Higher analogues of properness

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Forcing axioms

Developed in late 1960s early 1970s, initially to crystalize center points for applications of iterated forcing.

**Martin’s axiom** (MA, for \( \omega_1 \) antichains): for any c.c.c. poset \( P \) and any collection \( A \) of \( \omega_1 \) maximal antichains of \( P \), there is a filter on \( P \) which meets every antichain in \( A \).

Obtained through an iteration of enough c.c.c. posets. Can then be used axiomatically as a starting point for consistency proofs that would otherwise require an iteration of c.c.c. posets.

Key points in proving consistency of MA:

(a) Finite support iteration of c.c.c. posets does not collapse \( \omega_1 \), and in fact the iteration poset is itself c.c.c.

(b) Can “close off”, that is reach a point where enough c.c.c. posets have been hit to ensure MA.
Proper forcing

There are classes of posets other than c.c.c. which also preserve $\omega_1$.

Definition

Let $\mathbb{P}$ be a poset. Let $\kappa$ be large enough that $\mathbb{P} \in H(\kappa)$. $p \in \mathbb{P}$ is a master condition for $M \prec H(\kappa)$ if

1. $p$ forces that every maximal antichain $A$ of $\mathbb{P}$ that belongs to $M$ is met by the generic filter inside $M$.

Equivalently any of:

2. $p$ forces that $\dot{G} \cap \check{M}$ is generic over $M$.

3. $p$ forces that $M[\dot{G}] \prec H(\kappa)[\dot{G}]$ and $M[\dot{G}] \cap V = M$.

Definition

$\mathbb{P}$ is proper if for all large enough $\kappa$ and all countable $M \prec H(\kappa)$, every condition in $M$ extends to a master condition for $M$.

Proper posets do not collapse $\omega_1$; immediate from (3).
Proper forcing axiom (PFA): the parallel of MA for proper posets. Again used axiomatically as a starting point for consistency proofs.

Key points in consistency proof of PFA:

(a) Countable support iteration of proper posets does not collapse $\omega_1$, and is indeed proper.

(b) Can close off, assuming a supercompact cardinal.

For (b), fix a supercompact cardinal $\theta$. Iterate up to $\theta$ hitting proper posets given by a Laver function. At stage $\theta$, using properties of the Laver function and supercompactness, have covered enough posets to ensure PFA holds.

Obtained in late 1970s, Baumgartner, Shelah.
Consequences (some of many)

Compositions of $\text{Col}(\omega_1, \delta)$ and c.c.c. posets are proper.

Gives: Tree property at $\omega_2$; every tree of size and height $\omega_1$ has at most $\omega_1$ cofinal branches; any two $\omega_1$ dense subsets of $\mathbb{R}$ are order isomorphic; $\square_\kappa$ fails for $\kappa \geq \omega_1$.

Posets using finite sequences of countable models as side conditions to enforce properness.

Gives: Failure of $\square_\kappa$ for $\kappa \geq \omega_1$; P-ideal dichotomy; Open Coloring Axiom; rainbow Ramsey principle on $\omega_1$.

Mapping Reflection Principle (MRP).

Gives: Failure of $\square_\kappa$ for $\kappa \geq \omega_1$; SCH; wellordering of $\mathbb{R}$ of ordertype $\omega_2$ definable over $H(\omega_2)$ from parameter contained in $\omega_1$. 
Higher analogues?

In the case of MA, the forcing axiom has higher analogues, and in fact strengthenings.

For example it is consistent that for all c.c.c. posets, all maximal antichain in families of size $\omega_2$ can be simultaneously met by a filter.

Initial expectation was that similar analogues should exist for PFA.

Naive attempt: demand existence of master conditions also for models of size $\omega_1$.

Posets in the resulting class preserve $\omega_1$ and $\omega_2$ (certainly a necessary property for a higher analogue).

But preservation under iteration fails.

Search for higher analogues largely dormant.
Two-size nodes

For regular $\theta \geq \omega_2$ and $f : H(\theta)^{\lt \omega} \rightarrow H(\theta)$, let $C(\theta, f)$ consist of $M$ satisfying one of:

1. (Type $\omega_1$.) $|M| = \omega_1$, $M \prec H(\theta)$, internal on a club, closed under $f$.

2. (Countable type elementary.) $|M| = \omega$, $M \prec H(\theta)$, closed under $f$.

3. (Countable type tower.) $|M| \leq \omega$, $M \neq \emptyset$, linearly ordered by $\in$, every $N \in M$ satisfies (1), $(\forall N \in M)(M \cap N \in N)$.

Called nodes. Non-tower nodes are elementary.

Easy to check $\mathbb{P}$ proper iff $(\exists$ large enough $\theta, f)$ $(\forall \in$-increasing set $s$ of countable elementary nodes) $(\forall Q \in s)$ every $p \in \mathbb{P} \cap Q$ which is a master condition for all $M \in s \cap Q$ extends to a master condition for all $M \in s$.

