \kappa$. In particular, ν^+ has cofinality greater than λ in $V[G_\xi \mid \xi < \lambda]$, and all cardinals of V up to λ remain cardinals in $V[G_\xi \mid \xi < \lambda]$.

Let $b_j^\xi = \dot{b}_j[G_\xi]$. Since $\mathrm{cof}(\nu^+)$ is greater than λ in $V[G_\xi \mid \xi < \lambda]$, we can find $\gamma_0 < \nu^+$ so that for every ξ and j, $\mathrm{dom}(b_j^\xi) \subseteq \gamma_0$ whenever $\mathrm{dom}(b_j^\xi)$ is bounded in ν^+. Since by assumption the cofinal b_j^ξ do not belong to V, it follows by mutual genericity that whenever $\xi \neq \zeta$ and $\mathrm{dom}(b_j^\xi)$ and $\mathrm{dom}(b_j^\zeta)$ both have points above γ_0, then the branches b_j^ξ and b_j^ζ are distinct. Again using the fact that $\mathrm{cof}(\nu^+) > \lambda$ in $V[G_\xi \mid \xi < \lambda]$, we can find $\gamma_1 > \gamma_0$ so that whenever b_j^ξ and b_j^ζ both have $\alpha > \gamma_1$ in their domains, the two branches differ at a point below γ_1 (possibly because one is defined and the other is not). By Definition 3.2 and since $\alpha > \gamma_1$ this implies in particular that $b_j^\xi(\alpha) \neq b_j^\zeta(\alpha)$ (possibly because one is defined and the other is not) if both are branches through the same relation R_i.

Let $\alpha > \gamma_1$ belong to D. By Definition 3.2, for each $\xi < \lambda$ there is some j_ξ so that $\alpha \in \mathrm{dom}(b_{j_\xi}^\xi)$. Let $\delta_\xi = b_{j_\xi}^\xi(\alpha)$ and let i_ξ be such that $b_{j_\xi}^\xi$ is a branch through R_{i_ξ}. λ is greater than $|I| \cdot |J| \cdot \tau$, in V and hence also in $V[G_\xi \mid \xi < \lambda]$. So there must be $\xi \neq \zeta$ so that $j_\xi = j_\zeta$, $i_\xi = i_\zeta$, and $\delta_\xi = \delta_\zeta$. But then letting $j = j_\xi = j_\zeta$ and $i = i_\xi = i_\zeta$ we have $b_j^\xi(\alpha) = b_j^\zeta(\alpha)$, where b_j^ξ and b_j^ζ are both branches through the same relation R_i, contradicting the conclusion of the previous paragraph. ⊣

REMARK 3.4. Our proof of Lemma 3.3 makes it clear that assumption (2) of the lemma can be weakened to require only that there is a poset $\mathbb{P}^\lambda$ which adds

λ mutually generic filters for $\mathbb{P}$ without collapsing any cardinals $\leq \lambda$ and without reducing the cofinality of ν^+ to λ or below, where $\lambda = \max\{|I|, |J|, \tau\}^+$.

Lemma 3.5 (Sinapova [7] based on Magidor–Shelah [4]). *Let $\{R_i\}_{i \in I}$ be a system on $D \times \tau$ where D is cofinal in ν^+ and $\tau < \nu$. Suppose that forcing with $\mathbb{P}$ adds an elementary embedding $\pi\colon V \to V^*$, with $\mathrm{crit}(\pi) > \max\{\tau, |I|\}$ and $\pi(\nu^+) > \sup(\pi''\nu^+)$. Then forcing with $\mathbb{P}$ adds a system of branches $\{b_j\}_{j\in J}$ through $\{R_i\}_{i\in I}$, with $J = I \times \tau$.*

Proof. Let G generic for $\mathbb{P}$ over V. Let $\pi \in V[G]$ be an embedding as in the assumption of the lemma. Note that $\pi(\tau) = \tau$ as $\mathrm{crit}(\pi) > \tau$. Since $\mathrm{crit}(\pi) > |I|$ we may assume, modifying I if needed, that $\pi(I) = I$. So $\pi(\{R_i\}_{i\in I})$ is equal to $\{\pi(R_i)\}_{i\in I}$, and is a system on $\pi(D) \times \tau$ in V^*.

Let γ be an ordinal in $\pi(D)$ between $\sup(\pi''\nu^+)$ and $\pi(\nu^+)$. For each $\langle i, \delta\rangle \in I \times \tau$, let $b_{i,\delta}$ be the partial map sending $\alpha \in D$ to the unique $\xi < \tau$ so that $\langle \pi(\alpha), \xi\rangle\ \pi(R_i)\ \langle \gamma, \delta\rangle$ if such ξ exists. Uniqueness is guaranteed by condition (2) in Definition 3.1 since $\{\pi(R_i)\}_{i\in I}$ is a system. It is clear from the same definition, and elementarity, that $b_{i,\delta}$ is a branch of R_i.

Finally, to check condition (2) of Definition 3.2, fix $\alpha \in D$, and note that since $\{\pi(R_i)\}_{i\in I}$ is a system on $\pi(D) \times \tau$, there is by condition (3) of Definition 3.1 some $\xi, \delta < \tau$, and some $i \in I$, so that $\langle \pi(\alpha), \xi\rangle\ \pi(R_i)\ \langle \gamma, \delta\rangle$. Then $\alpha \in \mathrm{dom}(b_{i,\delta})$, as required. ⊣

Lemma 3.6. *Let κ_n, $2 \leq n < \omega$ be a strictly increasing sequence of regular cardinals cofinal in ν. Suppose that κ_2 is supercompact, and that for each $m \geq 2$ there is a generic embedding $\pi\colon V \to V^*$ added by a poset $\mathbb{P}$ so that:*

- $\sup(\pi''\nu^+) < \pi(\nu^+)$.
- $\mathrm{crit}(\pi) > \kappa_m$.
- *$\mathbb{P}$ is $<\kappa_m$ closed in a model $W \subseteq V$ so that V is a κ_m-c.c. extension of W.*

For each strong limit cardinal $\mu < \kappa_2$ of cofinality ω, let $\mathbb{L}(\mu)$ be the poset $\mathrm{Col}(\omega, \mu) \times \mathrm{Col}(\mu^+, <\kappa_2)$. Then there is $\mu < \kappa_2$ so that the extension of V by $\mathbb{L}(\mu)$ satisfies the tree property at ν^+.

Proof. Let κ denote κ_2. Suppose for contradiction that the tree property at ν^+ fails in all extensions of V by $\mathbb{L}(\mu)$ as μ ranges over strong limit cardinals of cofinality ω below κ. Fix $\mathbb{L}(\mu) = \mathrm{Col}(\omega, \mu) \times \mathrm{Col}(\mu^+, <\kappa)$ names $\dot{T}(\mu) \in V$ for trees forced to witness this.

Let $I = \{\langle a, b, \mu\rangle \mid \mu < \kappa$ is a singular strong limit of cofinality ω and $\langle a, b\rangle \in \mathrm{Col}(\omega, \mu) \times \mathrm{Col}(\mu^+, <\kappa)\}$. For $i = \langle a, b, \mu\rangle \in I$ let S_i be the relation $\langle \alpha, \xi\rangle\ S_i\ \langle \beta, \zeta\rangle$ iff $\langle a, b\rangle \Vdash \langle \alpha, \xi\rangle\ \dot{T}(\mu)\ \langle \beta, \zeta\rangle$. It is clear, using the fact that each $\dot{T}(\mu)$ is forced to be a ν^+ tree, that $\{S_i\}_{i\in I}$ is a system on $\nu^+ \times \nu$.

Using the supercompactness of κ, let $\pi\colon V \to V^*$ be elementary, with $\mathrm{crit}(\pi) = \kappa$, $\pi(\kappa) > \nu$, and V^* closed under sequences of length ν^+ in V. In particular $\pi''\nu^+ \in V^*$ and hence $\pi(\nu^+) > \sup(\pi''\nu^+)$.

Let $G^*_0 \times G^*_1$ be generic for $\mathrm{Col}(\omega, \nu)^{V^*} \times \mathrm{Col}(\nu, <\pi(\kappa))^{V^*}$ over V, hence also over V^*. Let $T^* = \pi(\dot{T})(\nu)[G^*_0 \times G^*_1]$, where $\dot{T}$ here denotes the map $\mu \mapsto \dot{T}(\mu)$. In $V[G^*_0 \times G^*_1]$, ν is collapsed to ω, and ν^+ is ω_1.

Let γ^* be an ordinal between $\sup(\pi''\nu^+)$ and $\pi(\nu^+)$. For each $\alpha < \nu^+$ let $\xi^* = \xi^*_\alpha$ be the unique ordinal so that $\langle \pi(\alpha), \xi^*\rangle\ T^*\ \langle \gamma^*, 0\rangle$. ξ^*_α is an ordinal

below $\pi(\nu) = \sup_{n<\omega} \pi(\kappa_n)$. For each α, let $n = n_\alpha$ be least so that $\xi^*_\alpha < \pi(\kappa_n)$. Let $\dot{\xi}^*_\alpha$ and $\dot{n}_\alpha$ in V be the canonical $\mathrm{Col}(\omega,\nu)^{V^*} \times \mathrm{Col}(\nu^+, {<}\pi(\kappa))^{V^*}$ names for ξ^*_α and n_α.

Since ν^+ is equal to ω_1 in $V[G^*_0 \times G^*_1]$, there is a cofinal $D^* \subseteq \nu^+$, and $n < \omega$, so that $n^*_\alpha = n$ for all $\alpha \in D^*$. The fact that $n^*_\alpha = n$ is forced by some condition $\langle a_\alpha, b_\alpha\rangle \in G^*_0 \times G^*_1$. a_α is an initial segment of G^*_0 and of finite length. Shrinking D^* we may therefore assume that there is a specific initial segment a so that $a_\alpha = a$ for all $\alpha \in D^*$. In particular then D^* can be determined using a without reference to the full generic G^*_0, and hence $D^* \in V[G^*_1]$.

Claim 3.7. *$\{S_i{\restriction}(D^* \times \kappa_n)\}_{i\in I}$ is a system.*

Proof. Conditions (1) and (2) of Definition 3.1 hold for $\{S_i{\restriction}(D^* \times \kappa_n)\}_{i\in I}$ because they hold for the system $\{S_i\}_{i\in I}$. We have to check condition (3).

Fix $\alpha < \beta$ both in D^*. Then ξ^*_α and ξ^*_β are both smaller than $\pi(\kappa_n)$. By the definitions above, $\langle\pi(\alpha), \xi^*_\alpha\rangle$ and $\langle\pi(\beta), \xi^*_\beta\rangle$ are both below $\langle\gamma, 0\rangle$ in the relation T^*, and in particular they are compatible. Hence there is a condition $\langle a^*, b^*\rangle \in G^*_0 \times G^*_1$ forcing that $\langle\pi(\alpha), \xi^*_\alpha\rangle\ \pi(\dot{T})(\nu)\ \langle\pi(\beta), \xi^*_\beta\rangle$.

By elementarity of π, it follows that there is $\mu < \kappa$, $\xi_\alpha, \xi_\beta < \kappa_n$, and a condition $\langle a, b\rangle$, so that $\langle a, b\rangle \Vdash \langle\alpha, \xi_\alpha\rangle\ \dot{T}(\mu)\ \langle\beta, \xi_\beta\rangle$. Then $\langle\alpha, \xi_\alpha\rangle$ and $\langle\beta, \xi_\beta\rangle$ are related in $S_{a,b,\mu}{\restriction}(D^* \times \kappa_n)$, witnessing condition (3) for the system $\{S_i{\restriction}(D^* \times \kappa_n)\}_{i\in I}$ at α and β. ⊣

Claim 3.8. *There is, in V, a cofinal set $D \subseteq \nu^+$ so that $\{S_i{\restriction}(D \times \kappa_n)\}_{i\in I}$ is a system.*

Proof. Let R be a large initial segment of V and let $X \prec R$ be an elementary substructure of size ν^+, with $\nu^+ \subseteq X$, closed under sequences of length $< \nu^+$, and containing all objects relevant to the constructions above. $\mathrm{Col}(\nu^+, {<}\pi(\kappa))^{V^*}$ is ${<}\nu^+$ closed in V^*, hence also in V, so working in V we can find, without any further forcing, $\bar{G}^*_1 \subseteq X$ which is generic for $\mathrm{Col}(\nu^+, {<}\pi(\kappa))^{V^*}$ over X.

By Claim 3.7, applied inside $X[\bar{G}^*_1]$, there is $\bar{D}^* \in X[\bar{G}^*_1]$, cofinal in ν^+, so that $X[\bar{G}^*_1]$ satisfies that $\{S_i{\restriction}(\bar{D}^* \times \kappa_n)\}_{i\in I}$ is a system.

Since being a system is absolute, $\{S_i{\restriction}(\bar{D}^* \times \kappa_n)\}_{i\in I}$ is a system in V. ⊣

We so far have $n < \omega$ and $D \subseteq \nu^+$ cofinal, so that $\{S_i{\restriction}(D \times \kappa_n)\}_{i\in I}$ is a system.

Let $m = n + 2$. By assumption of the lemma, there is a poset $\mathbb{P}$ adding an embedding π with $\mathrm{crit}(\pi) > \kappa_m$, $\pi(\nu^+) > \sup(\pi''\nu^+)$, and such $\mathbb{P}$ is ${<}\kappa_m$ closed in a model W so that V is a κ_m-c.c. extension of W.

By Lemma 3.5, forcing with $\mathbb{P}$ adds a system of branches $\{b_j\}_{j\in J}$ to $\{S_i{\restriction}(D \times \kappa_n)\}_{i\in I}$, with $J = I \times \kappa_n$, and in particular $|J|^+ < \kappa_{n+2} = \kappa_m$. By Lemma 3.3 there is $i \in I$ so that a cofinal branch through $S_i{\restriction}(D \times \kappa_n)$ exists already in V. Fix such i, and let $f \in V$ be the cofinal branch.

Let μ and $\langle a, b\rangle \in \mathrm{Col}(\omega, \mu)^V \times \mathrm{Col}(\mu^+, {<}\kappa)^V$ be such that $i = \langle a, b, \mu\rangle$. Then by definition of S_i, $\langle a, b\rangle \Vdash \langle\alpha, f(\alpha)\rangle\ \dot{T}(\mu)\ \langle\beta, f(\beta)\rangle$ for all $\alpha < \beta$ both in $\mathrm{dom}(f)$, which is cofinal in ν^+. Letting $G_0 \times G_1$ be generic with $\langle a, b\rangle \in G_0 \times G_1$, it follows that in $V[G_0 \times G_1]$, f determines a cofinal branch through $\dot{T}(\mu)[G_0 \times G_1]$. But this contradicts the fact that $\dot{T}(\mu)$ is forced to have no cofinal branches. This contradiction completes the proof of Lemma 3.6. ⊣

Corollary 3.9. *Let κ_n, $2 \leq n < \omega$, be an increasing sequence of indestructibly supercompact cardinals. Then there is a strong limit cardinal $\mu < \kappa_2$ of cofinality ω, so that in the extension of V by $\mathrm{Col}(\omega, \mu) \times \mathrm{Col}(\mu^+, {<}\kappa_2) \times \prod_{2 \leq n < \omega} \mathrm{Col}(\kappa_n, {<}\kappa_{n+1})$, the tree property holds at ν^+. ν^+ is equal to $\aleph_{\omega+1}$ of the extension.*

The indestructibility assumed in the corollary can be arranged by standard arguments, with a preparatory forcing, starting from ω supercompact cardinals.

Proof of Corollary 3.9. Let $H = \prod_{2 \leq n < \omega} H_n$ be generic for the poset $\prod_{2 \leq n < \omega} \mathrm{Col}(\kappa_n, {<}\kappa_{n+1})$. It is enough to prove that the assumptions of Lemma 3.6 hold in $V[H]$. Then, by the lemma, there is $\mu < \kappa_2$ so that in the further extension by $\mathrm{Col}(\omega, \mu) \times \mathrm{Col}(\mu^+, {<}\kappa_2)$, the tree property holds at ν^+.

The assumptions of the lemma are easy to verify in $V[H]$. $\kappa = \kappa_2$ is supercompact in $V[H]$, by indestructibility. For each $m \geq 2$, forcing over $V[H]$ with $\mathrm{Col}(\kappa_m, \gamma)^V$ for sufficiently large γ, adds an embedding $\pi \colon V[H] \to V^*[H^*]$ with critical point κ_{m+1} and $\pi(\nu^+) > \sup(\pi''\nu^+)$. (This is easy to see using indestructibility of κ_{m+1} and standard extension of embeddings.) $\mathrm{Col}(\kappa_m, \gamma)^V$ is ${<}\kappa_m$ closed in $W = V[H_m \times H_{m+1} \times \ldots]$, and $V[H]$ is a κ_m-c.c. extension of $V[H_m \times H_{m+1} \times \ldots]$. ⊣

The corollary produces a model where the supremum of ω supercompact cardinals is turned into $\aleph_\omega$, and the tree property holds at $\aleph_{\omega+1}$. For future arguments that involve securing the tree property also below $\aleph_\omega$, it is useful to notice that our assumptions in Lemma 3.6 can be weakened in a couple of ways, to produce a lemma that works in somewhat more general settings. The next lemma formalizes this.

