Inner models and ultrafilters in \(L(\mathbb{R})\)

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Part 2:
1. Review.
2. Iteration trees and directed systems.
3. Supercompactness measure on \(\mathcal{P}_{\omega_1}(\aleph_\omega)\).
4. Ultrafilter on \([\mathcal{P}(\omega_1)]^{<\omega_1}\).
5. Forcing over \(L(\mathbb{R})\) to collapse \(\aleph_\omega\) to \(\omega_1\).
Large cardinal assumption:

For each \( u \in \mathbb{R} \) there is a class model \( M \) s.th.
(1) \( u \in M \);
(2) \( M \) has \( \omega \) Woodin cardinals, say with sup \( \delta \);
(3) \( \mathcal{P}(\delta)^M \) is countable in \( V \); and
(4) \( M \) is iterable.

Any statement (with real parameters) \((*)\) forced to hold in the symmetric collapse of \( M \), holds in the true \( L(\mathbb{R}) \).

Ultrafilter on \( \omega_1 \):

\( a(M) = \) first measurable of \( M \).

\( C_M = \{a(P) \mid P \) a linear iterate of \( M \} \).

Used \((*)\) to show that for every \( X \subset \omega_1 \) in \( L(\mathbb{R}) \), have \( M \) so that either \( C_M \subset X \) or \( C_M \subset \neg X \).
Ultrafilter on \([\omega_1]^{<\omega_1}\):

\(\kappa\) the first measurable limit of measurables in \(M\).

\(\langle \tau_\xi \mid \xi < \gamma \rangle\) lists the measurables of \(M\) below \(\kappa\) in increasing order.

\(a(M) = \langle \tau_\xi \mid \xi < \gamma \rangle\).

\(C_M\) as before.

The sets \(C_M\) generate an ultrafilter. Used (\ast) to get bddness. Used bddness in forcing.

Next, aim to do the same with \(\aleph_\omega\) instead of \(\omega_1\).
In general iterations may be non-linear.

Non-linear iterations are called iteration trees.

Iteration trees involve some choices at limit stages. $M$ is iterable if these choices can be made in a way which secures well-foundedness.

A correct iteration tree on an iterable $M$ is one which follows the limit choices needed to secure wellfoundedness.
Already at the level of linear iterations there is an implicit notion of correctness: A linear iteration of “length $\alpha$” is “correct” if $\alpha$ is well-founded.

This “correctness” for linear iterations is $\Pi_1^1$. In the claim of boundedness last time it was the contribution of correctness to the complexity of the set

$$(\exists \text{ an iterate } P \text{ of } M)(\psi[a(P), x, u] \text{ holds in a symmetric collapse of } P).$$

that made it $\Sigma_2^1$.

For iteration trees the complexity of correctness is higher. How high depends on the large cardinals involved in the iteration.

Let $M$ be iterable and let $\tau \in M$ be least such that $L(M\parallel \tau) \models "\tau \text{ is Woodin.}"$

For iteration trees on $M$ using extenders from below $\tau$, correctness is roughly $\Pi_2^1$. 
Let $M$ now be *fine-structural* over a real $u$.

Any two correct iterates of $M$ can be *compared*.

In other words, for any two correct iterations $j_1: M \to P_1$ and $j_2: M \to P_2$ there are further iterations $h_1: P_1 \to Q_1$ and $h_2: P_2 \to Q_2$ so that $Q_1 = Q_2$.

Further, the embeddings given by correct iterations are unique, by the Dodd–Jensen lemma. $k_1: M \to Q$ and $k_2: M \to Q$ both iteration embeddings, then $k_1 = k_2$.

Use $\pi_{M,Q}$ for the iteration emb from $M$ to $Q$.

In the situation of the comparison above the uniqueness implies that $h_1 \circ j_1$ equals $h_2 \circ j_2$. 
By iteration from now on we mean a correct iteration of countable length.

Comparisons allow considering the directed system of all iterates of $M$.