Right-to-left direction immediate. Left-to-right by iterated applications of condition defining properness.
Two-size side conditions

A two-size side condition is a finite set of nodes, $\in$-increasing (each node belongs to its successor), and closed under intersections in the sense:

- If $N \in M$ of type $\omega_1$ and countable elementary both in $s$, then $M \cap N$ in $s$.
- If $N \in M$ of type $\omega_1$ and tower both in $s$, and $M \cap N \neq \emptyset$, then there is tower $\bar{M} \supseteq M \cap N$ occurring in $s$ before $N$.

Ordered in the natural way, reverse inclusion as sets.

For elementary $Q \in s$, the residue of $s$ in $Q$ is $s \cap Q$. Denoted $\text{res}_Q(s)$. Is itself a two-size side condition.

Lemma

If $Q \in s$ elementary and $t \in Q$ extends $\text{res}_Q(s)$, then $s$ and $t$ are compatible.

Gives strong properness for poset of two-size side conditions. Poset preserves $\omega_1, \omega_2$, collapses $H(\theta)$ to $\omega_2$. 
Two-size properness

Recall $P$ proper iff $(\exists$ large enough $\theta$, and $f$) 
($\forall$ $\in$-increasing set $s$ of countable elementary nodes) 
($\forall Q \in s$) every $p \in P \cap Q$ which is a m.c. for all $M \in s \cap Q$ extends to a m.c. for all $M \in s$.

Two-size properness (1st approx.): $(\exists$ large enough $\theta$, $f$) 
($\forall$ two-size side condition $s$) ($\forall Q \in s$ elementary) every $p \in P \cap Q$ which is a m.c. for all $M \in \text{res}_Q(s)$ extends to a m.c. for all $M \in s$.

(By m.c. for tower $M$ means m.c. for all $N \in M$.)

For added generality, replace “m.c. for $M$” with “$\in \text{mc}(M)$”, 
where $M \mapsto \text{mc}(M)$ abstracts essential properties of the function $M \mapsto \{\text{master conditions for } M\}$.

Some essential properties: every $q \in \text{mc}(M)$ is a m.c. for $M$; $\text{mc}(M \cap N) \supseteq \text{mc}(M) \cap \text{mc}(N)$; $\text{mc}(M)$ open in $P$; $\text{mc}(M) \subseteq \text{mc}(M')$ for $M' \subseteq M$ both tower.

Posets satisfying this (for some $\text{mc}$) are two-size proper.
Two-size proper forcing axiom

Two-size proper posets admit master conditions for countable models and models of size $\omega_1$. (But definition requires more.) Preserve $\omega_1$ and $\omega_2$.

Two-size proper forcing axiom: For every two-size proper $\mathbb{P}$, every collection $\mathcal{A}$ of $\omega_2$ maximal antichains of $\mathbb{P}$, there is a filter on $\mathbb{P}$ which meets every antichain in $\mathcal{A}$.

Theorem (N.) (2012 as stated, 2010 finer tower nodes)
Suppose $\theta$ is supercompact. Then there is a forcing extension satisfying the two-size proper forcing axiom.

Covers posets of two-size side conditions, in particular posets which collapse arbitrary $\delta \geq \omega_2$ to $\omega_2$.

Covers c.c.c. posets.

Class is closed under compositions.

Similar to classes for initial uses of PFA.
Relaxing

Recall two-size properness: \((\exists \text{ large enough } \theta, f)\) \((\forall \text{ two-size side condition } s) (\forall Q \in s \text{ elementary}) \) every \(p \in Q \cap \bigcap_{M \in \text{res}_Q(s)} \text{mc}(M)\) extends to \(q \in \bigcap_{M \in s} \text{mc}(M)\).

Relax the extension condition by placing restrictions on the configuration of \(s\) and \(Q\).

Only require condition to hold in following instances:

- \(\text{res}_Q(s) = \emptyset\).
- \(Q\) countable, \(p \in \text{mc}(U) \cap Q\) for some tower \(U \in Q\) which subsumes \(\text{res}_Q(s)\).

By \(U\) subsumes \(r\) mean every \(M \in r\) is either contained in \(U\) or belongs to \(U\). For tower \(U\), in particular implies \(r\) has only tower and type \(\omega_1\) nodes.

Resulting class is relaxed two-size proper.

**Theorem (N.) (2013)**

Suppose \(\theta\) is supercompact. There is a forcing extension satisfying the relaxed two-size proper forcing axiom.
Some words on the proof

Lifts new method for PFA consistency using finite support. Method relies on two-type side conditions (ctbl elem.; transitive) to preserve properness. N., building on Mitchell-Friedman posets for adding clubs in $\omega_2$ with finite conditions.

To generalize need three-type side conditions, preserve $\omega_1, \omega_2$, supercompact $\theta$ (which becomes $\omega_3$).

Requires introduction of non-elementary nodes, which give rise to tower node in two-size properness.

Initial version with fine, very technical, notion of non-elementary nodes 2010.

Around the same time, independently, Aspero-Mota used finite side condition with ctbl models to show weakenings of PFA for $\omega_2$-c.c. posets consistent with large continuum. Aspero-Mota class subsumed in $\omega_2$-c.c. relaxed two-size proper.
Square at $\omega_1$

Lemma (independently Krueger, N.)