Lemma 3.10. *Let κ_n, $2 \leq n < \omega$, be a strictly increasing sequence of regular cardinals cofinal in ν. Let* Index $\subseteq \kappa_2$ *and suppose that $\mathbb{L}(\mu)$ for each $\mu \in$* Index *is a poset of size $\leq \kappa_2$. Suppose that:*

1. *For each $m \geq 2$, there is a generic embedding $\pi \colon V \to V^*$ added by a poset $\mathbb{P}$ so that:*
   (a) $\sup(\pi''\nu^+) < \pi(\nu^+)$.
   (b) $\mathrm{crit}(\pi) > \kappa_m$.
   (c) *There is a κ_mth power of $\mathbb{P}$ that adds κ_m mutually generic filters for $\mathbb{P}$, without collapsing any cardinals $\leq \kappa_m$, and without reducing the cofinality of ν^+ to or below κ_m.*
2. *Let R be a large rank initial segment of V satisfying a large enough fragment of* ZFC. *For each $X \prec R$ that is closed under sequences of length ν, with $V_\nu \subseteq X$ and $|X| = \nu^+$, let $\bar{V} = \bar{V}_X$ be the transitive collapse of X. Then, for a stationary set of X, there exists a ν^+-Knaster poset $\mathbb{P} = \mathbb{P}_X$ forcing the existence of π and L so that:*
   (a) *$\pi \colon \bar{V} \to \bar{V}^*$ is elementary with $\sup(\pi''\nu^+) < \pi(\nu^+)$.*
   (b) *$\mathrm{crit}(\pi) = \kappa_2$, $\pi(\kappa_2) > \nu$, and $\nu \in \pi(\mathrm{Index})$.*
   (c) *L is generic over $\bar{V}^*$ for $\pi(\mathbb{L})(\nu)$.*

Then there is $\mu < \kappa_2$ so that the extension of V by $\mathbb{L}(\mu)$ satisfies the tree property at ν^+.

Recall that a poset $\mathbb{P}$ is ν^+-*Knaster* if every sequence of ν^+ conditions in the poset can be refined to a subsequence of the same size so that any two conditions in the subsequence are compatible. The poset $\mathrm{Col}(\omega, \nu)$ that was used in the proof of Lemma 3.6 is of course ν^+-Knaster.

Proof of Lemma 3.10. The proof is similar to that of Lemma 3.6. The main difference is in the use of the embeddings given by condition (2) as a replacement for the assumption that κ_2 is supercompact.

Suppose for contradiction that the tree property fails at ν^+ in all extensions by $\mathbb{L}(\mu)$, $\mu \in$ Index. Let $\dot{T}(\mu)$ be names witnessing this, meaning that $\dot{T}(\mu)$ is forced in $\mathbb{L}(\mu)$ to be a ν^+ tree with no cofinal branches. Let $I = \{\langle r, \mu\rangle \mid \mu \in \text{Index}$ and $r \in \mathbb{L}(\mu)\}$. For $i = \langle r, \mu\rangle \in I$ let S_i be the relation $\langle \alpha, \xi\rangle\ S_i\ \langle \beta, \zeta\rangle$ iff $r \Vdash_{\mathbb{L}(\mu)} \langle \alpha, \xi\rangle\ \dot{T}(\mu)\ \langle \beta, \zeta\rangle$. As in the proof of Lemma 3.6, $\{S_i\}_{i\in I}$ is a system on $\nu^+ \times \nu$, and our first goal is to show that its restriction to $D \times \kappa_n$ is a system, for some cofinal $D \subseteq \nu^+$ and $n < \omega$.

Let R, X, $\bar{V}$, and $\mathbb{P}$ be as in condition (2), with the function $\mu \mapsto \dot{T}(\mu)$ in X. Let G be generic for $\mathbb{P}$ over V, and let $\pi, L \in V[G]$ be as in condition (2). Let $T^* = \pi(\dot{T})(\nu)[L] \in V[G]$.

Let γ^* be an ordinal between $\sup(\pi''\nu^+)$ and $\pi(\nu^+)$. For each $\alpha < \nu^+$ let ξ^*_α be the unique ordinal so that $\langle \pi(\alpha), \xi^*_\alpha\rangle\ T^*\ \langle \gamma^*, 0\rangle$. ξ^*_α is an ordinal below $\pi(\nu) = \sup_{n<\omega} \pi(\kappa_n)$. For each α, let $n = n_\alpha$ be least so that $\xi^*_\alpha < \pi(\kappa_n)$. Let $\dot{n}_\alpha$ name n_α in the forcing $\mathbb{P}$.

For each $\alpha < \nu^+$, fix a condition $p_\alpha \in \mathbb{P}$ forcing a value for $\dot{n}_\alpha$. Note that this is done in V, with no reference to the generic G. (Our use of G above was just for notational convenience.)

Since $\mathbb{P}$ is ν^+-Knaster, there is a cofinal $D \subseteq \nu^+$ so that for any $\alpha, \beta \in D$, p_α and p_β are compatible in $\mathbb{P}$. Thinning the set D, but maintaining the fact that it is cofinal, we may assume that there is a fixed $n < \omega$ so that for each $\alpha \in D$, the value p_α forces for $\dot{n}_\alpha$ is n.

Claim 3.11. *$\{S_i \restriction (D \times \kappa_n)\}_{i\in I}$ is a system.*

Proof. Conditions (1) and (2) of Definition 3.1 are inherited from $\{S_i\}_{i\in I}$. We have to check condition (3). Fix $\alpha < \beta$ both in D. Then p_α and p_β are compatible. Let p extend both. Revising G, we may assume $p \in G$. Then by definitions, $\langle \alpha, \xi^*_\alpha\rangle$ and $\langle \beta, \xi^*_\beta\rangle$ are both below $\langle \gamma^*, 0\rangle$ in T^*, and hence they are compatible. It follows, again by definitions and since $n_\alpha = n_\beta = n$, that $\bar{V}^*$ satisfies "there exists $\mu \in \pi(\text{Index})$, $r \in \pi(\mathbb{L})(\mu)$, and $\xi, \zeta < \pi(\kappa_n)$, so that $r \Vdash_{\pi(\mathbb{L})(\mu)} \langle \pi(\alpha), \xi\rangle\ \pi(\dot{T})(\mu)\ \langle \pi(\beta), \zeta\rangle$". By elementarity of π, there exists $\mu \in$ Index, $r \in \mathbb{L}(\mu)$, and $\xi, \zeta < \kappa_n$, so that $r \Vdash_{\mathbb{L}(\mu)} \langle \alpha, \xi\rangle\ \dot{T}(\mu)\ \langle \beta, \zeta\rangle$. Then $\langle \alpha, \xi\rangle$ and $\langle \beta, \zeta\rangle$ are related in $S_{\langle r,\mu\rangle} \restriction (D \times \kappa_n)$, as required. ⊣

As in the proof of Lemma 3.6, an application of Lemma 3.5 now shows that forcing with the poset $\mathbb{P}$ given by condition (1) of the current lemma for $m = n + 1$, adds a system of branches $\{b_j\}_{j\in J}$ through $\{S_i\}_{i\in I}$, with $J = I \times \kappa_n$. An application of Lemma 3.3, in conjunction with Remark 3.4, then shows that there must be a cofinal branch through one of the relations $S_{\langle r,\mu\rangle}$, already in V. This gives a cofinal branch through an interpretation of one of the names $\dot{T}(\mu)$, completing the proof of Lemma 3.10. ⊣

§4. The tree property below $\aleph_\omega$. Let κ_n, $2 \leq n < \omega$ be an increasing sequence of supercompact cardinals. Let $\nu = \sup\{\kappa_n \mid n < \omega\}$. We describe a forcing extension in which κ_n becomes $\aleph_n$ and the tree property holds at $\aleph_n$ for all $n \geq 2$.

Our construction is a modification of the poset defined in Cummings–Foreman [2]. There are several differences between the two constructions. One difference is that we do not preserve $\aleph_1$. Instead we allow the poset to select a cardinal μ, from a specific index set that we define, whose successor is then turned by the forcing into $\aleph_1$. Other differences, throughout the poset's definition, make the poset more amenable to "reverse analysis", meaning analysis by splitting the poset into a *product* of an initial segment and a tail-end. These modifications are intended to bring the poset to a form that fits with Lemma 3.10 (although parts of the "reverse analysis" will be useful already before we get to that). We cannot literally reach a poset that splits into a product of an initial segment and a tail-end; some elements of the tail-end poset cannot be brought into V and so the split cannot be viewed as a product. But we take products where we can, and in cases where composition is necessary, we identify variants of the tail-end posets that exist in V.

Suppose that each κ_n is indestructibly supercompact, and suppose moreover that there is a partial function ϕ so that for each n, $\phi\restriction\kappa_n$ is an indestructible Laver function for κ_n. By this we mean that for each $A \in V$, ordinal γ, and $<\kappa_n$ directed closed forcing extension $V[E]$ of V, there is a γ supercompactness embedding π in $V[E]$ with critical point κ_n, so that $\pi\restriction\mathrm{Ord}$ belongs to V, $\pi(\phi)(\kappa_n) = A$, and the next point in $\mathrm{dom}(\pi(\phi))$ above κ_n is greater than γ. This situation can easily be arranged if V is obtained by the standard construction of indestructibility. Thinning the domain of ϕ we may also assume that for every $\alpha \in \mathrm{dom}(\phi)$, $\gamma \in \mathrm{dom}(\phi) \cap \alpha \rightarrow \phi(\gamma) \in V_\alpha$.

For $n \geq 2$, let $\mathbb{A}_n$ be the forcing $\mathrm{Add}(\kappa_n, \kappa_{n+2})$. Let κ_0 denote ω, and let $\mathbb{A}_0 = \mathrm{Add}(\omega, \kappa_2)$. Let $\mathbb{A}_1 = \Sigma_{\mu\in\mathrm{Index}}\,\mathrm{Add}(\mu^+, \kappa_3)$, where the sum is defined to be the disjoint union of the posets, with conditions in distinct posets of the union taken to be incompatible, so that a generic for $\mathbb{A}_1$ is simply a generic for one of the posets $\mathrm{Add}(\mu^+, \kappa_2)$. In contexts where we work with such a generic, μ is determined by the generic, and we use κ_1 to denote μ^+. We will define the set Index over which the sum is taken shortly. For now we just say that all elements of Index are limit cardinals of cofinality ω, below κ_2.

Let $\mathbb{A}$ be the full support product of the posets $\mathbb{A}_n$, $n < \omega$. We use $\mathbb{A}_{[n,m]}$ to denote the poset $\prod_{n\leq i\leq m} \mathbb{A}_i$, and similarly with open and half open intervals. We use $\mathbb{A}_n\restriction\gamma$, for $\gamma < \kappa_{n+2}$, to denote the obvious restriction of $\mathbb{A}_n$, and use similar notation for generic objects and conditions, so that, for example, if G is generic for $\mathrm{Add}(\kappa_n, \kappa_{n+2})$, then $G\restriction\gamma$ consists of the first γ subsets of κ_n added by G, and is generic for $\mathrm{Add}(\kappa_n, \gamma) = \mathrm{Add}(\kappa_n, \kappa_{n+2})\restriction\gamma$.

Let $\mathbb{A}$ denote $\mathbb{A}_{[0,\omega)}$. By $\mathbb{A}\restriction\alpha$ we mean the poset $\mathbb{A}_{[0,n)} \times \mathbb{A}_n\restriction\alpha$ where n is least so that $\alpha \leq \kappa_{n+2}$.

Definition 4.1. Define a poset $\mathbb{B}$ in V and a poset $\mathbb{U}$ in the extension of V by $\mathbb{A}$, simultaneously as follows. (For notational convenience, fix A generic for

$\mathbb{A}$ over V. $\mathbb{U}$ is described in $V[A]$, and this translates naturally to a definition of a name $\dot{\mathbb{U}} \in V$ for this poset.)

1. All conditions p in $\mathbb{B}$ are functions so that $\mathrm{dom}(p) \subseteq \nu$, and for every inaccessible cardinal α, $|\mathrm{dom}(p) \cap \alpha| < \alpha$. (This parallels Easton support.) In particular, $|\mathrm{dom}(p) \cap \kappa_{n+2}| < \kappa_{n+2}$ for each n.
2. If $\alpha \in \mathrm{dom}(p)$ then α is an inaccessible cardinal, α is not equal to any κ_n, $\alpha \in \mathrm{dom}(\phi)$, and $\phi(\alpha)$ is an $(\mathbb{A}\restriction\alpha) * (\dot{\mathbb{U}}\restriction\alpha)$ name for a poset forced to be $<\!\alpha$ directed closed.
3. $p(\alpha)$ is an $(\mathbb{A}\restriction\alpha) * (\dot{\mathbb{U}}\restriction\alpha)$ name for a condition in $\phi(\alpha)$.
4. $p^* \leq p$ in $\mathbb{B}$ iff $\mathrm{dom}(p^*) \supseteq \mathrm{dom}(p)$ and for each $\alpha \in \mathrm{dom}(p)$, $\langle \emptyset, p^*\restriction\alpha\rangle$ forces in $(\mathbb{A}\restriction\alpha) * (\dot{\mathbb{U}}\restriction\alpha)$ that $p^*(\alpha) \leq p(\alpha)$.
5. $\mathbb{U} = \dot{\mathbb{U}}[A]$ has the same conditions as $\mathbb{B}$, but the richer order given by $p^* \leq p$ iff $\mathrm{dom}(p^*) \supseteq \mathrm{dom}(p)$ and there exists a condition $a^* \in A$ so that for every $\alpha \in \mathrm{dom}(p)$, $\langle a^*\restriction\alpha, p^*\restriction\alpha\rangle \Vdash_{(\mathbb{A}\restriction\alpha)*(\dot{\mathbb{U}}\restriction\alpha)} p^*(\alpha) \leq p(\alpha)$.

Remark 4.2. The condition defining the order in (5) is equivalent to the seemingly weaker condition that $\mathrm{dom}(p^*) \supseteq \mathrm{dom}(p)$ and for every $\alpha \in \mathrm{dom}(p)$ there exists $a \in A\restriction\alpha$ so that $\langle a, p^*\restriction\alpha\rangle \Vdash_{(\mathbb{A}\restriction\alpha)*(\dot{\mathbb{U}}\restriction\alpha)} p^*(\alpha) \leq p(\alpha)$. To see that the two are equivalent, suppose the seemingly weaker condition holds, and let $a^* \in A$ force this fact about p^* and p over V. Then a^* witnesses that the condition in (5) holds.

When taking a filter in $\mathbb{A} * \dot{\mathbb{U}}$, we always assume that it is strong enough on the $\mathbb{A}$ coordinate, to be generated by a set of pairs $\langle a, \check{b}\rangle$ where $a \in \mathbb{A}$ and $b \in \mathbb{B}$. (Any generic filter has this property, since any condition $\langle a, \dot{b}\rangle$ in the filter can be strengthened on the $\mathbb{A}$ coordinate to force a value for $\dot{b}$.)

Definition 4.3. Let $\beta < \nu$, and let $F \subseteq \mathbb{A} * \dot{\mathbb{U}}\restriction\beta$ be a filter. Define $\mathbb{B}^{+F}\restriction[\beta, \nu)$ to consist of conditions $p \in \mathbb{B}$ with $\mathrm{dom}(p) \subseteq [\beta, \nu)$ ordered as follows: $p^* \leq p$ iff $\mathrm{dom}(p^*) \supseteq \mathrm{dom}(p)$ and there exists $\langle a, b\rangle \in F$ so that for every $\alpha \in \mathrm{dom}(p)$, $\langle a\restriction\alpha, b \cup p^*\restriction\alpha\rangle$ forces $p^*(\alpha) \leq p(\alpha)$.

Our main initial uses of Definition 4.3 are in cases where F is generic for $\mathbb{A}\restriction\beta * \dot{\mathbb{U}}\restriction\beta$. Other uses will include situations where $F = \{\langle \emptyset, b\rangle \mid b \in B_\beta\}$ with B_β generic for $\mathbb{B}\restriction\beta$. We will also have hybrids of these two forms, where a part of F is of the first form above, and another part is of the second.

$\mathbb{U}$ itself can be viewed as a use of Definition 4.3. Let A be generic for $\mathbb{A}$ over V. Then $F = \{\langle a, \emptyset\rangle \mid a \in A\}$ is a filter contained in $\mathbb{A} * \dot{\mathbb{U}}\restriction 0$. It is easy to check that the poset $\mathbb{B}^{+F}\restriction[0, \nu)$ in this case is simply the poset $\mathbb{U}$.

Similarly, if U_β is generic for $\mathbb{U}\restriction\beta$ over $V[A]$, then $F = \{\langle a, u\rangle \mid a \in A, u \in U_\beta\}$ is a filter contained in $\mathbb{A} * \dot{\mathbb{U}}\restriction\beta$. $\mathbb{B}^{+F}\restriction[\beta, \nu)$ is a poset in $V[A][U_\beta]$. We denote it by $\mathbb{U}\restriction[\beta, \nu)$. More generally, $\mathbb{U}\restriction[\beta, \gamma)$ denotes the poset $\mathbb{B}^{+A*U_\beta}\restriction[\beta, \gamma)$. The poset belongs to the extension of V by $\mathbb{A} * \mathbb{U}\restriction\beta$. The generic $A * U_\beta$ is omitted in the notation $\mathbb{U}\restriction[\beta, \gamma)$, and is understood from the context.

Let $\mathbb{U}_0 = \mathbb{U}\restriction\kappa_2$, and for $n > 0$ let $\mathbb{U}_n = \mathbb{U}\restriction[\kappa_{n+1}, \kappa_{n+2})$. Define $\mathbb{U}_{[0,n]} = \mathbb{U}\restriction\kappa_{n+2}$, and define other interval posets similarly. $\mathbb{U}_{[0,n]}$ is a poset in $V[A_{[0,n]}]$. For $n > 0$, $\mathbb{U}_n$ is a poset in $V[A_{[0,n]} * U_{[0,n)}]$.

