Inner models and ultrafilters in $\mathsf{L}(\mathbb{R})$

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Part 2:

1. Review.
2. Iteration trees and directed systems.
3. Supercompactness measure on $\mathcal{P}_{\omega_1}(\aleph_\omega)$.
4. Ultrafilter on $[\mathcal{P}(\omega_1)]^{<\omega_1}$.
5. Forcing over $\mathsf{L}(\mathbb{R})$ to collapse $\aleph_\omega$ to ω_1.

Large cardinal assumption:

For each $u \in \mathbb{R}$ there is a class model M s.th.
(1) $u \in M$;
(2) M has ω Woodin cardinals, say with sup δ;
(3) $\mathcal{P}(\delta)^M$ is countable in V; and
(4) M is iterable.

Any statement (with real parameters) $(*)$
forced to hold in the symmetric collapse
of M, holds in the true L($\mathbb{R}$).

Ultrafilter on ω_1:

$a(M) =$ first measurable of M.

$C_M = \{a(P) \mid P$ a linear iterate of $M\}$.

Used $(*)$ to show that for every $X \subset \omega_1$ in L($\mathbb{R}$), have M so that either $C_M \subset X$ or $C_M \subset \neg X$.

Ultrafilter on $[\omega_1]^{<\omega_1}$:

κ the first measurable limit of measurables in M.

$\langle \tau_\xi \mid \xi < \gamma \rangle$ lists the measurables of M below κ in increasing order.

$a(M) = \langle \tau_\xi \mid \xi < \gamma \rangle$.

C_M as before.

The sets C_M generate an ultrafilter. Used $(*)$ to get bddness. Used bddness in forcing.

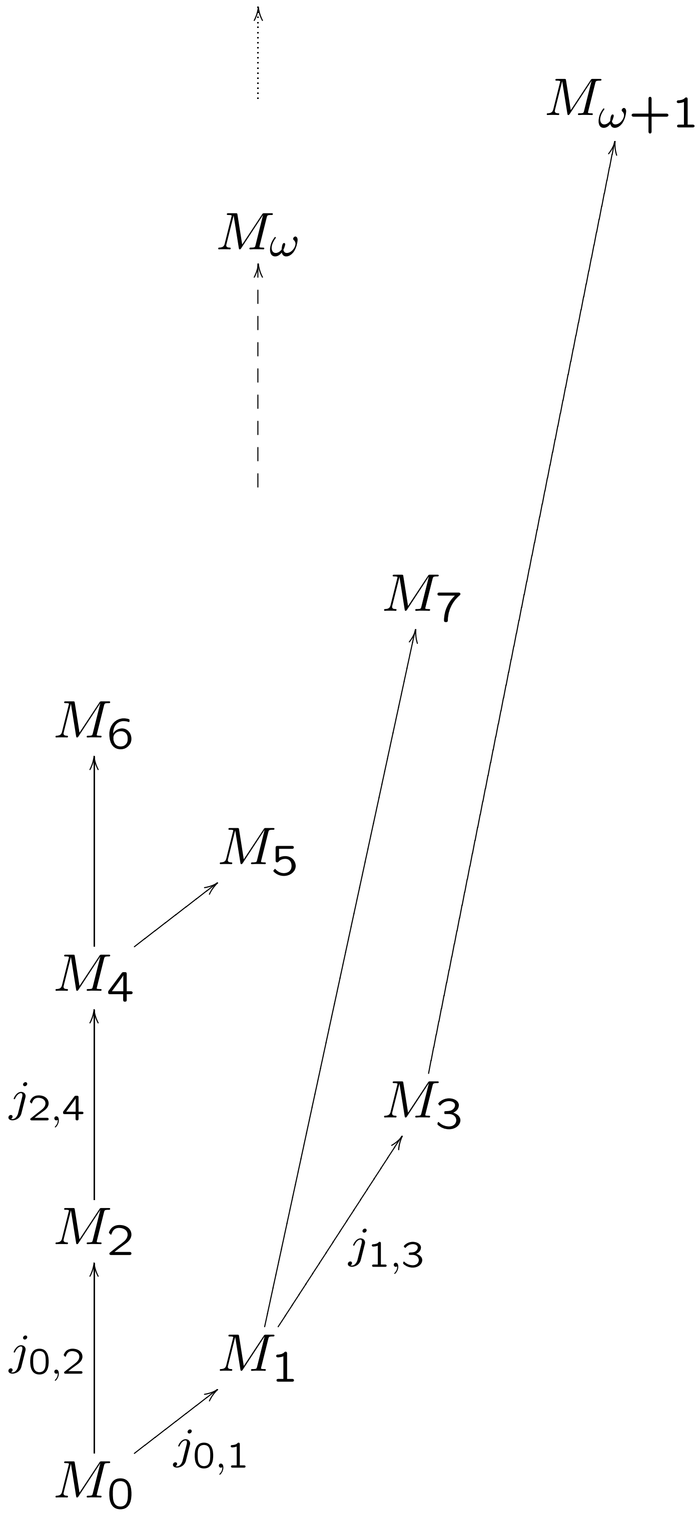
Next, aim to do the same with $\aleph_\omega$ instead of ω_1.