Let $\mathcal{D}$ be the set of pairs $\langle P, x \rangle$ so that $P$ is an iterate of $M$, and $x$ belongs to $P$.

For $\langle P, x \rangle$ and $\langle P', x' \rangle$ both in $\mathcal{D}$ set $\langle P, x \rangle \sim \langle P', x' \rangle$ iff in the comparison of $P$ and $P'$ get $h(x) = h'(x')$.

$\sim$ is an equivalence relation on $\mathcal{D}$.

Define further $\langle P, x \rangle \in^* \langle P', x' \rangle$ iff in the comparison of $P$ and $P'$ get $h(x) \in h(x')$.

$\in^*$ induces a wellfounded relation on $\mathcal{D}/\sim$. Set $M_\infty = \text{transitive collapse of } (\mathcal{D}/\sim ; \in^*)$.

$M_\infty$ is the direct limit of all (countable) iterates of $M$. Have $\pi_{M,\infty}$ from $M$ into $M_\infty$ defined by $\pi_{M,\infty}(x) = \text{equivalence class of } \langle M, x \rangle$. 
We are working in $L(R)$. $\omega_2$ is equal to $\delta_2^1$. $\omega_3$, $\omega_4$, etc. are all singular cardinals of cofinality $\omega_2$. $\aleph_\omega$ is the size of a homogeneous tree for $\Pi^1_2$ sets. $\aleph_{\omega+1}$ is equal to $\delta^1_3$.

Suppose $M$ is iterable, $\tau = \tau(M)$ is least such that $L(M \parallel \tau) \models \text{"}\tau \text{ is Woodin,"} \text{ and } \tau \text{ is countable in } V$.

**Theorem (Woodin):** $\pi_{M,\infty}(\tau)$ is equal to $\aleph_\omega$.

This is connected to the fact that correctness for trees below $\tau$ is roughly $\Pi^1_2$. 
Recall our scheme for getting ultrafilters:

Define $a(M)$ somehow.

Set $C_M = \{a(P) \mid P$ is an iterate of $M\}$.

Use the $C_M$s to generate an ultrafilter.

Here we want an ultrafilter on $\mathcal{P}_{\omega_1}(\aleph_\omega)$. So we need $a(M) \in \mathcal{P}_{\omega_1}(\aleph_\omega)$.

Natural attempt: set $a(M) = \pi_{M,\infty''} \tau(M)$.

Then $a(M) \in \mathcal{P}_{\omega_1}(\aleph_\omega)$ and $C_M \subset \mathcal{P}_{\omega_1}(\aleph_\omega)$.

As before the sets $C_M$ generate an ultrafilter.* It’s the supercompactness measure on $\mathcal{P}_{\omega_1}(\aleph_\omega)$.

*The finite intersection property takes more work here than in the case of $\omega_1$. More on this in the next talk.
Proof of normality:

Fix $f \in L(\mathbb{R})$ on $\mathcal{P}_{\omega_1}(\mathbb{N})$ such that $f(X) \in X$ for all $X$.

Wlog, $f$ is definable from a real $u$. Fix $\varphi$ so that $f(X) = \alpha$ iff $L(\mathbb{R}) \models \varphi[u, X, \alpha]$.

Take $M$ satisfying L.C. assumption with $u \in M$. Let $\tau$ be least so that $L(M \parallel \tau) \models \tau$ is Woodin.”

Look at $\alpha = f(a(M))$.

$\alpha$ belongs to $a(M) = \pi_{M, \infty}'' \tau$. 
Have $\bar{\alpha} < \tau$ in $M$ so that

$$f(\pi_{M,\infty''}(\tau)) = \pi_{M,\infty}(\bar{\alpha}).$$

This statement (about $M$, $\tau$, and $\bar{\alpha}$) is true in $L(\mathbb{R})$; hence true in the symmetric collapse of $M$; hence true in the symmetric collapse of every iterate $P$ of $M$, about $P$, $\pi_{M,P}(\tau)$, and $\pi_{M,P}(\bar{\alpha})$; hence true in $L(\mathbb{R})$ about $P$, $\pi_{M,P}(\tau)$, and $\pi_{M,P}(\bar{\alpha})$.