We sometimes use the notation $\mathbb{B}^{+F}_{[\beta,\nu)}$ for $\mathbb{B}^{+F}\restriction[\beta, \nu)$, and similarly with $\mathbb{U}$.

Recall that we left the exact definition of the set Index used in the definition of $\mathbb{A}_1$ unspecified. We now discharge our obligation to specify the set. Its definition refers to $\mathbb{A}_0$ and $\dot{\mathbb{U}}_0$, but these are both known before any use of $\mathbb{A}_1$.

Definition 4.4. Define Index to consist of all $\mu < \kappa_2$ so that:

1. μ is a strong limit cardinal of cofinality ω and $\mathrm{dom}(\phi)$ has a largest point λ below μ.
2. Over any extension $V[E]$ of V by a μ closed poset, the further extension by $\mathbb{A}_0 \restriction \lambda * \dot{\mathbb{U}}_0 \restriction \lambda + 1$ does not collapse $(\mu^+)^V$.
3. $\mathbb{A}_0 \restriction \lambda * \dot{\mathbb{U}}_0 \restriction \lambda + 1$ has size at most μ^+.

There are many μ satisfying the requirements of the definition. For example any strong limit cardinal $\mu < \kappa_2$ of cofinality ω, with largest point λ below μ in $\mathrm{dom}(\phi)$ and so that $|\phi(\lambda)| < \mu$, satisfies the requirements, as the poset $\mathbb{A} \restriction \lambda * \dot{\mathbb{U}} \restriction \lambda + 1$ in this case has size less than μ, and in particular cannot collapse μ^+ over any model. Our forcing constructions will use a slightly different situation, where $|\phi(\lambda)| = \mu^+$, but forcing with $\mathbb{A} \restriction \lambda * \dot{\mathbb{U}} \restriction \lambda + 1$ still preserves μ^+.

Claim 4.5. *Let $\bar{F} \subseteq F$ both be filters for $\mathbb{A} * \dot{\mathbb{U}} \restriction \beta$. Let $\bar{G}$ be generic for $\mathbb{B}^{+\bar{F}} \restriction [\beta, \nu)$ over some model containing $\bar{F}$ and F. Then the upward closure of $\bar{G}$ in $\mathbb{B}^{+F} \restriction [\beta, \nu)$ is generic for $\mathbb{B}^{+F} \restriction [\beta, \nu)$ over the same model.*

Proof. Note to begin with that $\mathbb{B}^{+\bar{F}} \restriction [\beta, \nu)$ and $\mathbb{B}^{+F} \restriction [\beta, \nu)$ have the same conditions, and that the latter has a richer order, immediately by their definitions. So the upward closure of $\bar{G}$ in $\mathbb{B}^{+F} \restriction [\beta, \nu)$ makes sense, and is a filter.

It is easy to check that if $q \leq_{\mathbb{B}^{+F} \restriction [\beta,\nu)} p$, then there is $r \leq_{\mathbb{B}^{+F} \restriction [\beta,\nu)} q$ so that $r \leq_{\mathbb{B}} p$. (Let $\langle a, u \rangle \in F$ witness that $q \leq_{\mathbb{B}^{+F} \restriction [\beta,\nu)} p$. Define r with the same domain as q as follows. If $\alpha \notin \mathrm{dom}(p)$, set $r(\alpha) = q(\alpha)$. For $\alpha \in \mathrm{dom}(p)$, set $r(\alpha)$ to be a name forced equal to $q(\alpha)$ by $\langle a \restriction \alpha, u \cup r \restriction \alpha \rangle$, and forced equal to $p(\alpha)$ by all conditions of $\mathbb{A} \restriction \alpha * \dot{\mathbb{U}} \restriction \alpha$ that are incompatible with $\langle a \restriction \alpha, u \cup r \restriction \alpha \rangle$.)

So every dense open set in $\mathbb{B}^{+F} \restriction [\beta, \nu)$ is dense in $\mathbb{B}^{+\emptyset} \restriction [\beta, \nu)$, hence also in $\mathbb{B}^{+\bar{F}} \restriction [\beta, \nu)$. The claim follows. ⊣

Remark 4.6. The converse of Claim 4.5 may fail in general. A generic G for $\mathbb{B}^{+F} \restriction [\beta, \nu)$ may contain conditions which are incompatible in $\mathbb{B}^{+\bar{F}} \restriction [\beta, \nu)$, and in particular it is not a filter in the latter poset, let alone a generic filter. However, by standard forcing arguments using Claim 4.5, one can force to add a refinement $\bar{G} \subseteq G$ which is a generic filter for $\mathbb{B}^{+\bar{F}} \restriction [\beta, \nu)$, and so that G is the upward closure of $\bar{G}$.

We refer to the forcing refining a generic G for $\mathbb{B}^{+F} \restriction [\beta, \nu)$ to a generic $\bar{G} \subseteq G$ for $\mathbb{B}^{+\bar{F}} \restriction [\beta, \nu)$ as the *factor forcing*. The forcing is simply the restriction of $\mathbb{B}^{+\bar{F}} \restriction [\beta, \nu)$ to conditions in G.

Claim 4.7. *Let $\bar{\beta} \leq \beta$. Suppose that $\bar{F}$ is generic for $\mathbb{A} \restriction \bar{\beta} * \dot{\mathbb{U}} \restriction \bar{\beta}$ over V. Then $\mathbb{B}^{+\bar{F}} \restriction [\beta, \nu)$ is $<\beta$ directed closed in $V[\bar{F}]$.*

Proof. Let $\tau \in V$ name a sequence in $V[\bar{F}]$ of conditions in $\mathbb{B}^{+\bar{F}} \restriction [\beta, \nu)$, of length $\delta < \beta$, that form a directed set. Without loss of generality suppose that the fact that the set is directed is forced by the empty condition in $\mathbb{A} \restriction \bar{\beta} * \dot{\mathbb{U}} \restriction \bar{\beta}$.

Let D be the union of all possible values forced for $\mathrm{dom}(\tau)_\xi$, $\xi < \delta$. β is the smallest possible element of D, and for every $\alpha > \beta$, $D \cap \alpha$ is the union of fewer than α sets which each satisfy the support requirements of condition (1) of Definition 4.1 at α. It follows that $D \cap \alpha$ too satisfies these requirements.

We now define a condition p, with domain D, that is forced to be a lower bound for all conditions τ_ξ. The definition is by induction on $\alpha \in D$. Working in V, let $p(\alpha)$ be an $\mathbb{A}{\restriction}\alpha * \dot{\mathbb{U}}{\restriction}\alpha$ name forced by $\langle \emptyset, p{\restriction}\alpha\rangle$ to be a lower bound in $\phi(\alpha)[F_\alpha]$ for the conditions $\tau_\xi[F_\alpha{\restriction}\bar{\beta}](\alpha)[F_\alpha]$. ($F_\alpha$ here indicates a generic for $\mathbb{A}{\restriction}\alpha * \dot{\mathbb{U}}{\restriction}\alpha$.) Such a name exists since by condition (2) of Definition 4.1, $\phi(\alpha)$ is forced in $\mathbb{A}{\restriction}\alpha * \dot{\mathbb{U}}{\restriction}\alpha$ to be $<\alpha$ directed closed, and, using induction and the initial assumption about τ, $\langle \emptyset, p{\restriction}\alpha\rangle$ forces $\tau_\xi[F_\alpha{\restriction}\bar{\beta}](\alpha)[F_\alpha]$ to be directed.

Then p is a lower bound in $\mathbb{B}^{+\bar{F}}{\restriction}[\beta, \nu)$ for the conditions $\tau_\xi[\bar{F}]$. $\dashv$

Remark 4.8. Let $\alpha < \kappa_{n+1}$ be a successor point of $\mathrm{dom}(\mathbb{B})$, above κ_n. (If $n \geq 1$, the set of such α is cofinal in κ_{n+1}.) The poset $(\mathbb{A}_{[0,n]} * \dot{\mathbb{U}}{\restriction}\alpha) \times \mathbb{B}^{+\emptyset}{\restriction}[\alpha, \kappa_{n+2})$ is a product of an α-c.c. poset with a $<\alpha$ closed poset. (The first factor is α-c.c. since $\mathbb{A}_{[0,n]}$ is κ_n^+-c.c. in V and $\mathbb{U}{\restriction}\alpha$ has size less than α. The second factor is $<\alpha$ closed by Claim 4.7.) By Claim 2.5, it does not collapse α. Forcing with $(\mathbb{A}_{[0,n]} * \dot{\mathbb{U}}{\restriction}\alpha) \times \mathbb{B}^{+\emptyset}{\restriction}[\alpha, \kappa_{n+2})$ subsumes forcing with $\mathbb{A}_{[0,n]} * \dot{\mathbb{U}}_{[0,n]}$, since, by Claim 4.5, the upward closure of a generic for $\mathbb{B}^{+\emptyset}{\restriction}[\alpha, \kappa_{n+2})$ provides a generic for $\mathbb{U}{\restriction}[\alpha, \kappa_{n+2})$. Hence forcing with $\mathbb{A}_{[0,n]} * \dot{\mathbb{U}}_{[0,n]}$ does not collapse α. If $n \geq 1$, this is true for cofinally many $\alpha < \kappa_{n+1}$, so forcing with $\mathbb{A}_{[0,n]} * \dot{\mathbb{U}}_{[0,n]}$ does not collapse κ_{n+1}.

Claim 4.9. *Let $A * U$ be generic for $\mathbb{A} * \dot{\mathbb{U}}$ over V. Let $\beta < \nu$ and let $F = A{\restriction}\beta * U{\restriction}\beta$. Then, in the factor poset to add a generic G for $\mathbb{B}^{+F}{\restriction}[\beta, \nu)$ that refines $U{\restriction}[\beta, \nu)$, every decreasing sequence that belongs to $V[F]$ and has length $< \beta$, has a lower bound.*

Proof. Let $\vec{p} = \langle p_\xi \mid \xi < \delta\rangle$ in $V[F]$ be a descending sequence of length $\delta < \beta$ in the factor poset, meaning that the sequence is descending in $\mathbb{B}^{+F}{\restriction}[\beta, \nu)$, and the conditions p_ξ all belong to $U{\restriction}[\beta, \nu)$.

Let $\langle a, u\rangle \in A_{[\beta,\nu)} * U_{[\beta,\nu)}$ force, over $V[F]$, that $(\forall \xi < \delta) p_\xi \in U{\restriction}[\beta, \nu)$. Then $u \leq p_\xi$ in $\mathbb{U}{\restriction}[\beta, \nu)$ for all ξ, and this is forced by a. Extending $\langle a, u\rangle$ if needed we may assume it also forces that $\vec{p}$ has no lower bound in the factor poset. In other words it forces that no lower bound for $\vec{p}$ in $\mathbb{B}^{+F}{\restriction}[\beta, \nu)$ belongs to $U{\restriction}[\beta, \nu)$.

By Claim 4.7 and since the sequence $\vec{p}$ belongs to $V[F]$, there is p which is a lower bound for $\vec{p}$ in $\mathbb{B}^{+F}{\restriction}[\beta, \nu)$.

An argument similar to that in the proof of Claim 4.5 now produces a condition $r \leq_{\mathbb{U}{\restriction}[\beta,\nu)} u$ so that $r \leq_{\mathbb{B}^{+F}{\restriction}[\beta,\nu)} p_\xi$ for all ξ. (Define r so that for each $\alpha \in \mathrm{dom}(r)$, $\langle a{\restriction}\alpha, u{\restriction}\alpha\rangle$ forces $r(\alpha) = u(\alpha)$, and all conditions incompatible with $\langle a{\restriction}\alpha, u{\restriction}\alpha\rangle$ force $r(\alpha) = p(\alpha)$.)

But then r is a lower bound for $\vec{p}$ in $\mathbb{B}^{+F}{\restriction}[\beta, \nu)$, and since $r \leq_{\mathbb{U}{\restriction}[\beta,\nu)} u$, $\langle a, u\rangle$ does not force r outside $U{\restriction}[\beta, \nu)$, contradicting the choice of $\langle a, u\rangle$. $\dashv$

Definition 4.10. Let $V[E]$ be an extension of V by a poset $\mathbb{E}$, and let $\mathbb{P} = \dot{\mathbb{P}}[E]$ be a poset in $V[E]$. Define the poset $\hat{\mathbb{P}}$ in V to consist of canonical names $\dot{p}$ forced to be elements of $\dot{\mathbb{P}}$, with $\dot{p}^* \leq_{\hat{\mathbb{P}}} \dot{p}$ iff $\Vdash^V_{\mathbb{E}} \dot{p}^* \leq \dot{p}$.

Claim 4.11. *Let $\dot{\mathbb{P}}$ and $\hat{\mathbb{P}}$ be as in Definition 4.10.*

1. *If $\dot{\mathbb{P}}$ is forced to be $<\alpha$ directed closed, then $\hat{\mathbb{P}}$ is $<\alpha$ directed closed in V.*
2. *Let $\hat{G}$ be generic for $\hat{\mathbb{P}}$ over a model that contains $V[E]$. Then the upward closure of $\{\dot{p}[E] \mid \dot{p} \in \hat{G}\}$ in $\dot{\mathbb{P}}[E]$ is generic for $\mathbb{P}$ over the same model.*

Proof. Similar to the proofs of Claims 4.7 and 4.5. ⊣

Lemma 4.12. *Let $n < \omega$. Let $A * U_{[0,n]}$ be generic for $\mathbb{A} * \dot{\mathbb{U}}_{[0,n]}$ over V. Then in $V[A][U_{[0,n]}]$, κ_{n+2} is generically supercompact, and this supercompactness is indestructible under forcing with posets in $V[A_{[0,n]}][U_{[0,n]}]$ that are $<\kappa_{n+2}$ directed closed in $V[A_{[0,n]}][U_{[0,n]}]$.*

The forcing notion producing the generic supercompactness embedding is isomorphic to $\mathrm{Add}(\kappa_n, \pi(\kappa_{n+2}))^V \times \mathrm{Add}(\kappa_{n+1}, \pi(\kappa_{n+3}))^V$*, where π is the embedding produced.*

Precisely the statement of the lemma means the following. Let $\mathbb{P}$ be $<\kappa_{n+2}$ directed closed in $V[A_{[0,n]}][U_{[0,n]}]$. Let G be generic for $\mathbb{P}$ over $V[A][U_{[0,n]}]$. Then for each γ there is, in an extension of $V[A][U_{[0,n]}][G]$, an elementary embedding $\pi\colon V[A][U_{[0,n]}][G] \to V^*[A^*][U^*_{[0,n]}][G^*]$ so that $\mathrm{crit}(\pi) = \kappa_{n+2}$, $\pi(\kappa_{n+2}) > \gamma$, $\pi\restriction\mathrm{Ord}$ belongs to V, and $V^*[A^*][U^*_{[0,n]}][G^*]$ is γ closed in the generic extension producing the embedding. The generic extension producing the embedding is an extension of $V[A][U_{[0,n]}][G]$ by $\mathrm{Add}(\kappa_n, \pi(\kappa_{n+2}))^V \times \mathrm{Add}(\kappa_{n+1}, \pi(\kappa_{n+3}))^V$.

Proof of Lemma 4.12. Fix γ. Let $\dot{\mathbb{P}} \in V$ name $\mathbb{P}$. Using the fact that ϕ is an indestructible Laver function, find a γ supercompactness embedding $\pi\colon V[A_{[n+2,\omega)}] \to V^*[A^*_{[n+2,\omega)}]$, in $V[A_{[n+2,\omega)}]$, with $\pi\restriction\mathrm{Ord}$ in V, $\mathrm{crit}(\pi) = \kappa_{n+2}$, and $\pi(\phi)(\kappa_{n+2}) = \dot{\mathbb{P}}$.

Increasing γ if needed, We may pick π so that γ^{++} is a fixed point of the embedding. In particular then the set $\{$dense subsets of $\pi(\mathbb{B})^{+\emptyset}\restriction(\kappa_{n+2}, \pi(\kappa_{n+2}))$ that belong to $V^*\}$ has cardinality γ^+ in V. Using this, the fact that the first point in $\mathrm{dom}(\pi(\phi))$ above κ_{n+2} is greater than γ, and the closure given by Claim 4.7, one can construct, in $V[A_{[n+2,\omega)}]$, a filter $\hat{B}$ which is generic for $\pi(\mathbb{B})^{+\emptyset}\restriction(\kappa_{n+2}, \pi(\kappa_{n+2}))$ over $V^*[A^*_{[n+2,\omega)}]$. (Claim 4.7 is applied in V^* with $\bar{\beta} = 0$. It shows that the poset $\pi(\mathbb{B})^{+\emptyset}\restriction(\kappa_{n+2}, \nu)$ is γ closed in V^*, and therefore so is $\pi(\mathbb{B})^{+\emptyset}\restriction(\kappa_{n+2}, \pi(\kappa_{n+2}))$. This closure transfers to $V[A_{[n+2,\omega)}]$, since V^* is itself γ closed in this model. In $V[A_{[n+2,\omega)}]$ one can then enumerate the dense sets that belong to $V^*[A^*_{[n+2,\omega)}]$ and meet all of them through a construction of length γ^+.)

Let $\hat{\mathbb{P}}$ be the forcing notion associated to $\dot{\mathbb{P}}$ by Definition 4.10. By Claim 4.11, $\hat{\mathbb{P}}$ is $<\kappa_{n+2}$ directed closed in V. By elementarity of π it follows that $\pi(\hat{\mathbb{P}})$ is $<\pi(\kappa_{n+2})$ directed closed in V^*, and this implies that it is γ closed in $V[A_{[n+2,\omega)}]$. Working in $V[A_{[n+2,\omega)}]$ we can therefore find $\hat{G}^*$ which is generic for $\pi(\hat{\mathbb{P}})$ over $V^*[A^*_{[n+2,\omega)}][\hat{B}]$. We build $\hat{G}^*$ below a specific condition $\dot{p}^*_0$ in $\hat{\mathbb{P}}$. We will say what this condition is later on.