There is a finite conditions poset, strongly proper for countable and size $\omega_1$ nodes, forcing $\square_{\omega_1}$.

Earlier work on forcing $\square_{\omega_1}$ with finite conditions by Dolinar-Dzamonja, but clubs for the square sequence added with ctbl fragments. Not strongly proper.

Poset in lemma not relaxed two-size proper; extension condition fails. Variant (N.) for $\square_{\omega_1, \text{fin}}$ is.

Corollary

The relaxed two-size proper forcing axiom implies $\square_{\omega_1, \text{fin}}$.

Not necessarily a good thing; may create too much structure on $\omega_2$. (Non-relaxed) two-size proper forcing axiom does not imply $\square_{\omega_1, \text{fin}}$. Suggests some applications may require restricting forcing class—seems to weaken axiom, but may give extra preservation on $\omega_2$. 
Analogue of MRP

Fix $X$. $\Sigma \subseteq \mathcal{P}(X)$ is open if for every $A \in \Sigma$ there is finite $a \subseteq A$ so that $a \subseteq B \subseteq A \rightarrow B \in \Sigma$.

$\Sigma \subseteq \mathcal{P}(X)$ is $\mathcal{N}$-stationary on size $\kappa$ if $\forall f : X^{<\omega} \cup \kappa \rightarrow X$ in $\mathcal{N}$, there is $A \in \mathcal{N} \cap \Sigma$ closed under $f$ and containing $f''\kappa$.

Map $\Sigma$ into $\mathcal{P}(X)$ is open, $\kappa$-stationary if for every $\mathcal{N} \in \text{dom}(\Sigma)$, $\Sigma(\mathcal{N})$ open, $\mathcal{N}$-stationary on size $\kappa$.

Work with sequences $\langle M_\xi | \xi < \kappa^+ \rangle$, $\in$-linear, continuous, $M_\xi$ of size $\kappa$, $\kappa \subseteq M_\xi$.

$\alpha < \kappa^+$ is a $\Sigma$ reflection point if $(\forall$ large enough $\xi < \alpha$ of cofinality $\kappa)$ $M_\xi \cap X \in \Sigma(M_\alpha)$.

Mapping Reflection Principle (Moore): for $\omega$-stationary open map $\Sigma$ on club of $\text{ctbl} \ N \prec H(\theta)$, exists $\langle M_\xi | \xi < \omega_1 \rangle$ with club of $\Sigma$ reflection points.

Follows from PFA. Foundationally important consequences: $\neg \Box_\lambda$ for $\lambda \geq \omega_1$; wo of $\mathbb{R}$ of ordertype $\omega_2$, definable over $H(\omega_2)$ from parameter $\subseteq \omega_1$; SCH.
Analogue of MRP (cont.)

For ctbl $P$, $\text{fatten}(P) = P \cup \bigcup \{Z \in P \mid |Z| = \omega_1\}$.

$\Sigma \subseteq \mathcal{P}(X)$ is $N$-amenable, for $N$ of size $\omega_1$ internal on club, if for club of ctbl $P \subseteq N$, $\Sigma(N) \cap \text{fatten}(P) \in N$.

Map $\Sigma$ is amenable if $\forall N \in \text{dom}(\Sigma)$, $\Sigma$ is $N$-amenable.

Let $\mathcal{P}_{ic-\omega_1}(H) = \{N \subseteq H \mid |N| = \omega_1, N \text{ internal on club}\}$.

**Lemma (N.)**

Let $\Sigma$ be amenable $\omega_1$-stationary open map, with $\text{dom}(\Sigma)$ containing a club relative to $\mathcal{P}_{ic-\omega_1}(H(\theta))$. Then there is a relaxed two-size forcing adding $\langle M_\xi \mid \xi < \omega_2 \rangle$ with stationary set of $\Sigma$ reflection points.

**Corollary**

Consistent that for every amenable $\omega_1$-stationary open map $\Sigma$ with domain containing a club relative to $\mathcal{P}_{ic-\omega_1}(H(\theta))$, exists $\langle M_\xi \mid \xi < \omega_2 \rangle$ with stationary set of $\Sigma$ reflection points.
Analogue of MRP (cont.)

Enough to imply failure of $\square_\lambda$ at $\lambda \geq \omega_2$, through analogue of MRP antithreading argument. (Antithreading can also be done directly, and suggests the MRP analogue.)

Not enough for analogue of coding of reals, in MRP argument for wellordering of $\mathbb{R}$.

In MRP argument, given real coded by $\sup(\text{Ord} \cap \bigcup M_\xi)$. Here there is also a dependence on a stationary $S \subseteq \omega_2$.

If $\langle M_\xi \mid \xi < \omega_2 \rangle$ actually generic (not pseudo generic), outside $S$ behavior is generic and does not code any real. So $x$ uniquely determined from $\sup(\text{Ord} \cap \bigcup M_\xi)$.

Possible that by restricting forcing class, can preserve “non-coding” through an iteration.

Would allow strengthening thm to add this property to $S$.

Would then get wellordering of $\mathbb{R}$ of ordertype $\omega_3$ definable over $H(\omega_3)$ from parameter $\subseteq \omega_2$. 