So

$$f(a(P)) = f(\pi_{P,\infty''}\pi_{M,P}(\tau)) = \pi_{P,\infty}(\pi_{M,P}(\bar{\alpha})) = \pi_{M,\infty}(\bar{\alpha}) = \alpha$$

for every iterate $P$ of $M$.

In other words, $f(X) = \alpha$ for all $X \in C_M$. 
An ultrafilter on \([\mathcal{P}_{\omega_1}(\aleph_\omega)]^{<\omega_1}\):

Let \(M\) be an iterable fine-structural model over a real \(u\).

Say that \(\tau \in M\) is **good** if \(M \models \tau \models \text{“}\tau\text{ is Woodin.”}\)

Suppose \(M\) has a measurable limit of good cardinals, and let \(\kappa = \kappa(M)\) be the least such.

Suppose \(\kappa\) is countable in \(V\).

Let \(\langle \tau_\xi \mid \xi < \gamma \rangle\) list the good cardinals of \(M\) below \(\kappa\), in increasing order.

For each \(\alpha < \gamma\) let \(g_\alpha\) be generic over \(M\) for collapsing \(\sup\{\tau_\xi \mid \xi < \alpha\}\). Let \(M_\alpha\) denote \(M[g_\alpha]\).

\(\tau_\alpha\) is the first good cardinal of \(M_\alpha = M[g_\alpha]\).

Set \(a_\alpha = \pi_{M_\alpha, \infty}'' \tau_\alpha\), and \(a(M) = \langle a_\alpha \mid \alpha < \gamma \rangle\).

Then each \(a_\alpha\) belongs to \(\mathcal{P}_{\omega_1}(\aleph_\omega)\), and \(a(M)\) belongs to \([\mathcal{P}_{\omega_1}(\aleph_\omega)]^{<\omega_1}\).
Set $C_M = \{a(P) \mid P \text{ is an iterate of } M\}$.

Our earlier proofs all carry over to the current settings.

The sets $C_M$ generate an ultrafilter on $[\mathcal{P}_{\omega_1}(\aleph_\omega)]^{<\omega_1}$, call it $\mathcal{F}$.

The ultrafilter concentrates on long sequences.

The projection of $\mathcal{F}$ to $[\mathcal{P}_{\omega_1}(\aleph_\omega)]^1$ is precisely our earlier filter, namely the supercompactness measure, on $\mathcal{P}_{\omega_1}(\aleph_\omega)$.

The projection of $\mathcal{F}$ to $[\mathcal{P}_{\omega_1}(\aleph_\omega)]^\alpha$ is the $\alpha$-length iteration of the supercompactness measure.
The proof of boundedness for the filter on \([\omega_1]<\omega_1\) also carries over to current settings.

Recall that in that proof we defined \(E\) to be the set of reals \(x\) so that:

\[
(\exists \text{ an iterate } P \text{ of } M)(\psi[a(P), x, u] \text{ holds in a symmetric collapse of } P).
\]

\(E\) was \(\Sigma^1_2\), and this allowed proving boundedness for functions into \(\omega_2 = \delta^1_2\).

In the current settings being a (correct) iterate is \(\Pi^1_2\). \(E\) is therefore \(\Sigma^1_3\), and the proof of boundedness works for \(\delta^1_3 = \aleph_{\omega+1}\). We get:

**Claim:** Let \(g: [\mathcal{P}_{\omega_1}(\aleph_\omega)]^{<\omega_1} \to \aleph_{\omega+1}\). Then there is a set \(X \in F\) so that \(g|X\) is bounded below \(\aleph_{\omega+1}\).

For the s.c. measure on \(\mathcal{P}_{\omega_1}(\aleph_\omega)\) (as opposed to the iterated measure on \([\mathcal{P}_{\omega_1}(\aleph_1)]^{<\omega_1}\)) boundedness is due to Becker (1979) by classical methods.
An application to forcing over $L(\mathbb{R})$:

Recall: can use $\mathcal{F}$ to define a forcing notion.