Let $\hat{A}_n$ be generic for $\pi(\mathbb{A}_n)\restriction[\kappa_{n+2}, \pi(\kappa_{n+2}))$ over $V[A][U_{[0,n]}][G]$. Similarly let $\hat{A}_{n+1}$ be generic for $\pi(\mathbb{A}_{n+1})\restriction(\pi(\kappa_{n+3}) - \pi''\kappa_{n+3}))$ over $V[A][U_{[0,n]}][G][\hat{A}_n]$.

(These posets are isomorphic to $\mathrm{Add}(\kappa_n, \pi(\kappa_{n+2}))^V$ and $\mathrm{Add}(\kappa_{n+1}, \pi(\kappa_{n+3}))^V$ respectively.)

Then A_n and $\hat{A}_n$ can be joined to form a generic A^*_n for $\pi(\mathbb{A}_n)$, and similarly A_{n+1} and $\hat{A}_{n+1}$ can be joined to form a generic A^*_{n+1} for $\pi(\mathbb{A}_{n+1})$. Let $A^*_{[0,n+1]}$ be the resulting sequence $\langle A_0, \dots, A_{n-1}, A^*_n, A^*_{n+1}\rangle$. It is clear that $\pi\colon V[A_{[n+2,\omega)}] \to V^*[A^*_{[n+2,\omega)}]$ now extends to an embedding, which we also denote π, from $V[A]$ to $V^*[A^*]$.

$U_{[0,n]}$, G, and the upward closure of $\hat{B}$ in $\pi(\mathbb{U})\restriction(\kappa_{n+2}, \pi(\kappa_{n+2}))$ can be joined to form a generic $U^*_{[0,n]}$ for $\pi(\mathbb{U}\restriction\kappa_{n+2})$. It is clear that π extends further, to an embedding of $V[A][U_{[0,n]}]$ to $V^*[A^*][U^*_{[0,n]}]$.

Since $\pi''\restriction V$ belongs to $V[A_{[n+2,\omega)}]$, V^* is γ closed in $V[A_{[n_2,\omega)}]$, and G is part of the generic $U^*_{[0,n]}$, $\pi''G$ belongs to $V^*[A^*_{[0,n]}][U^*_{[0,n]}]$. It follows from this and the directed closure of $\pi(\mathbb{P})$ in $V^*[A^*_{[0,n]}][U^*_{[0,n]}]$ that $\pi''G$ has a lower bound in $\pi(\mathbb{P})$. Let $\dot{p}^*_0 \in V^*$ name such a lower bound. Note that $\dot{p}^*_0$ can be defined without reference to $A^*_{[0,n]}$ and $U^*_{[0,n]}$, and in particular with no reference to $\hat{A}_n$ and $\hat{A}_{n+1}$, so it could have defined earlier in the proof, before fixing $\hat{G}^*$. We may therefore assume that $\dot{p}^*_0$ belongs to $\hat{G}^*$.

So far we extended π to an embedding of $V[A][U_{[0,n]}]$ into $V^*[A^*][U^*_{[0,n]}]$. $\hat{G}^*$ is generic for $\pi(\hat{\mathbb{P}})$ over $V^*[A^*_{[n+2,\omega)}][\hat{B}]$. From this and the genericity of $A^*_{[0,n+1]}$, $U_{[0,n]}$, and G over $V^*[A^*_{[n+2,\omega)}][\hat{B}][\hat{G}^*]$ (indeed these objects are generic over $V[A_{[n+2,\omega)}]$, which contains $V^*[A^*_{[n+2,\omega)}][\hat{B}][\hat{G}^*]$), it follows that $\hat{G}^*$ is generic over $V^*[A^*_{[n+2,\omega)}][\hat{B}][A^*_{[0,n+1]}][U_{[0,n]}][G]$. Hence $\hat{G}^*$ is generic also over (the smaller model) $V^*[A^*][U^*_{[0,n]}]$.

By Claim 4.11 it follows that the upward closure of $\{\dot{p}[A^*_{[0,n]}][U^*_{[0,n]}] \mid \dot{p} \in \hat{G}^*\}$ is generic for $\pi(\mathbb{P})$ over $V^*[A^*][U^*_{[0,n]}]$. Let G^* denote this upward closure. Since $\dot{p}^*_0[A^*_{[0,n]}][U^*_{[0,n]}]$ is a lower bound for $\pi''G$, G^* contains $\pi''G$. So π extends, finally, to an embedding of $V[A][U_{[0,n]}][G]$ into $V^*[A^*][U^*_{[0,n]}][G^*]$. $\dashv$

The definition of $\mathbb{B}$ and $\mathbb{U}$ was designed specifically to lead to Lemma 4.12. We continue now with similar definitions of posets that collapse all cardinals between κ_{n+1} and κ_{n+2} to κ_{n+1}, and do so in a way that allows us to argue for the tree property at κ_{n+2}.

Definition 4.13. For each $n < \omega$ define a poset $\mathbb{C}_n$ in V as follows. Conditions in $\mathbb{C}_n$ are functions p so that:

1. $\mathrm{dom}(p)$ is contained in the interval $[\kappa_{n+1}, \kappa_{n+2})$, and $|\mathrm{dom}(p)| < \kappa_{n+1}$.
2. For each $\alpha \in \mathrm{dom}(p)$, $p(\alpha)$ is an $(\mathbb{A}\restriction\alpha) * \dot{\mathbb{U}}\restriction\kappa_{n+1}$ name for a condition in the poset $\mathrm{Add}(\kappa_{n+1}, 1)$ of the extension by $(\mathbb{A}\restriction\alpha) * \dot{\mathbb{U}}\restriction\kappa_{n+1}$.

Conditions are ordered as follows: $p^* \leq p$ iff $\mathrm{dom}(p^*) \supseteq \mathrm{dom}(p)$, and for each $\alpha \in \mathrm{dom}(p)$, it is forced (by the empty condition) in $(\mathbb{A}\restriction\alpha) * \dot{\mathbb{U}}\restriction\kappa_{n+1}$ that $p^*(\alpha) \leq p(\alpha)$.

If $n \geq 1$, then $\mathbb{U}\restriction\kappa_{n+1}$ is simply $\mathbb{U}_{[0,n)}$. In this case the poset $(\mathbb{A}\restriction\alpha) * \dot{\mathbb{U}}\restriction\kappa_{n+1}$ used in the definition can also be written as $(\mathbb{A}_{[0,n)} * \dot{\mathbb{U}}_{[0,n)}) \times \mathbb{A}_n\restriction\alpha$. (If $n = 0$ this is not quite a precise match, since $\mathbb{U}\restriction\kappa_1$ is part of $\mathbb{U}_0$.)

Let $\mathbb{C}$ be the full support product of the posets $\mathbb{C}_n$. We use interval notation in the usual way, so that for example, $\mathbb{C}\restriction[\kappa_{n+1}, \kappa_{n+2})$ is $\mathbb{C}_n$, and $\mathbb{C}\restriction[\kappa_{n+1}, \nu)$ is $\mathbb{C}_{[n,\omega)}$.

Definition 4.14. For a filter $F \subseteq \mathbb{A} * \dot{\mathbb{U}}$ define the *enrichment* of $\mathbb{C}$ to F, denoted $\mathbb{C}^{+F}$, to be the poset with the same conditions as $\mathbb{C}$, but the richer order given by $p^* \leq p$ iff there exists a condition $\langle a, u\rangle \in F$ so that for each $\alpha \in \mathrm{dom}(p)$, $\langle a\restriction\alpha, u\restriction\kappa_i\rangle \Vdash_{\mathbb{A}\restriction\alpha*\dot{\mathbb{U}}\restriction\kappa_i} p^*(\alpha) \leq p(\alpha)$, where i is largest so that $\kappa_i \leq \alpha$.

The poset we intend to use is the enrichment $\mathbb{C}^{+A*U}$ where $A * U$ is generic for $\mathbb{A} * \dot{\mathbb{U}}$ over V. We will refer to intervals of this poset, for example $\mathbb{C}_n^{+A*U} = \mathbb{C}^{+A*U}\restriction[\kappa_{n+1}, \kappa_{n+2})$. In such references only $A\restriction\kappa_{n+2} * U\restriction\kappa_{n+1}$ is relevant to the enrichment, but to reduce notational clutter we still use the superscript $+A * U$.

The definition of $\mathbb{C}$ and $\mathbb{C}^{+A*U}$ is similar to the corresponding the definition of $\mathbb{B}$ and $\mathbb{U}$, except that the underlying posets used at each coordinate α are different, the support is different, and there is no self-reference, meaning that the ordering at coordinate α does not rely on the restriction of the conditions ordered to α. The definition of $\mathbb{C}$ is simpler than the simultaneous definition of $\mathbb{B}$ and $\mathbb{U}$, because there is no need to deal with self-reference here.

Note that the definition of $\mathbb{C}_0$ makes a reference to κ_1. In contexts where we have a generic A_1 for $\mathbb{A}_1$, κ_1 is determined by this generic. In other contexts, κ_1 is a parameter in the definition of $\mathbb{C}_0$. We sometimes refer to the poset as $\mathbb{C}_0(\kappa_1)$, when κ_1 is not understood from the context.

Claim 4.15.

1. *Let F be generic for $\mathbb{A}\restriction\beta * \dot{\mathbb{U}}\restriction\beta$ for $\beta \leq \kappa_{n+1}$. Then the poset $\mathbb{C}^{+F}\restriction[\kappa_{n+1}, \nu)$ is $<\kappa_{n+1}$ directed closed in $V[F]$.*
2. *Let $\beta \in (\kappa_{n+1}, \kappa_{n+2})$ and let F be generic for $\mathbb{A}\restriction\beta * \dot{\mathbb{U}}\restriction\kappa_{n+1}$. Then the poset $\mathbb{C}^{+F}\restriction[\beta, \kappa_{n+2})$ is $<\kappa_{n+1}$ directed closed in $V[F]$.*

Proof. Similar to Claim 4.7, except that (a) the amount of closure here in condition (2) is lower, because the underlying poset at each coordinate α is only forced to be $<\kappa_{n+1}$ directed closed; and (b) the domain D of the lower bound must be defined more carefully, since it is required here to have size $< \kappa_{n+1}$, a stricter demand than the support restrictions in the case of Claim 4.7.

We indicate how to obtain the domain D, in the harder of the two cases of the claim, case (2), and leave the remaining details to the reader.

Let $\tau \in V$ name a sequence in $V[F]$ of conditions in $\mathbb{C}^{+F}\restriction[\beta, \kappa_{n+2})$, of length $\delta < \kappa_{n+1}$, that form a directed set.

Suppose to begin with that $n \geq 1$. By Remark 4.8, κ_{n+1} is not collapsed in $V[F]$. Since τ is forced to be a sequence of length $< \kappa_{n+1}$ it follows that there is $\eta < \kappa_{n+1}$ so that the restriction of F to $\mathbb{A}\restriction\beta * \dot{\mathbb{U}}\restriction\eta$ is sufficient to interpret τ. Let $\langle a, u\rangle$ in F force this.

Increasing η we may assume it is a successor point in $\mathrm{dom}(\phi)$ and greater than κ_n, so that $\mathbb{A}\restriction\beta * \dot{\mathbb{U}}\restriction\eta$ is η-c.c. in V. Below $\langle a, u\rangle$, there are then fewer than η possible values for the domain of τ_ξ for each $\xi < \delta$. The set D equal to the union of these possible values over all $\xi < \delta$ then has size less than κ_{n+1}.

Suppose next that $n = 0$. By Definition 4.4 and since $\kappa_1 = \mu^+$ for some $\mu \in \mathrm{Index}$, $\mathbb{A}\restriction\lambda * \dot{\mathbb{U}}\restriction\lambda + 1$ has size at most κ_1, and does not collapse κ_1, where

λ is the largest point of $\mathrm{dom}(\phi)$ below μ. $\mathbb{U}\restriction\kappa_1$ is equal to $\mathbb{U}\restriction\lambda+1$, so the full poset $\mathbb{A}\restriction\beta * \dot{\mathbb{U}}\restriction\kappa_1$ is equal to $(\mathbb{A}\restriction\lambda * \dot{\mathbb{U}}\restriction\lambda+1) \times \mathrm{Add}(\omega, [\lambda, \beta))$. Since $\mathrm{Add}(\omega, [\lambda, \beta))$ is κ_1-c.c. over any model that preserves κ_1, the full poset does not collapse κ_1. As in the case of $n > 0$ it now follows that $\tau[F]$ can be determined from the restriction of F to $\mathrm{Add}(\omega, [\lambda, \beta))$ and some part of $\mathbb{A}\restriction\lambda * \dot{\mathbb{U}}\restriction\lambda + 1$ of size $\mu < \kappa_1$. Again as in the case of $n > 0$, this allows bounding the union of possible domains for τ_ξ by a set of size $< \kappa_1$. ⊣

Claim 4.16. *Let $A * U$ be generic for $\mathbb{A} * \dot{\mathbb{U}}$ over V, and let S be generic for $\mathbb{C}^{+A*U}$ over $V[A * U]$. Let $n < \omega$ and let $F = A\restriction\kappa_{n+2} * U\restriction\kappa_{n+2}$. Then, in the factor poset to add a generic G for $\mathbb{C}^{+F}\restriction[\kappa_{n+2}, \nu)$ that refines $S\restriction[\kappa_{n+2}, \nu)$, every decreasing sequence that belongs to $V[F]$ and has length $< \kappa_{n+2}$, has a lower bound.*

Proof. Similar to Claim 4.9 (with $\beta = \kappa_{n+2}$). ⊣

Claim 4.17. *Let $A\restriction\kappa_{n+2} * U\restriction\kappa_{n+1}$ be generic for $\mathbb{A}\restriction\kappa_{n+2} * \mathbb{U}\restriction\kappa_{n+1}$ over V. Then forcing with $\mathbb{C}^{+A\restriction\kappa_{n+2}*U\restriction\kappa_{n+1}}\restriction[\kappa_{n+1}, \kappa_{n+2})$ over $V[A\restriction\kappa_{n+2} * U\restriction\kappa_{n+1}]$ collapses all cardinals between κ_{n+1} and κ_{n+2} to κ_{n+1}.*

Proof. Let S_n be generic for $\mathbb{C}^{+A\restriction\kappa_{n+2}*U\restriction\kappa_{n+1}}\restriction[\kappa_{n+1}, \kappa_{n+2})$. For each $\alpha \in [\kappa_{n+1}, \kappa_{n+2})$, let $x_\alpha = \bigcup_{p\in S_n} p(\alpha)[A\restriction\alpha * U\restriction\kappa_{n+1}]$. Then by the definition of $\mathbb{C}^{+A\restriction\kappa_{n+2}*U\restriction\kappa_{n+1}}$ and genericity, x_α is a subset of κ_{n+1}, added generically using bounded initial segments that belong to $V[A\restriction\alpha * U\restriction\kappa_{n+1}]$. In $V[A\restriction\alpha]$ there are at least α subsets of κ_n. (This is because $\mathbb{A}_n\restriction\alpha = \mathrm{Add}(\kappa_n, \alpha)$.) By genericity, each of these occurs as a segment of x_α. Since x_α is a subset of κ_{n+1}, it follows that α is collapsed to κ_{n+1}. ⊣

Let A be generic for $\mathbb{A}$ over V, let U be generic for $\mathbb{U}$ over $V[A]$, and let S be generic for $\mathbb{C}^{+A*U}$ over $V[A][U]$. Let e be generic over $V[A][U][S]$ for the poset $\mathrm{Col}(\omega, \mu)^V$. (Recall that the generic A_1 selects μ. A_1 is generic for $\mathrm{Add}(\kappa_1, \kappa_3)$ where $\kappa_1 = \mu^+$.) We intend to show that in the extension $V[A][U][S][e]$, κ_n is $\aleph_n$ for each n, and the tree property holds at κ_{n+2}.

We begin by determining the cardinals of the model. For this we use a reverse analysis of the forcing. Let C refine S to a generic for $\mathbb{C}$ over $V[A][U][e]$. Let B refine U to a generic for the product $\mathbb{B}\restriction\kappa_1 \times \prod \mathbb{B}^{+\emptyset}\restriction[\kappa_{n+1}, \kappa_{n+2})$ over $V[A][C][e]$. (This product is not the same as $\mathbb{B}$, since $\mathbb{B}$ is not a product of its coordinates.) Then $V[A][U][S][e] \subseteq V[A][B][C][e]$. (Indeed, U is the upward closure of B in $\mathbb{U}$, and S is the upward closure of C in $\mathbb{C}^{+A*U}$.) $V[A][B][C][e]$ is a product of its segments between successive κ_ns, rather than a composition, and therefore easier to analyze.

Claim 4.18. *Let $n < \omega$. Then $V_{n+2} = V[A_{[n+2,\omega)} \times B\restriction[\kappa_{n+2}, \nu) \times C\restriction[\kappa_{n+2}, \nu)]$ is a $<\kappa_{n+2}$ closed extension of V.*

Proof. Closure is clear for $\mathbb{A}_{[n+2,\omega)}$, holds by Claim 4.7 (with $\bar{\beta} = 0$) for $\mathbb{B}^{+\emptyset}\restriction[\kappa_{n+2}, \nu)$, and by part (1) of Claim 4.15 (with $\beta = 0$) for $\mathbb{C}\restriction[\kappa_{n+2}, \nu)$. ⊣

Let $\mathbb{B}_n$ denote $\mathbb{B}^{+\emptyset}\restriction[\kappa_{n+1}, \kappa_{n+2})$, and let $B_n = B\restriction[\kappa_{n+1}, \kappa_{n+2})$, so that B_n is generic for $\mathbb{B}_n$, and $B = B\restriction\kappa_1 \times \prod B_n$. (With this indexing, U_0 is an upward closure of $B\restriction\kappa_1 \times B_0$, and for $n \geq 1$, U_n is an upward closure of B_n.)

Claim 4.19. *κ_{n+2} is an inaccessible cardinal in $V_{n+2}[B_n]$. Moreover V_{n+2} has the $<\tau$ covering property in $V_{n+2}[B_n]$, for every cardinal $\tau \geq \kappa_{n+2}$ of V_{n+2}. In particular the extension does not collapse any cardinals above κ_{n+2}.*

Proof. For any successor point $\alpha \in \mathrm{dom}(B_n)$, $\mathbb{B}^{+\emptyset} \restriction [\kappa_{n+1}, \alpha) \times \mathbb{B}^{+\emptyset} \restriction [\alpha, \kappa_{n+2})$ subsumes $\mathbb{B}_n$ by Claim 4.5. (Given a generic $B_{[\kappa_{n+1},\alpha)} \times \bar{G}$ for the product, the claim is used to convert $\bar{G}$ to a generic for $\mathbb{B}^{+B_{[\kappa_{n+1},\alpha)}} \restriction [\alpha, \kappa_{n+2})$ that can then be appended to $B_{[\kappa_{n+1},\alpha)}$.) This is a product of a poset which has size less than α (because α is a successor point in $\mathrm{dom}(B_n)$), with a poset which is $<\alpha$ closed (by Claim 4.7). It follows that α remains a cardinal in the extension by this product, that the cofinality of κ_{n+2} is not changed to be smaller than α, and that there are at most α bounded subsets of α in the extension. It also follows that every subset of V_{n+2} of size $< \alpha$ in the extension is contained in a set of size less than α in V_{n+2}. Since the product subsumes $\mathbb{B}_n$, all these claims hold also for the extension by B_n. Taken together for all successor points $\alpha \in \mathrm{dom}(B_n)$ they imply that κ_{n+2} is inaccessible in $V_{n+2}[B_n]$, and that V_{n+2} has the $<\kappa_{n+2}$ covering property in $V_{n+2}[B_n]$. Finally, since the forcing notion adding B_n has size κ_{n+2}, every subset of V_{n+2} of size $\lambda \geq \kappa_{n+2}$ in $V_{n+2}[B_n]$ is contained in a set of the same size in V_{n+2} $\dashv$

Claim 4.20. *$\mathbb{C}_n$ is κ_{n+2}-c.c. in $V_{n+2}[B_n]$. In particular, no cardinals $\geq \kappa_{n+2}$ are collapsed in the extension of $V_{n+2}[B_n]$ by C_n, and $V_{n+2}[B_n]$ has the $<\tau$ covering property in the extension, for every cardinal $\tau \geq \kappa_{n+2}$ of $V_{n+2}[B_n]$.*

Proof. By Claims 4.18 and 4.19, V has the $<\kappa_{n+2}$ covering property in $V_{n+2}[B_n]$. Since all conditions in $\mathbb{C}_n$ are functions in V with domain of size $< \kappa_{n+1}$ in V, it follows by Claim 2.2 that any antichain of $\mathbb{C}_n$ of size κ_{n+2} in $V_{n+2}[B_n]$ can be refined to an antichain of the same size, with conditions whose domains form a Δ system. Letting r be the root of the system this implies that there are κ_{n+2} pairwise incompatible conditions in $\mathbb{C}_n$ with domain r. But as $\sup(r) < \kappa_{n+2}$, this contradicts the definition of $\mathbb{C}_n$ and the fact that κ_{n+2} is inaccessible in $V_{n+2}[B_n]$. $\dashv$

Claim 4.21. *$\mathbb{A}_{n+1}$ is κ_{n+2}-c.c. in $V_{n+2}[B_n][C_n]$. In particular no cardinals $\geq \kappa_{n+2}$ are collapsed in the extension of $V_{n+2}[B_n][C_n]$ by A_{n+1}, and the model has the $<\tau$ covering property in the extension, for every cardinal $\tau \geq \kappa_{n+2}$ of the model.*

Proof. Similar to the proof of Claim 4.20, using the fact that, by Claims 4.18, 4.19, and 4.19, V has the $<\kappa_{n+2}$ covering property in $V_{n+2}[B_n][C_n]$. $\dashv$

Corollary 4.22. *For $n \geq 2$, κ_n is a cardinal in $V_1 = V[A_{[1,\omega)}][B \restriction [\kappa_1, \nu)][C]$, and V has the $<\kappa_n$ covering property in V_1.*

Proof. Immediate working through the extensions in reverse, using Claims 4.18, 4.19, 4.20, and 4.21. $\dashv$

Claim 4.23. *$V_1 = V[A_{[1,\omega)}][B \restriction [\kappa_1, \nu)][C]$ is a $<\kappa_1$ closed extension of V, and in particular κ_1 is a cardinal in this extension.*

Proof. $V_2 = V[A_{[2,\omega)}][B_{[1,\omega)}][C_{[1,\omega)}]$ is a $<\kappa_2$ closed extension of V by Claim 4.18. The posets $\mathbb{A}_1$, $\mathbb{B}_0$, and $\mathbb{C}_0$ are $<\kappa_1$ closed in V, hence also in V_2, so $V[A_{[1,\omega)}][B_{[\kappa_1,\nu)}][C_{[0,\omega)}]$ is a $<\kappa_1$ closed extension of V. $\dashv$

Lemma 4.24. *In the extension* $V[A][U][S][e]$, $\kappa_n = \aleph_n$ *for each* n, *and* V *has the* $<\!\kappa_n$ *covering property for all* $n \geq 2$. *The same is true in the larger extension* $V[A][U\restriction\kappa_1][B\restriction[\kappa_1,\nu)][C][e]$.

Proof. By Corollary 4.22, κ_n remains a cardinal in $V[A_{[1,\omega)}][B\restriction[\kappa_1,\nu)][C]$ for each $n \geq 2$, and V has the $<\!\kappa_n$ covering property in this model. Since $\mathbb{A}_0 = \mathrm{Add}(\omega,\kappa_{n+2})^V$ is ω_1-c.c. in this model, the poset $\mathrm{Col}(\omega,\mu)$ leading to e has size $\mu < \kappa_1$, and the poset $\mathbb{U}\restriction\kappa_1$ has size at most κ_1 by the requirements in Definition 4.4, the same is true of the model $V[A_{[1,\omega)}][B\restriction[\kappa_1,\nu)][C][A_0][e][U\restriction\kappa_1] = V[A][B\restriction[\kappa_1,\nu)][C][e][U\restriction\kappa_1]$.

By Claim 4.23, $V[A_{[1,\omega)}][B\restriction[\kappa_1,\nu)][C]$ is a $<\!\kappa_1$ closed extension of V, and in particular κ_1 is a cardinal in this extension. Recall that $\kappa_1 = \mu^+$ for some μ which belongs to the set Index given in Definition 4.4. By definition of Index, this implies that there is a largest point λ in $\mathrm{dom}(\phi)$ below μ (equivalently largest below κ_1, as $\mathrm{dom}(\phi)$ includes only inaccessible cardinals), so that forcing with $\mathbb{A}\restriction\lambda * \dot{\mathbb{U}}\restriction\lambda+1$ over any $<\!\kappa_1$ closed extension of V, does not collapse κ_1. So κ_1 remains a cardinal in $V[A_{[1,\omega)}][B\restriction[\kappa_1,\nu)][C][A\restriction\lambda][U\restriction\lambda+1]$. Since there are no points in the domains of conditions of B between λ and μ, $U\restriction\lambda+1$ is the same as $U\restriction\kappa_1$. Since $\mathbb{A}_0$ and $\mathrm{Col}(\omega,\mu)$ are κ_1-c.c. in any model where κ_1 is a cardinal, the addition of $A_0\restriction[\lambda,\kappa_2)$ and e does not collapse κ_1. It follows that κ_1 is a cardinal in $V[A_{[1,\omega)}][B\restriction[\kappa_1,\nu)][C][e][A\restriction\kappa_2][U\restriction\kappa_1] = V[A][B\restriction[\kappa_1,\nu)][C][e][U\restriction\kappa_1]$. $\kappa_0 = \omega$ is of course a cardinal in the model too. The addition of e, $A\restriction\kappa_2$, and $U\restriction\kappa_1$ does not destroy the $<\!\tau$ covering property for any $\tau \geq \kappa_2$, since these objects are added by posets which are κ_1-c.c. or of size κ_1.

We showed so far that κ_n is a cardinal in $V[A][B\restriction[\kappa_1,\nu)][C][e][U\restriction\kappa_1]$ for all n, and that V has the $<\!\kappa_n$ covering property in this model for $n \geq 2$. These properties transfer to the smaller model $V[A][U\restriction\kappa_1][U\restriction[\kappa_1,\nu)][S][e] = V[A][U][S][e]$. To complete the proof of the lemma, it is enough to show that for every $n < \omega$, all cardinals between κ_n and κ_{n+1} are collapsed to κ_n in this model. For $n \geq 1$ this is true by Claim 4.17, and for $n = 0$ it is true because e collapses μ, the predecessor of κ_1, to $\omega = \kappa_0$. $\dashv$

Remark 4.25. Recall that $\nu = \sup \kappa_n$. It follows from Lemma 4.24 that ν^+ is not collapsed in the extensions $V[A][U][S][e]$ and $V[A][U\restriction\kappa_1][B\restriction[\kappa_1,\nu)][C][e]$. Since the posets leading to these extensions have size ν^+, no greater cardinals are collapsed either. Note that the proof of Lemma 4.24 and the claims leading to it could be repeated over any ν closed extension $V[E]$ of V, with no change. It follows that for any such E, ν^+ is not collapsed by the forcing to add $A*U*S*e$ over $V[E]$, and similarly it is not collapsed by the forcing to add $A*U\restriction\kappa_1 * B\restriction[\kappa_1,\nu) * C * e$ over $V[E]$.

Lemma 4.26. *For each* $n < \omega$, *all sequences of ordinals of length* $< \kappa_{n+1}$ *in* $V[A][U][S][e]$ *belong to* $V[A\restriction\kappa_{n+2}][U\restriction\kappa_{n+1}][S\restriction\kappa_{n+1}][e]$.

Proof. Let f be a sequence of ordinals of length $< \kappa_{n+1}$ in $V[A][U][S][e]$. Then f belongs to $V[A][U\restriction\kappa_{n+1}][B\restriction[\kappa_{n+1},\nu)][S\restriction\kappa_{n+1}][C\restriction[\kappa_{n+1},\nu)][e]$, in other words to $V_{n+1}[A_{[0,n]}][U\restriction\kappa_{n+1}][S\restriction\kappa_{n+1}][e]$. Let $\dot{f} \in V_{n+1}$ be a name so that $f = \dot{f}[A_{[0,n]}][U\restriction\kappa_{n+1}][S\restriction\kappa_{n+1}][e]$.

Since κ_{n+1} is a cardinal in $V_{n+1}[A_{[0,n]}][U\restriction\kappa_{n+1}][S\restriction\kappa_{n+1}][e]$, and since the length of f is smaller than κ_{n+1}, there is $\delta < \kappa_{n+1}$ so that the parts of $U\restriction\kappa_{n+1}$ and $S\restriction\kappa_{n+1}$ needed to interpret $\dot{f}$ are just $U\restriction\delta$ and $S\restriction\delta$. (In case $n = 0$, where $\delta < \kappa_1$, $U\restriction\delta$ means the restriction of U to a subset of $\dot{\mathbb{U}}$ of size μ. We can find such a restriction, which still suffices to interpret $\dot{f}$, because of the properties of $\kappa_1 = \mu^+$ given by the definition of the set Index, specifically condition (3) in Definition 4.4.)

Since $\mathbb{A}_{[0,n]}$ is κ_{n+1}-c.c. in V_{n+1}, and the poset giving rise to e, $\mathrm{Col}(\omega,\mu)$, has size $\mu < \kappa_{n+1}$, it follows using these restrictions that $\dot{f}$ can be replaced by a name of size $< \kappa_{n+1}$ in V_{n+1}. By Claim 4.18, or Claim 4.23 if $n = 0$, it follows that $\dot{f}$ belongs to V. Hence f belongs to $V[A_{[0,n]}][U\restriction\kappa_{n+1})][S\restriction\kappa_{n+1}][e]$. $\dashv$

Remark 4.27. It follows from the proof of Lemma 4.26 that if $\mathbb{Q} \in V$ is $<\kappa_{n+1}$ closed in V, then forcing with $\mathbb{Q}$ over $V[A][U][S][e]$ does not add sequences of ordinals of length $< \kappa_{n+1}$. So see this, let Q be generic for $\mathbb{Q}$ over $V[A][U][S][e]$, and repeat the proof of Lemma 4.26 using $V_{n+1}[Q]$ instead of V_{n+1} throughout. ($V_{n+1}[Q]$ is a $<\kappa_{n+1}$ closed extension of V, by the closure of $\mathbb{Q}$, and this is all that the proof required.) The proof shows that any sequence of ordinals of length $< \kappa_{n+1}$ in $V[A][U][S][e][Q]$ belongs to $V[A\restriction\kappa_{n+2}][U\restriction\kappa_{n+1}][S\restriction\kappa_{n+1}][e]$, and in particular it belongs to $V[A][U][S][e]$.

Claim 4.28. *In $V[A][U][S][e]$, $2^{\kappa_n} = \kappa_{n+2}$ for each n.*

Proof. It is clear that $2^{\kappa_n} \geq \kappa_{n+2}$, since A_n adds κ_{n+2} subsets of κ_n. For the reverse direction, it is enough by Lemma 4.26 to show that $2^{\kappa_n} \leq \kappa_{n+2}$ in the extension $V[A\restriction\kappa_{n+2}][U\restriction\kappa_{n+1}][S\restriction\kappa_{n+1}][e]$.

The extension $V[A\restriction\kappa_{n+1}][U\restriction\kappa_{n+1}][S\restriction\kappa_{n+1}][e]$ is obtained through a poset of size κ_{n+1}, leaving κ_{n+2} an inaccessible cardinal. A standard counting of names shows that in the further extension by $A\restriction[\kappa_{n+1},\kappa_{n+2})$, $2^{\kappa_n} \leq \kappa_{n+2}$. $\dashv$

Lemma 4.29. *In $V[A][U][S][e]$, the tree property holds at κ_{n+2} for each n.*

Proof. Fix n. Let T be a κ_{n+2} tree, in other words an $\aleph_{n+2}$ tree, in $V[A][U][S][e]$. We intend to produce, in a generic extension of $V[A][U][S][e]$ by some poset $\mathbb{P}$, an elementary embedding $\pi\colon V[A][U][S][e] \to V[A^*][U^*][S^*][e]$ with critical point κ_{n+2}. Then since T is a κ_{n+2} tree, $\pi(T)\restriction\kappa_{n+2}$ is simply T itself. Any node on level κ_{n+2} of $\pi(T)$ determines a cofinal branch through $\pi(T)\restriction\kappa_{n+2}$, hence through T. So T has cofinal branches in the extension producing π, namely the extension by $\mathbb{P}$. We will end the proof by showing that $\mathbb{P}$ is a forcing notion that does not add new cofinal branches to T, so T must already have cofinal branches in $V[A][U][S][e]$.

We begin by producing π, while keeping track of the forcing notions needed to obtain it.

Let $F = A\restriction\kappa_{n+2} * U\restriction\kappa_{n+2}$. Let $\mathbb{P}_1$ be the forcing notion refining $U\restriction[\kappa_{n+2},\nu)$ to a generic G_1 for $\mathbb{B}^{+F}\restriction[\kappa_{n+2},\nu)$. Let $\mathbb{P}_2$ be the forcing notion refining $S\restriction[\kappa_{n+2},\nu)$ to a generic G_2 for $\mathbb{C}^{+F}\restriction[\kappa_{n+2},\nu)$.

Claim 4.30. *$\mathbb{P}_1$ and $\mathbb{P}_2$ are $<\kappa_{n+1}$ closed in $V[A][U][S\restriction[\kappa_{n+1},\nu)]$.*

Proof. By Claim 4.9, every decreasing sequence of $\mathbb{P}_1$ that has length $< \kappa_{n+2}$ and belongs to $V[F]$, has a lower bound in $\mathbb{P}_1$. By Lemma 4.26, every decreasing

sequence of length $< \kappa_{n+1}$ in $\mathbb{P}_1$ that belongs to $V[A][U][S{\restriction}[\kappa_{n+1},\nu)]$, belongs already to $V[A{\restriction}\kappa_{n+2}][U{\restriction}\kappa_{n+1}] \subseteq V[F]$. (A direct application of the lemma gives that the sequence belongs to $V[A{\restriction}\kappa_{n+2}][U{\restriction}\kappa_{n+1}][S{\restriction}\kappa_{n+1}]$. The sequence is assumed to belong to $V[A][U][S{\restriction}[\kappa_{n+1},\nu)]$. These two models are mutually generic extensions of $V[A{\restriction}\kappa_{n+2}][U{\restriction}\kappa_{n+1}]$. Since the sequence belongs to both, it must belong to $V[A{\restriction}\kappa_{n+2}][U{\restriction}\kappa_{n+1}]$.)