Conditions are pairs $(t, Y)$ where: $t$ belongs to $[\mathcal{P}_{\omega_1}(\mathbb{N}_\omega)]^{<\omega_1}$; $Y$ is a set of extensions of $t$; and \{s \mid t \upharpoonright s \in Y\} is nice.

$(X \subset [\mathcal{P}_{\omega_1}(\mathbb{N}_\omega)]^{<\omega_1}$ is nice if: $X \in \mathcal{F}$; $X$ is ctbly closed; and \{r \mid s \upharpoonright r \in X\} \in \mathcal{F}$ for each $s \in X$.)

The order on conditions is defined in the natural way: $(t', Y') < (t, Y)$ if $t'$ extends $t$, $Y' \subset Y$, and $t' \in Y$.

Let $\mathbb{A}$ be this poset. Let $H$ be $\mathbb{A}$–generic over $L(\mathbb{R})$.

$\mathbb{A}$ is countably closed. So it does not add reals. It follows $\omega_1$ is not collapsed by $\mathbb{A}$, and that $\delta^1_3$ is not changed.
\( H \) introduces a sequence \( \langle a_\xi \mid \xi < \omega_1 \rangle \), with each \( a_\xi \) a countable subset of \( \kappa_\omega \).

The genericity of \( H \) implies that \( \bigcup_{\xi < \omega_1} a_\xi = \kappa_\omega \). Thus, \( H \) collapses \( \kappa_\omega \) to \( \omega_1 \).

**Boundedness** implies that \( \kappa_{\omega + 1} \) is not collapsed.

So \( \kappa_{\omega + 1} \) becomes \( \omega_2 \) in the generic extension.

\[
\begin{array}{c|c|c|c|c}
\kappa_{\omega + 1} & \Delta_3^1 & \omega_2 & \Delta_3^1 \\
\kappa_\omega & & & \\
\vdots & & & \\
\omega_3 & \Delta_2^1 & \omega_1 & \Delta_2^1 \\
\omega_2 & & & \\
\omega_1 & & & \\
 L(\mathbb{R}) & & & L(\mathbb{R})[H] \\
\end{array}
\]

Since \( \Delta_3^1 \) does not change, we have:

\( L(\mathbb{R})[H] \models "\Delta_3^1 = \omega_2." \)
Steel–VanWesep–Woodin (≈1980) show how to force over $L(\mathbb{R})$ and introduce the axiom of choice without collapsing $\omega_2$. Their methods adapt to forcing over $L(\mathbb{R})[H]$, giving:

**Theorem** (N., Woodin independently): It is consistent with ZFC (and $AD^L(\mathbb{R})$) that $\delta^1_3 = \omega_2$.

Same argument works for higher levels.

Can get the s.c. measure on $\mathcal{P}_{\omega_1}(\lambda)$ for any $\lambda \leq \delta^1_1$.

Can collapse $\alpha < \delta^1_n$ to $\omega_1$ without collapsing $\delta^1_n$. Get the consistency of ZFC + $AD^L(\mathbb{R}) + \delta^1_n = \omega_2$. 

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With a modification, can recover results by Becker–Jackson on the supercompactness of the $\delta_n^1$s.

For example, to get the supercompactness measure on $\mathcal{P}_{\omega_2}(\aleph_\omega)$:

Let $M$ be a model with a cardinal $\tau$ so that $L(M\|\tau) \models \"\tau$ is Woodin.\"" Define

$$a(M) = \bigcup_{Q \text{ an iterate of } M \text{ via a tree in } L(M\|\tau) \text{ (except for final branch)}} \pi_{Q,\infty}''\pi_{M,Q}(\tau).$$

Then define $C_M$ as before.

$a(M)$ here has size $\aleph_1$. Get an ultrafilter on $\mathcal{P}_{\omega_2}(\aleph_\omega)$. 