It follows that all decreasing sequence of length $< \kappa_{n+1}$ in $\mathbb{P}_1$ that belongs to $V[A][U][S{\restriction}[\kappa_{n+1},\nu)]$ have lower bounds in $\mathbb{P}_1$. A Similar argument using Claim 4.16 applies to $\mathbb{P}_2$. ⊣

The posets $\mathbb{B}^{+F}{\restriction}[\kappa_{n+2},\nu)$ and $\mathbb{C}^{+F}{\restriction}[\kappa_{n+2},\nu)$ belong to $V[F]$ and are $<\kappa_{n+2}$ directed closed in this model, by Claims 4.7 and 4.15. We can therefore apply Lemma 4.12, using specifically the indestructibility of the generic supercompactness of κ_{n+2} under forcing with the product of these two posets. Applying the lemma we obtain an elementary embedding $\pi\colon V[A][U_{[0,n]}][G_1][G_2] \to V^*[A^*][U^*_{[0,n]}][G^*_1][G^*_2]$.

By Lemma 4.12, π is obtained in the extension of $V[A][U_{[0,n]}][G_1][G_2]$ by the posets $\mathrm{Add}(\kappa_n,\pi(\kappa_{n+2}))^V \times \mathrm{Add}(\kappa_{n+1},\pi(\kappa_{n+3}))^V$. Let $\hat{A}_n$ and $\hat{A}_{n+1}$ be the corresponding generics.

Let $U^*_{[n+1,\omega)}$ be the upward closure of G^*_1 in $\pi(\mathbb{U}_{[n+1,\omega)})$. Then $U^*_{[n+1,\omega)}$ is generic for $\pi(\mathbb{U}_{[n+1,\omega)})$ over $V^*[A^*][U^*_{[0,n]}][G^*_2]$. Letting U^* be the sequence obtained by joining $U^*_{[0,n]}$ and $U^*_{[n+1,\omega)}$, it follows that U^* is generic for $\pi(\mathbb{U})$ over $V^*[A^*]$, and that G^*_2 is generic over $V^*[A^*][U^*]$. Moreover, π restricts to an elementary embedding, which we also denote π, from $V[A][U][G_2]$ to $V^*[A^*][U^*][G^*_2]$.

Let $S^*_{[n+1,\omega)}$ be the upward closure of G^*_2 in $\pi(\mathbb{C})^{+A^**U^*}_{[n+1,\omega)}$. As in the previous paragraph, π restricts further, to an elementary embedding of $V[A][U][S_{[n+1,\omega)}]$ into $V^*[A^*][U^*][S^*_{[n+1,\omega)}]$. Since e and $S_{[0,n-1]}$ are generic for posets of size less than $\kappa_{n+2} = \mathrm{crit}(\pi)$, this embedding in turn extends to an elementary embedding of $V[A][U][S_{[n+1,\omega)}][S_{[0,n-1]}][e]$ into $V^*[A^*][U^*][S^*_{[n+1,\omega)}][S_{[0,n-1]}][e]$.

Finally, let G_3 be generic for $\mathbb{P}_3 = \pi(\mathbb{C})^{+A{\restriction}\kappa_{n+2}*U{\restriction}\kappa_{n+1}}{\restriction}[\kappa_{n+2},\pi(\kappa_{n+2}))$. Let G^+_3 be the upward closure of G_3 in $\pi(\mathbb{C}_n)^{+A^**U^*}{\restriction}[\kappa_{n+2},\pi(\kappa_{n+2}))$. (Note $A^* * U^*$ extends $A{\restriction}\kappa_{n+2} * U{\restriction}\kappa_{n+1}$.) Let $S^*_n = S_n \times G^+_3$. Then S^*_n is generic for $\pi(\mathbb{C}^{+A*U}_n) = \pi(\mathbb{C}_n)^{+A^**U^*}$, and π extends to an embedding of $V[A][U][S][e] = V[A][U][S_{[n+1,\omega)}][S_{[0,n-1]}][e][S_n]$ into $V^*[A^*][U^*][S^*_{[n+1,\omega)}][S_{[0,n-1]}][e][S^*_n]$ which is equal to $V^*[A^*][U^*][S^*][e]$.

Claim 4.31. *The poset* $\mathbb{P}_3 = \pi(\mathbb{C})^{+A{\restriction}\kappa_{n+2}*U{\restriction}\kappa_{n+1}}{\restriction}[\kappa_{n+2},\pi(\kappa_{n+2}))$ *used to add* G_3 *is* $<\kappa_{n+1}$ *closed in* $V[A][U][S{\restriction}[\kappa_{n+1},\nu)]$.

Proof. By part (2) of Claim 4.15, applied in $V^*[A^*][U^*]$, the poset is $<\kappa_{n+1}$ closed in $V^*[A^*{\restriction}\kappa_{n+2}][U^*{\restriction}\kappa_{n+1}] = V^*[A{\restriction}\kappa_{n+2}][U{\restriction}\kappa_{n+1}]$.

V^* is κ_{n+2} closed in $V[A_{[n+2,\omega)}]$, and hence $V^*[A{\restriction}\kappa_{n+2}][U{\restriction}\kappa_{n+1}]$ is $<\kappa_{n+2}$ closed in $V[A_{[n+2,\omega)}][A{\restriction}\kappa_{n+2}][U{\restriction}\kappa_{n+1}]$. By the previous paragraph then, the poset is $<\kappa_{n+1}$ closed in $V[A_{[n+2,\omega)}][A{\restriction}\kappa_{n+2}][U{\restriction}\kappa_{n+1}]$.

By Lemma 4.26, any sequence of ordinals of length $< \kappa_{n+1}$ that belongs to $V[A][U][S{\restriction}[\kappa_{n+1},\nu)]$, belongs already to $V[A{\restriction}\kappa_{n+2}][U{\restriction}\kappa_{n+1}]$, and hence belongs

to $V[A_{[n+2,\omega)}][A\restriction\kappa_{n+2}][U\restriction\kappa_{n+1}]$. It follows that any descending chain of length $< \kappa_{n+1}$ in the poset, that belongs to $V[A][U][S\restriction[\kappa_{n+1},\nu)]$, belongs already to $V[A_{[n+2,\omega)}][A\restriction\kappa_{n+2}][U\restriction\kappa_{n+1}]$, and has a lower bound using the closure in the previous paragraph. $\dashv$

We have so far produced an elementary embedding π on $V[A][U][S][e]$, with critical point κ_{n+2}. Since $\pi(T)$ determines cofinal branches through $T = \pi(T)\restriction\kappa_2$, the model containing the embedding has such branches. The embedding exists in a generic extension of $V[A][U][S][e]$ by the product of $\mathbb{P}_1$, $\mathbb{P}_2$, $\mathrm{Add}(\kappa_n,\pi(\kappa_{n+2}))^V$, $\mathrm{Add}(\kappa_{n+1},\pi(\kappa_{n+3}))^V$, and $\mathbb{P}_3$. The generics added by these posets are G_1, G_2, $\hat{A}_n$, $\hat{A}_{n+1}$, and G_3. It remains to see that the extension by these objects does not add new cofinal branches to T. Since there are cofinal branches in the extension, this implies that there are cofinal branches through T already in $V[A][U][S][e]$.

Note that all the posets involved in the extension belong to $V[A][U][S][e]$. We may therefore consider them in any order we wish. We will add $\hat{A}_{n+1}$ first, followed by $G_1 \times G_2 \times G_3$, followed finally by $\hat{A}_n$.

By Lemma 4.24, V has the $<\kappa_{n+2}$ covering property in $V[A][U][S][e]$. It follows using Claim 2.2 that the poset $\mathrm{Add}(\kappa_{n+1},\pi(\kappa_{n+3}))^V$ adding $\hat{A}_{n+1}$ is κ_{n+2}-c.c. in $V[A][U][S][e]$. Hence by Claim 2.3, the extension by $\hat{A}_{n+1}$ does not add new cofinal branches to T.

The extension by $\hat{A}_{n+1}$, being κ_{n+2}-c.c., does not collapse any cardinals at or above κ_{n+2}. By Remark 4.27 it does not add any sequences of ordinals of length $< \kappa_{n+1}$ (hence it does not collapse cardinals below κ_{n+2} either). By Claims 4.30 and 4.31 it follows that $\mathbb{P}_1$, $\mathbb{P}_2$, and $\mathbb{P}_3$ are $<\kappa_{n+1}$ closed in $V[A][U][S\restriction[\kappa_{n+1},\nu)][\hat{A}_{n+1}]$. Let W denote this model. Note that $2^{\kappa_n} = \kappa_{n+2}$ in W, and $V[A][U][S][e][\hat{A}_{n+1}]$ is an extension of W by the poset $\mathbb{C}^{+A*U}_{[0,n)} \times \mathrm{Col}(\omega,\mu)$.

Claim 4.32. *The poset* $\mathbb{C}^{+A*U}_{[0,n)} \times \mathrm{Col}(\omega,\mu)$ *is* κ_{n+1}*-c.c. in* W.

Proof. Since $\mu < \kappa_{n+1}$, and the posets $\mathbb{C}^{+A*U}_i$ for $i < n-1$ have size $< \kappa_{n+1}$, it is enough to check that (if $n \geq 1$) $\mathbb{C}^{+A*U}_{n-1}$ is κ_{n+1}-c.c. in W.

Recall that $\hat{A}_{n+1}$ does not add sequences of ordinals of length $< \kappa_{n+1}$. From this, the fact that $n \geq 1$, and Lemma 4.24, it follows that V has the $<\kappa_{n+1}$ covering property in $V[A][U][S][e][\hat{A}_{n+1}]$, and therefore also in W. By an argument similar to that of Claim 4.20, this implies that $\mathbb{C}^{+A*U}_{n-1}$ is κ_{n+1}-c.c. in W. $\dashv$

Since $\mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3$ belongs to W and is $<\kappa_{n+1}$ closed in W, $2^{\kappa_n} = \kappa_{n+2}$ in W, and $V[A][U][S][e][\hat{A}_{n+1}]$ is a κ_{n+1}-c.c. extension of W, it follows using Claim 2.4 that forcing with $\mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3$ over $V[A][U][S][e][\hat{A}_{n+1}]$, to add $G_1 \times G_2 \times G_3$, does not add any new cofinal branches to T.

It remains to show that forcing to add $\hat{A}_n$ over $V[A][U][S][e][\hat{A}_{n+1}][G_1 \times G_2 \times G_3]$ does not add new cofinal branches to T.

By Claim 2.5, the extension by $G_1 \times G_2 \times G_3$ does not add any sequences of ordinals of length $< \kappa_{n+1}$. If $n \geq 1$ it follows from this and Lemma 4.24 that V has the $<\kappa_{n+1}$ covering property in $V[A][U][S][e][\hat{A}_{n+1}][G_1 \times G_2 \times G_3]$. This in turn implies that $\mathrm{Add}(\kappa_n,\pi(\kappa_{n+2}))^V$, the poset adding $\hat{A}_n$, is κ_{n+1}-c.c. in this model, and indeed so is $\mathrm{Add}(\kappa_n,\pi(\kappa_{n+2})\cdot\lambda)^V$ for any cardinal λ. The same

conclusion is true for $n = 0$, because the poset is $\mathrm{Add}(\omega, \pi(\kappa_{n_2}) \cdot \lambda)^V$ in this case, and this poset is κ_1-c.c. in any model where κ_1 is a cardinal.

κ_{n+2} is collapsed in the extension by $G_1 \times G_2 \times G_3$. But since the extension does not add sequences of ordinals of length $< \kappa_{n+1}$, the cofinality of κ_{n+2} in the extension is at least κ_{n+1}. The poset $\mathrm{Add}(\kappa_n, \pi(\kappa_{n+2}) \cdot \lambda)^V$ is a λth power of the poset $\mathrm{Add}(\kappa_n, \pi(\kappa_{n+2}))^V$ adding $\hat{A}_n$, meaning that it adds λ mutually generic filters for $\mathrm{Add}(\kappa_n, \pi(\kappa_{n+2}))^V$. Using the fact that this poset is κ_{n+1}-c.c. it now follows by Claim 2.3 that the final extension, by $\hat{A}_n$ over the model $V[A][U][S][e][\hat{A}_{n+1}][G_\times G_2 \times G_3]$, does not add new cofinal branches to T. This completes the proof of Lemma 4.29. $\dashv$

We showed as part of the proof of Lemma 4.29 that for any $m \geq 2$, there are generic elementary supercompactness embeddings on $V[A][U][S][e]$, with critical point κ_m. The next claim summarizes some properties of these embeddings and the posets used to obtain them.

CLAIM 4.33. *Let $\lambda < \nu$, and let $n \geq 2$ be large enough that $\kappa_n > \lambda$. Then there is a poset $\mathbb{P}$ in $V[A][U][S][e]$, and a λth power of this poset, $\mathbb{P}^\lambda$, so that:*

1. *Forcing with $\mathbb{P}$ over $V[A][U][S][e]$ adds an elementary $\pi \colon V[A][U][S][e] \to V^*[A^*][U^*][S^*][e]$, with $\mathrm{crit}(\pi) = \kappa_{n+2}$ and $\sup(\pi''\nu^+) < \pi(\nu^+)$.*
2. *Forcing with $\mathbb{P}^\lambda$ over $V[A][U][S][e]$ does not add any sequences of ordinals of length $< \kappa_n$. In particular no cardinals $\leq \kappa_n$ are collapsed, and the cofinality of ν^+ is not reduced below κ_n.*

By a λth power of $\mathbb{P}$ here we mean a poset adding λ mutually generic filters for $\mathbb{P}$.

PROOF. $\mathbb{P}$ is the product of $\mathrm{Add}(\kappa_n, \pi(\kappa_{n+2}))^V$, $\mathrm{Add}(\kappa_{n+1}, \pi(\kappa_{n+3}))^V$, $\mathbb{P}_1$, $\mathbb{P}_2$, and $\mathbb{P}_3$, used in the proof of Lemma 4.29 to extend the embedding π. Starting with an embedding which is at least ν^+ supercompact we then immediately get condition (1). It remains to define $\mathbb{P}^\lambda$ and prove condition (2).

Let $\mathbb{P}^\lambda$ be the product of the posets $\mathrm{Add}(\kappa_n, \pi(\kappa_{n+2}))^V$, $\mathrm{Add}(\kappa_{n+1}, \pi(\kappa_{n+3}))^V$, $\mathbb{P}_1^\lambda$, $\mathbb{P}_2^\lambda$, and $\mathbb{P}_3^\lambda$, where the powers of $\mathbb{P}_1$, $\mathbb{P}_2$, and $\mathbb{P}_3$ are taken with full support in the model $V[A][U][S{\restriction}[\kappa_{n+1}, \nu)]$. By Claims 4.30 and 4.31, $\mathbb{P}_1$, $\mathbb{P}_2$, and $\mathbb{P}_3$ are $<\kappa_{n+1}$ closed in this model, and therefore so are their full support λth powers.

Since each of $\mathrm{Add}(\kappa_n, \pi(\kappa_{n+2}))^V$ and $\mathrm{Add}(\kappa_{n+1}, \pi(\kappa_{n+3}))^V$ is isomorphic to a λth power of itself, $\mathbb{P}^\lambda$ adds λ mutually generic filters for $\mathbb{P}$.

By Remark 4.27, forcing with $\mathrm{Add}(\kappa_n, \pi(\kappa_{n+2}))^V \times \mathrm{Add}(\kappa_{n+1}, \pi(\kappa_{n+3}))^V$ to add $\hat{A}_n$ and $\hat{A}_{n+1}$ does not add sequences of ordinals of length $< \kappa_n$. As in the proof of Lemma 4.29, $\mathbb{P}_1^\lambda \times \mathbb{P}_2^\lambda \times \mathbb{P}_3^\lambda$ is $<\kappa_{n+1}$ closed in $V[A][U][S{\restriction}[\kappa_{n+1}, \nu)][\hat{A}_{n+1}]$, and using Claim 2.5 this implies that forcing with this poset over the model $V[A][U][S][\hat{A}_{n+1}][\hat{A}_n]$ does not add sequences of ordinals of length $< \kappa_{n+1}$. $\dashv$

§5. Further analysis. We showed in the last section that in $V[A][U][S][e]$, $\kappa_n = \aleph_n$ for each $n < \omega$, and the tree property holds at κ_n for $n \geq 2$. In this section we explore the model obtained by "removing" A_1, e, and S_0. We use this model later on. For now we just collect results on generic elementary embeddings acting on the model.

Since $\mathbb{U}_{[1,\omega)}$ and $\mathbb{C}^{+A*U}_{[1,\omega)}$ rely on A_1 in their definitions, we have to pass to coarser generics, in posets that do not rely on A_1, before we can remove A_1. We begin by defining the relevant posets.

Recall that whenever we work with a filter F on $\mathbb{A}\restriction\beta*\dot{\mathbb{U}}\restriction\beta$, we assume without saying that the filter is rich enough that every condition in F can be strengthened inside F to a condition of the form $\langle a, \check{u}\rangle$ (which abusing notation we refer to as $\langle a, u\rangle$). This assumption holds for generic filters.

Definition 5.1. Let F be a filter on $\mathbb{A}\restriction\beta * \dot{\mathbb{U}}\restriction\beta$. Let $\gamma \leq \nu$ and let B be a filter on $\mathbb{B}^{+F}\restriction[\beta,\gamma)$. Define $F+B$ to be $\{\langle a,u\rangle \mid \langle a, u\restriction\beta\rangle \in F$ and $u\restriction[\beta,\gamma) \in B\}$. For a condition $b \in \mathbb{B}^{+F}\restriction[\beta,\gamma)$, define $F+b$ to be $\{\langle a,u\rangle \mid \langle a,u\restriction\beta\rangle \in F$ and $b \leq_{\mathbb{B}^{+F}\restriction[\beta,\gamma)} u\restriction[\beta,\gamma)\}$.

It is easy to check that $F+B$ is a filter on $\mathbb{A}\restriction\gamma*\dot{\mathbb{U}}\restriction\gamma$, and similarly with $F+b$.

Claim 5.2. *Suppose that F is generic for $\mathbb{A}\restriction\beta*\dot{\mathbb{U}}\restriction\beta$ over V, and B is generic for $\mathbb{B}^{+F}\restriction[\beta,\gamma)$ over $V[F]$. Then:*

1. *$\mathbb{B}^{+F+B}\restriction[\gamma,\nu)$ is $<\gamma$ directed closed in $V[F][B]$.*
2. *If $\gamma \leq \kappa_{n+2}$ then $\mathbb{C}^{+F+B}\restriction[\kappa_{n+2},\nu)$ is $<\kappa_{n+2}$ directed closed in $V[F][B]$.*
3. *If $\beta \leq \kappa_{n+2}$ then $\mathbb{C}^{+F+B}\restriction[\kappa_{n+2},\nu)$ is $<\kappa_{n+2}$ directed closed in $V[F][B]$.*

Proof. Conditions (1) and (2) are similar to Claims 4.7 and 4.15 respectively. For condition (3), note that any set of size $< \kappa_{n+2}$ in $V[F][B]$ belongs to $V[F][B\restriction\kappa_{n+2}]$ since by condition (1), $B\restriction[\kappa_{n+2},\gamma)$ is added by a $<\kappa_{n+2}$ closed forcing over $V[F][B\restriction\kappa_{n+2}]$. Thus it is enough to show that directed sets of size $< \kappa_{n+2}$ in $\mathbb{C}^{+F+B}\restriction[\kappa_{n+2},\nu)$ that belong to $V[F][B\restriction\kappa_{n+2}]$ have lower bounds. This again can be done by arguments similar to those in the proofs of Claims 4.7 and 4.15. ⊣

Definition 5.3. Let F be a filter on $\mathbb{A}\restriction\beta * \dot{\mathbb{U}}\restriction\beta$. Define $\mathbb{Q}(\beta,F)$ to be the poset consisting of pairs $\langle b,c\rangle \in \mathbb{B}\restriction[\beta,\nu) \times \mathbb{C}\restriction[\beta,\nu)$, ordered by $\langle b^*,c^*\rangle \leq \langle b,c\rangle$ iff b^* extends b in $\mathbb{B}^{+F}\restriction[\beta,\nu)$, and c^* extends c in $\mathbb{C}^{+F+b^*}\restriction[\beta,\nu)$.

The poset $\mathbb{Q}(\beta,F)$ is forcing isomorphic to the composition of $\mathbb{B}^{+F}\restriction[\beta,\nu)$ followed by $\mathbb{C}^{+F+B}\restriction[\beta,\nu)$, where B is the generic added by the first stage of the composition. Indeed, the restriction of the composition to conditions of the form $\langle b,\check{c}\rangle$ (as opposed to the more general $\langle b,\dot{c}\rangle$) is isomorphic to $\mathbb{Q}(\beta,F)$.

Claim 5.4. *Let $\beta = \kappa_{n+2}$, and let F be generic for $\mathbb{A}\restriction\beta * \dot{\mathbb{U}}\restriction\beta$ over V. Then $\mathbb{Q}(\beta,F)$ is $<\kappa_{n+2}$ directed closed in $V[F]$.*

Proof. Immediate from Claim 5.2, viewing $\mathbb{Q}(\beta,F)$ as a composition. Condition (1) of the claim implies that the first stage $\mathbb{B}^{+F}\restriction[\kappa_{n+2},\nu)$ is $<\kappa_{n+2}$ directed closed in $V[F]$, and condition (3) implies that the second stage $\mathbb{C}^{+F+B}\restriction[\kappa_{n+2},\nu)$ is $<\kappa_{n+2}$ directed closed in $V[F][B]$. ⊣

Let $\dot{\mathbb{Q}}(\beta) \in V$ name the poset $\mathbb{Q}(\beta,F)$ in the extension by $\mathbb{A}\restriction\beta*\dot{\mathbb{U}}\restriction\beta$ to add F. Let $\hat{\mathbb{Q}}(\beta)$ be the forcing associated to $\dot{\mathbb{Q}}(\beta)$ by Remark 4.10. Conditions in $\hat{\mathbb{Q}}(\beta)$ are $\mathbb{A}\restriction\beta*\dot{\mathbb{U}}\restriction\beta$ names for elements of $\dot{\mathbb{Q}}(\beta)$, with the ordering $p^* \leq p$ iff this is forced by the empty condition in $\mathbb{A}\restriction\beta*\dot{\mathbb{U}}\restriction\beta$. For a filter $\bar{F} \subseteq \mathbb{A}\restriction\beta*\dot{\mathbb{U}}\restriction\beta$, $\hat{\mathbb{Q}}(\beta)^{+\bar{F}}$ is the enriched poset with the same conditions but richer order given by $p^* \leq p$ iff this is forced by some condition in $\bar{F}$.

Claim 5.5. *Let $A_0 * U_0$ be generic for $\mathbb{A}_0 * \dot{\mathbb{U}}_0$ over V, and let $\bar{B}$ be generic for $\mathbb{B}^{+A_0*U_0}\restriction[\kappa_2,\kappa_{n+2})$ over $V[A_0 * U_0]$. Let $\bar{F} = A_0 * U_0 + \bar{B}$. Then $\hat{\mathbb{Q}}(\kappa_{n+2})^{+\bar{F}}$ is isomorphic to $\mathbb{Q}(\kappa_{n+2},\bar{F})$.*

Proof. $\mathbb{Q}(\kappa_{n+2},\bar{F})$ is, immediately from the definitions, isomorphic to the restriction of $\hat{\mathbb{Q}}(\kappa_{n+2})^{+\bar{F}}$ to "check names", that is, conditions of the form $\langle \check{b},\check{c}\rangle$ rather than the more general form $\langle \dot{b},\dot{c}\rangle$. The isomorphism witnessing this is the map $\langle b,c\rangle \mapsto \langle \check{b},\check{c}\rangle$.

Thus, it is enough to prove that every condition in $\hat{\mathbb{Q}}(\kappa_{n+2})^{+\bar{F}}$ is equivalent to a check name.

Let $\langle \dot{b},\dot{c}\rangle$ be a condition in $\hat{\mathbb{Q}}(\kappa_{n+2})$. Then $\dot{b}$ is an $\mathbb{A}\restriction\kappa_{n+2} * \dot{\mathbb{U}}\restriction\kappa_{n+2}$ name for an element of $\mathbb{B}_{[\kappa_{n+2},\nu)}$. Similarly $\dot{c}$ is an $\mathbb{A}\restriction\kappa_{n+2} * \dot{\mathbb{U}}\restriction\kappa_{n+2}$ name for an element of $\mathbb{C}_{[\kappa_{n+2},\nu)}$.

Let $D_{\dot{b}}$ be the set of $\alpha \in [\kappa_{n+2},\nu)$ which can be forced into the domain of $\dot{b}$. Since these points are all inaccessible cardinals greater than κ_{n+2}, and since $\mathbb{A}\restriction\kappa_{n+2}*\dot{\mathbb{U}}\restriction\kappa_{n+2}$ has size κ_{n+2}, $D_{\dot{b}}$ satisfies the support requirements in condition (1) of Definition 4.1.

Define $b \in \mathbb{B}_{[\kappa_{n+2},\nu)}$, with $\mathrm{dom}(b) = D_{\dot{b}}$, as follows. For each $\alpha \in D_{\dot{b}}$, let $b(\alpha)$ be the canonical $\mathbb{A}\restriction\alpha * \dot{\mathbb{U}}\restriction\alpha$ name for $\dot{b}[A\restriction\kappa_{n+2} * U\restriction\kappa_{n+2}](\alpha)[A\restriction\alpha * U\restriction\alpha]$, where this is understood to be the empty condition if $\alpha \notin \mathrm{dom}(\dot{b}[A\restriction\kappa_{n+2} * U\restriction\kappa_{n+2}])$.

Define $c \in \mathbb{C}_{[\kappa_{n+2},\nu)}$ similarly, using the name $\dot{c}$, except that $D_{\dot{c}}$ must be defined more carefully, since the support restrictions in Definition 4.13 are more stringent: $D_{\dot{c}} \cap [\kappa_{n+2},\kappa_{n+3})$ must have size $< \kappa_{n+2}$. $D_{\dot{c}}$ satisfying this can be obtained as in the proof of Claim 4.15.

One can now check that $\langle \dot{b},\dot{c}\rangle \leq \langle \check{b},\check{c}\rangle$ in $\hat{\mathbb{Q}}(\kappa_{n+2})^{+\bar{F}}$, and vice versa. ⊣

Let A_0*U_0 be generic for $\mathbb{A}_0*\dot{\mathbb{U}}_0$ over V, let $B_{[1,\omega)}$ be generic for $\mathbb{B}^{+A_0*U_0}\restriction[\kappa_2,\nu)$ over $V[A_0 * U_0]$, and let $C_{[1,\omega)}$ be generic for $\mathbb{C}^{+A_0*U_0+B}\restriction[\kappa_2,\nu)$ over $V[A_0 * U_0][B_{[1,\omega)}]$. (Equivalently for the last two extensions, $B_{[1,\omega)} * C_{[1,\omega)}$ is generic for $\mathbb{Q}(\kappa_2, A_0 * U_0)$.) Let $A_{[2,\omega)}$ be generic for $\mathbb{A}_{[2,\omega)}$ over $V[A_0 * U_0][B_{[1,\omega)}][C_{[1,\omega)}]$.

Let M denote the model $V[A_{[2,\omega)}][A_0 * U_0][B_{[1,\omega)}][C_{[1,\omega)}]$. We work with this model for the rest of the section. Let $F = A_0 * U_0 + B_{[1,\omega)}$.

Lemma 5.6. *Let $n \geq 3$. Let $\lambda < \kappa_n$. Then there is a poset $\mathbb{P}$ in M, and a λth power $\mathbb{P}^\lambda$ of $\mathbb{P}$, so that, over M:*

1. *Forcing with $\mathbb{P}$ adds an elementary embedding $\pi\colon M \to M^*$ with $\mathrm{crit}(\pi) = \kappa_{n+2}$, and $\sup(\pi''\nu^+) < \pi(\nu^+)$.*
2. *Forcing with $\mathbb{P}^\lambda$ does not add sequences of ordinals of length $< \kappa_n$.*

Proof. This is similar to a combination of Lemma 4.12 and the construction of π in the proof of Lemma 4.29, but various changes have to be made to account for the fact that we are working with the filter $F\restriction\kappa_{n+2}$ rather than a full generic filter on $\mathbb{A}\restriction\kappa_{n+2} * \dot{\mathbb{U}}\restriction\kappa_{n+2}$.

Let $\pi\colon V[A_{[n+2,\omega)}] \to V^*[A^*_{[n+2,\omega)}]$ be a γ supercompactness embedding for some $\gamma > \nu^+$, in $V[A_{[n+2,\omega)}]$, with $\mathrm{crit}(\pi) = \kappa_{n+2}$, $\pi\restriction\mathrm{Ord}$ in V, and such that $\pi(\phi)(\kappa_{n+2}) = \dot{\mathbb{Q}}(\kappa_{n+2})$ and the next element of $\mathrm{dom}(\pi(\phi))$ above κ_{n+2} is greater than γ. Such an embedding can be found using the indestructibility properties of κ_{n+2} and ϕ. We can also arrange that γ^{++} is a fixed point of the embedding,

so that the various posets that come up in the construction below have at most γ^+ dense subsets that belong to the appropriate extensions of V^*.

Since $A_0 * U_0$ is added by a small forcing relative to κ_{n+2}, π extends to an embedding of $V[A_{[n+2,\omega)}][A_0 * U_0]$ to $V^*[A^*_{[n+2,\omega)}][A_0 * U_0]$.

Let $G = B_{[1,\omega)}\restriction[\kappa_{n+2}, \nu) * C_{[1,\omega)}\restriction[\kappa_{n+2}, \nu)$. Let $\bar{F} = F\restriction\kappa_{n+2} = A_0 * U_0 + B_{[1,\omega)}\restriction\kappa_{n+2}$. Then by definitions, G is generic for $\mathbb{Q}(\kappa_{n+2}, \bar{F})$ over $V[A_{[2,\omega)}][\bar{F}]$. By Claim 5.5, G (more precisely its isomorphic image) is generic for $\hat{\mathbb{Q}}(\kappa_{n+2})^{+\bar{F}}$. It follows from this and the choice of π that $B_{[1,\omega)}\restriction\kappa_{n+2}$ and G join to form a generic filter for $\pi(\mathbb{B})^{+A_0 * U_0}\restriction[\kappa_2, \kappa_{n+2}+1)$. Denote this generic by $B^*_{[1,\omega)}\restriction\kappa_{n+2}+1$. (It consists of conditions u so that $u\restriction\kappa_{n+2} \in B_{[1,\omega)}$, and $i(u(\kappa_{n+2})) \in G$ where i is the isomorphism given by Claim 5.5.)

Let $\mathbb{B}^*_{\mathrm{top}}$ denote the poset $\pi(\mathbb{B})^{+A_0 * U_0 + B^*_{[1,\omega)}\restriction\kappa_{n+2}+1}\restriction[\kappa_{n+2}+1, \pi(\nu))$. By Claim 5.2, this poset is $<\alpha$ closed in $V^*[A_0 * U_0][B_{[1,\omega)}\restriction\kappa_{n+2}][G]$ where α is the first point in $\mathrm{dom}(\pi(\phi))$ above κ_{n+2}. By choice of π, α is greater than γ and $V^*[A_0 * U_0][B_{[1,\omega)}\restriction\kappa_{n+2}][G]$ is γ closed in $V[A_{[2,\omega)}][A_0 * U_0][B_{[1,\omega)}\restriction\kappa_{n+2}][G]$, which is equal to $V[A_{[2,\omega)}][A_0 * U_0][B_{[1,\omega)}][C_{[1,\omega)}\restriction[\kappa_{n+2}, \nu)]$. Working inside this model we can therefore find a filter H^*_{top} on $\mathbb{B}^*_{\mathrm{top}}$ which meets all dense sets that belong to $V^*[A_0 * U_0][B_{[1,\omega)}\restriction\kappa_{n+2}][G]$. Since $\pi'' B_{[1,\omega)}\restriction[\kappa_{n+2}, \nu)$ belongs to $V^*[A_0 * U_0][B_{[1,\omega)}\restriction\kappa_{n+2}][G]$ (it can be computed from G using π), and by directed closure has a lower bound in $\mathbb{B}^*_{\mathrm{top}}$, we can build H^*_{top} so that it contains a lower bound for $\pi'' B_{[1,\omega)}\restriction[\kappa_{n+2}, \nu)$. Then $B^*_{[1,\omega)}\restriction\kappa_{n+2}+1$ and H^*_{top} join to form a generic $B^*_{[1,\omega)}$ for $\pi(\mathbb{B})^{+A_0 * U_0}\restriction[\kappa_2, \pi(\nu))$, and π extends to an embedding of $V[A_{[n+2,\omega)}][A_0 * U_0][B_{[1,\omega)}]$ to $V^*[A^*_{[n+2,\omega)}][A_0 * U_0][B^*_{[1,\omega)}]$.

A similar argument, using the fact that $C_{[1,\omega)}\restriction[\kappa_{n+2}, \nu)$ is also part of G, allows finding, still inside the model $V[A_{[2,\omega)}][A_0 * U_0][B_{[1,\omega)}][C_{[1,\omega)}\restriction[\kappa_{n+2}, \nu)]$, a filter $C^*_{[1,\omega)}\restriction[\pi(\kappa_{n+2}), \nu)$, so that π extends to an embedding of $V[A_{[n+2,\omega)}][A_0 * U_0][B_{[1,\omega)}][C_{[1,\omega)}\restriction[\kappa_{n+2}, \nu)]$ to $V^*[A^*_{[n+2,\omega)}][A_0 * U_0][B^*_{[1,\omega)}][C^*_{[1,\omega)}\restriction[\kappa_{n+2}, \nu)]$.

Since $A_{[2,n)}$ and $C_{[1,\omega)}\restriction\kappa_{n+1}$ are added by small forcing, the embedding trivially extends to absorb these generics too.

Finally, standard arguments allow extending π further, to absorb also A_n, A_{n+1}, and $C_{[1,\omega)}\restriction[\kappa_{n+1}, \kappa_{n+2})$. These extensions require further forcing, with the posets $\mathrm{Add}(\kappa_n, \pi(\kappa_{n+2}))^V$, $\mathrm{Add}(\kappa_{n+1}, \pi(\kappa_{n+3}))^V$, and $\pi(\mathbb{C})^{+\bar{F}}\restriction[\kappa_{n+2}, \pi(\kappa_{n+2}))$.

This completes the proof of part (1) of the lemma. The poset $\mathbb{P}$ needed to produce the final extension of π is the product of the three posets in the previous paragraph. The proof of part (2) for this poset is similar to the corresponding proof in Claim 4.33. Let us only note that $\mathbb{P}^\lambda$ is taken to be the product of $\mathrm{Add}(\kappa_n, \pi(\kappa_{n+2}))^V$, $\mathrm{Add}(\kappa_{n+1}, \pi(\kappa_{n+3}))^V$, and the full support λth power of $\pi(\mathbb{C})^{+\bar{F}}\restriction[\kappa_{n+2}, \pi(\kappa_{n+2}))$ in $V^*[A_0 * U_0][B_{[1,\omega)}\restriction\kappa_{n+2}]$, where this poset is $<\kappa_{n+1}$ closed. $\dashv$

Lemma 5.7. *In $V[A_{[2,\omega)}]$ there is a ν^+ supercompactness embedding π from $V[A_{[2,\omega)}]$ into $V^*[A^*_{[2,\omega)}]$, so that:*

1. $\mathrm{crit}(\pi) = \kappa_2$, $\pi(\kappa_2) > \nu$, $|\pi(\kappa_2)| = \nu^{++}$, *and* $\pi\restriction\mathrm{Ord}$ *belongs to* V. *(ν^+ supercompactness implies that also* $\sup(\pi''\nu^+) < \pi(\nu^+)$.*)*

2. *In any extension $M[\hat{A}_0]$ of M by the poset* $\mathrm{Add}(\omega, [\kappa_2, \pi(\kappa_2)))^{V^*}$*, π extends to an elementary embedding $\pi\colon M \to M^*$, with $\nu \in \pi(\mathrm{Index})$.*

Since $|\pi(\kappa_2)| = \nu^{++}$, the poset in condition (2) is isomorphic, in V and hence also in M, to $\mathrm{Add}(\omega, \nu^{++})$.

Proof. This is an application of Lemma 4.12, or more precisely its proof, but without A_1. Generic supercompactness is used in the extension of $V[A_{[2,\omega)}][A_0 * U_0]$ by $\mathbb{Q}(\kappa_2, A_0 * U_0)$.

The properties of π in condition (1) follow directly from the construction of π, as does the fact that π extends to act on M given the additional generic $\hat{A}_0$. We leave the details of the construction of the embedding to the reader, noting only that because M omits A_1, there is no need to force to add the filter $\hat{A}_1$ appearing in the proof of Lemma 4.12.

It remains to verify that, with the extended π, $\nu \in \pi(\mathrm{Index})$, that is ν satisfies the requirements of Definition 4.4 over V^*. Condition (1) of the definition is immediate, as ν is a strong limit in V^*, and the largest point below ν in $\mathrm{dom}(\pi(\phi))$ is κ_2. Condition (3) holds because $\mathbb{A}^*\restriction\kappa_2 * \dot{\mathbb{U}}^*\restriction\kappa_2 + 1$ is the poset $\mathbb{A}\restriction\kappa_2 * \dot{\mathbb{U}}\restriction\kappa_2$ composed with $\pi(\phi)(\kappa_2)[A_0 * U_0]$, which in this case is equal to $\mathbb{Q}(\kappa_2, A_0 * U_0)$, and has size ν^+. Finally, condition (2) of the definition holds because forcing with $\mathbb{A}\restriction\kappa_2 * \dot{\mathbb{U}}\restriction\kappa_2$ composed with $\mathbb{Q}(\kappa_2, A_0 * U_0)$ does not collapse ν^+, over any ν closed extension $V^*[E^*]$ of V^*. This follows from Remark 4.25, as $V^*[E^*]$ can be subsumed by $V[E][A_{[2,\omega)}]$ where $V[E]$ is a ν closed extension of V, and the further extension by $\mathbb{A}\restriction\kappa_2 * \dot{\mathbb{U}}\restriction\kappa_2$ composed with $\mathbb{Q}(\kappa_2, A_0 * U_0)$ is then subsumed by the forcing in the remark to add $A * U\restriction\kappa_1 * B\restriction[\kappa_1, \nu) * C * e$ over $V[E]$. $\dashv$

For each $\mu < \kappa_2$ that belongs to the set Index of Definition 4.4 (defined over V), let $\mathbb{L}(\mu)$ be the poset $\mathrm{Add}(\mu^+, \kappa_3)^V \times \mathbb{C}_0(\mu^+)^{+A_0 * U_0} \times \mathrm{Col}(\omega, \mu)$. (Recall that κ_1 is a parameter in the definition of $\mathbb{C}_0$, and $\mathbb{C}_0(\mu^+)$ denotes the poset defined relative to the parameter $\kappa_1 = \mu^+$.)

Lemma 5.8. *Let R be a rank initial segment of the universe, large enough to contain all relevant objects. Let $\bar{M} = \bar{V}[A_{[2,\omega)}][A_0 * U_0][B_{[1,\omega)}][C_{[1,\omega)}]$ where $\bar{V}$ is the transitive collapse of $X \prec R$ with $X \in V$, $V_\nu \subseteq X$, $|X| = \nu^+$, and X closed under sequences of length ν in V. Let $\hat{A}_0$ be generic for* $\mathrm{Add}(\omega, (\nu^{++})^{\bar{M}})$ *over M, hence also over $\bar{M}$. Let $\bar{\pi}\colon \bar{M} \to \bar{M}^*$ be the embedding given by Lemma 5.7, applied in $\bar{M}[\hat{A}_0]$. Finally, let e be generic for* $\mathrm{Col}(\omega, \nu)$ *over $M[\hat{A}_0]$.*

Then there are, in $M[\hat{A}_0][e]$, filters A_1^* and S_0^* so that $A_1^* \times S_0^* \times e$ is generic for $\pi(\mathbb{L})(\nu)$ over $\bar{M}^*$.

Proof. Let $\bar{V}^*$, $\bar{A}^*_{[2,\omega)}$, $\bar{B}^*_{[1,\omega)}$, and $\bar{C}^*_{[1,\omega)}$ be such that $\bar{M}^*$ is the model $\bar{V}^*[\bar{A}^*_{[2,\omega)}][A_0 * U_0][\bar{B}^*_{[1,\omega)}][\bar{C}^*_{[1,\omega)}]$. By Lemma 5.7, $\bar{V}^*$ belongs to $\bar{V}[A_{[2,\omega)}]$, and hence also to $V[A_{[2,\omega)}]$. Using the closure properties given by the lemma, the closure of $\bar{V}$ itself, and the fact that any $\mathbb{A}_{[2,\omega)}$ name for a sequence of ordinals of length ν can be thinned below some condition in $A_{[2,\omega)}$ to a name of size ν, $\bar{V}^*$ is closed under sequences of length ν in $V[A_{[2,\omega)}]$.

It is enough to produce a generic $A_1^* \times C_0^*$ for $\mathrm{Add}(\nu^+, \pi(\kappa_3))^{\bar{V}^*} \times \pi(\mathbb{C}_0)(\nu^+)$ over $\bar{M}^*[e]$, in $M[\hat{A}_0]$. The upward closure of C_0^* in $\pi(\mathbb{C}_0)(\nu^+)^{+A_0^* * U_0^*}$ then yields the necessary S_0^*.

Since $\bar{M}^*$ is contained in $\bar{M}[\hat{A}_0]$, it is enough to ensure that $A_1^* \times C_0^*$ is generic over $\bar{M}[\hat{A}_0][e] = \bar{V}[A_{[2,\omega)}][A_0 * U_0][B_{[1,\omega)}][C_{[1,\omega)}][\hat{A}_0][e]$. Since the poset adding $A_1^* \times C_0^*$ belongs to $\bar{V}^* \subseteq \bar{V}[A_{[2,\omega)}]$, it is enough to construct $A_1^* \times C_0^*$ so that it is generic over $\bar{V}[A_{[2,\omega)}]$, and so that $A_0 * U_0 * B_{[1,\omega)} * C_{[1,\omega)} \times \hat{A}_0 \times e$ is generic over $\bar{V}[A_{[2,\omega)}][A_1^* \times C_0^*]$. This in turn holds automatically if $A_1^* \times C_0^*$ belongs to $V[A_{[2,\omega)}]$, as $A_0 * U_0 * B_{[1,\omega)} * C_{[1,\omega)} \times \hat{A}_0 \times e$ is generic over $V[A_{[2,\omega)}]$.

So, it is enough to construct $A_1^* \times C_0^*$, generic for $\mathrm{Add}(\nu^+, \pi(\kappa_3))^{\bar{V}^*} \times \pi(\mathbb{C}_0)(\nu^+)$ over $\bar{V}[A_{[2,\omega)}]$, inside $V[A_{[2,\omega)}]$.

The poset $\mathrm{Add}(\nu^+, \pi(\kappa_3))^{\bar{V}^*} \times \pi(\mathbb{C}_0)(\nu^+)$ is ν closed in $\bar{V}^*$, and hence also ν closed in $V[A_{[2,\omega)}]$. Since $\bar{V}[A_{[2,\omega)}]$ has size ν^+, a generic over this model can be constructed in $V[A_{[2,\omega)}]$ by enumerating all dense sets in $\bar{V}[A_{[2,\omega)}]$ in order type ν^+, and meeting them one by one. ⊣

Lemma 5.9. *Let $\mu \in \mathrm{Index}$ and let $A_1 \times S_0 \times e$ be generic for $\mathbb{L}(\mu)$ over M. Let $A = A_0 \times A_1 \times A_{[2,\omega)}$, let $U = U_0 * U_{[1,\omega)}$ where $U_{[1,\omega)}$ is the upward closure of $B_{[1,\omega)}$ in $\mathbb{U}_{[1,\omega)}$, and let $S = S_0 \times S_{[1,\omega)}$ where $S_{[1,\omega)}$ is the upward closure of $C_{[1,\omega)}$ in $\mathbb{C}^{+A*U}_{[1,\omega)}$. Let N denote $V[A][U][S]$.*

Then $B_{[1,\omega)}$ and $C_{[1,\omega)}$ belong to a forcing extension $N[G]$ of N by a poset which is μ closed in N. Moreover G is generic also over $N[e]$, and ν and ν^+ are not collapsed in $N[e][G]$. (In fact none of the κ_ns is collapsed.)

Proof. $B_{[1,\omega)}$ and $C_{[1,\omega)}$ belong to the extension of $V[A][U][S][e]$ by the product of the factor poset refining $U_{[1,\omega)}$ to a filter for $\mathbb{B}^{+A_0*U_0}\restriction[\kappa_2,\nu)$, and the factor poset refining $S_{[1,\omega)}$ to a filter for $\mathbb{C}^{+A_0*U_0}\restriction[\kappa_2,\nu)$. ($B_{[1,\omega)}$ is itself generic for the former; $C_{[1,\omega)}$ is the upward closure in $\mathbb{C}^{+A_0*U_0+B_{[1,\omega)}}\restriction[\kappa_2,\nu)$ of a generic for the latter.) The factor posets belong to $V[A][U][S]$. By Claims 4.9 and 4.16, descending sequences of length $< \kappa_2$ in these posets, that belong to $V[A_0 * U_0]$, have lower bounds. By Lemma 4.26, all descending sequences of length $< \kappa_1$ in these posets that belong to $V[A][U][S]$, belong to $V[A\restriction\kappa_2][U\restriction\kappa_1]$, and in particular they belong to $V[A_0 * U_0]$. So the factor posets are $<\kappa_1$ closed, in other words μ closed, in $N = V[A][U][S]$. The model resulting from the extension of $N[e]$ by the factor posets is contained in the model $V[A][U\restriction\kappa_1][B\restriction[\kappa_1,\nu)][C][e]$ of Lemma 4.24 and Remark 4.25, and it follows from the lemma and remark that ν^+ remains a cardinal in the extension, as does each κ_n, and hence so does ν. ⊣

§6. The tree property up to $\aleph_{\omega+1}$. In this section we combine the ingredients given by the previous sections into a construction of a model where the tree property holds both at all $\aleph_n$ for $2 \leq n < \omega$, and at $\aleph_{\omega+1}$. A direct combination of these ingredients will yield the tree property at $\aleph_{\omega+1}$ not in the model we construct, but in a forcing extension of this model. The following preservation lemma from Magidor–Shelah [4] will allow us to then pull the necessary branches back to the original model. (The posets we refer to as μ closed, in the lemma and throughout the paper, are called μ^+ closed in Magidor–Shelah [4].)

Lemma 6.1 (Magidor–Shelah [4, Theorem 5.2]). *Suppose ν is a strong limit cardinal of cofinality ω. Let $N \subseteq N[G]$ where $N[G]$ is a μ closed forcing extension*

of N, for some $\mu < \nu$. Let e be generic over $N[G]$ for a poset $\mathbb{E} \in N$ of size μ. Let T be a ν^+ tree in $N[e]$. Then any cofinal branch of T in $N[e][G]$ belongs already to $N[e]$.

THEOREM 6.2. *Suppose there are ω supercompact cardinals, and let κ_n, $2 \leq n < \omega$, enumerate them in increasing order. Let $\nu = \sup\{\kappa_n \mid 2 \leq n < \omega\}$. Then there is a forcing extension in which $\kappa_n = \aleph_n$, $(\nu^+)^V = \aleph_{\omega+1}$, and the tree property holds at each successor cardinal in the interval $[\aleph_2, \aleph_{\omega+1}]$.*

PROOF. Using a preparatory forcing for indestructibility, we may assume that each κ_n, $2 \leq n < \omega$, is indestructibly supercompact. We may also assume that each κ_n carries an indestructible Laver function in the sense of Section 4. We begin the construction of the model witnessing Theorem 6.2 as in Section 5. Let $A_0 * U_0$ be generic for $\mathbb{A}_0 * \dot{\mathbb{U}}_0$ over V. Let $B_{[1,\omega)}$ be generic for $\mathbb{B}^{+A_0 * U_0} \restriction [\kappa_2, \nu)$ over $V[A_0 * U_0]$. Let $C_{[1,\omega)}$ be generic for $\mathbb{C}^{+A_0 * U_0 + B} \restriction [\kappa_2, \nu)$ over $V[A_0 * U_0][B_{[1,\omega)}]$. Let $A_{[2,\omega)}$ be generic for $\mathbb{A}_{[2,\omega)}$ over $V[A_0 * U_0][B_{[1,\omega)}][C_{[1,\omega)}]$.

Let M denote the model $V[A_{[2,\omega)}][A_0 * U_0][B_{[1,\omega)}][C_{[1,\omega)}]$. Let Index $\subseteq \kappa_2$ be the set given by Definition 4.4. (The definition refers to A_0 and U_0.)

For each $\mu \in$ Index, let $\mathbb{L}(\mu)$ be the poset $\text{Add}(\mu^+, \kappa_3)^V \times \mathbb{C}_0(\mu^+)^{+A_0 * U_0} \times \text{Col}(\omega, \mu)$.

CLAIM 6.3. *There is $\mu \in$ Index, so that in the extension of M by $\mathbb{L}(\mu)$, the tree property holds at ν^+.*

PROOF. It is enough to check that the assumptions of Lemma 3.10 hold in M. The claim then follows by an application of the lemma.

Assumption (1) of the lemma holds over M by Lemma 5.6, used with $n = m+1$ and $\lambda = \kappa_m$.

Assumption (2) of the lemma holds over M by Lemma 5.8, and the properties of the embedding π given by Lemma 5.7. The poset $\mathbb{P} = \mathbb{P}_X$ needed to introduce the embedding π (acting on $\bar{M}$) and the generic L is the poset $\text{Add}(\omega, \nu^+) \times \text{Col}(\omega, \nu)$. (This poset is isomorphic to the one used in Lemma 5.8, as $(\nu^{++})^{\bar{M}}$ has cardinality ν^+ in M.) It is ν^+-Knaster by standard arguments using a Δ-system for the component $\text{Add}(\omega, \nu^+)$ and the fact that $\text{Col}(\omega, \nu)$ has size less than ν^+. $\dashv$

Let μ be given by Claim 6.3, and let $A_1 \times S_0 \times e$ be generic for $\mathbb{L}(\mu)$ over M. Let $\kappa_1 = \mu^+$.

Let $A = A_0 \times A_1 \times A_{[2,\omega)}$. Let $U_{[1,\omega)}$ be the upward closure of $B_{[1,\omega)}$ in $\mathbb{U}_{[1,\omega)}$, and let $U = U_0 * U_{[1,\omega)}$. Similarly, let $S_{[1,\omega)}$ be the upward closure of $C_{[1,\omega)}$ in $\mathbb{C}^{+A*U}_{[1,\omega)}$, and let $S = S_0 * S_{[1,\omega)}$. Let N be the model $V[A][U][S]$.

CLAIM 6.4. *In $N[e]$, $\kappa_n = \aleph_n$, and the tree property holds at $\aleph_n$ for $n \geq 2$. $N[e]$ and V have the same cardinals from ν upward.*

PROOF. These are simply the results of Section 4, including in particular Lemmas 4.29 and 4.24, and Remark 4.25. $\dashv$

CLAIM 6.5. *In $N[e]$, the tree property holds at ν^+.*

PROOF. Let $T \in N[e]$ be a ν^+ tree. Then T belongs to $M[A_1 \times S_0 \times e]$ (as N was defined in $M[A_1 \times S_0]$). By Claim 6.3 and the subsequent choice of μ, this

model satisfies the tree property at ν^+, and therefore T has a cofinal branch in the model.

By Lemma 5.9, there is a μ closed forcing extension $N[G]$ of N, so that $B_{[1,\omega)}$ and $C_{[1,\omega)}$ belong to $N[G]$, G is generic also over $N[e]$, and ν and ν^+ remain cardinals in $N[e][G]$.

Since $B_{[1,\omega)}$ and $C_{[1,\omega)}$ belong to $N[G]$, and since $A_0 * U_0$, A_1, $A_{[2,\omega)}$, and S_0 belong to N, the entire model $M[A_1 \times S_0 \times e]$ is contained in $N[e][G]$. Since T has a cofinal branch in $M[A_1 \times S_0 \times e]$, it has a cofinal branch in $N[e][G]$.

An application of Lemma 6.1 now shows that T has a cofinal branch already in $N[e]$. $\dashv$

Claims 6.4 and 6.5 establish that in $N[e]$, the tree property holds at $\aleph_n$ (which is equal to κ_n) for $2 \leq n < \omega$, *and* at $\aleph_{\omega+1}$ (which is equal to ν^+). This completes the proof of Theorem 6.2. $\dashv$

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