In general iterations may be *non*-linear.

Non-linear iterations are called **iteration trees**.

Iteration trees involve some choices at limit stages. M is **iterable** if these choices can be made in a way which secures wellfoundedness.

A **correct** iteration tree on an iterable M is one which follows the limit choices needed to secure wellfoundedness.



Already at the level of linear iterations there is an implicit notion of correctness: A linear iteration of "length α" is "correct" if α is well-founded.

This "correctness" for linear iterations is $\mathbf{\Pi}^1_1$. In the claim of boundedness last time it was the contribution of correctness to the complexity of the set

$$(\exists \text{ an iterate } P \text{ of } M)(\psi[a(P), x, u] \text{ holds in a symmetric collapse of } P).$$

that made it $\mathbf{\Sigma}^1_2$.

For iteration trees the complexity of correctness is higher. How high depends on the large cardinals involved in the iteration.

Let M be iterable and let $\tau \in M$ be least such that $\mathsf{L}(M \| \tau) \models$ "τ is Woodin."

For iteration trees on M using extenders from below τ, correctness is roughly $\mathbf{\Pi}^1_2$.

Let M now be *fine-structural* over a real u.

Any two correct iterates of M can be *compared*.

In other words, for any two correct iterations $j_1\colon M \to P_1$ and $j_2\colon M \to P_2$ there are further iterations $h_1\colon P_1 \to Q_1$ and $h_2\colon P_2 \to Q_2$ so that $Q_1 = Q_2$.

$$\begin{tikzcd}
 & Q_1 = Q_2 & \\
P_1 \arrow[ur, "h_1"] & & P_2 \arrow[ul, "h_2"'] \\
 & M \arrow[ul, "j_1"] \arrow[ur, "j_2"'] &
\end{tikzcd}$$

Further, the embeddings given by correct iterations are unique, by the Dodd–Jensen lemma. $k_1\colon M \to Q$ and $k_2\colon M \to Q$ both iteration embeddings, then $k_1 = k_2$.

Use $\pi_{M,Q}$ for the iteration emb from M to Q.

In the situation of the comparison above the uniqueness implies that $h_1 \circ j_1$ equals $h_2 \circ j_2$.

By iteration from now on we mean a correct iteration of countable length.

Comparisons allow considering the directed system of all iterates of M.

Let $\mathcal{D}$ be the set of pairs $\langle P, x\rangle$ so that P is a an iterate of M, and x belongs to P.

For $\langle P, x\rangle$ and $\langle P', x'\rangle$ both in $\mathcal{D}$ set $\langle P, x\rangle \sim \langle P', x'\rangle$ iff in the comparison of P and P' get $h(x) = h'(x')$.

$\sim$ is an equivalence relation on $\mathcal{D}$.

Define further $\langle P, x\rangle \in^* \langle P', x'\rangle$ iff in the comparison of P and P' get $h(x) \in h(x')$.

$\in^*$ induces a wellfounded relation on $\mathcal{D}/\!\sim$. Set $M_\infty =$ transitive collapse of $(\mathcal{D}/\!\sim\,;\ \in^*)$.

M_∞ is the direct limit of all (countable) iterates of M. Have $\pi_{M,\infty}$ from M into M_∞ defined by $\pi_{M,\infty}(x) =$ equivalence class of $\langle M, x\rangle$.

We are working in $\mathsf{L}(\mathbb{R})$. ω_2 is equal to $\boldsymbol{\delta}^1_2$. ω_3, ω_4, etc. are all singular cardinals of cofinality ω_2. $\aleph_\omega$ is the size of a homogeneous tree for $\boldsymbol{\Pi}^1_2$ sets. $\aleph_{\omega+1}$ is equal to $\boldsymbol{\delta}^1_3$.

Suppose M is iterable, $\tau = \tau(M)$ is least such that $\mathsf{L}(M\| \tau) \models$ "τ is Woodin," and τ is countable in V.

Theorem (Woodin): $\pi_{M,\infty}(\tau)$ is equal to $\aleph_\omega$.

This is connected to the fact that correctness for trees below τ is roughly $\boldsymbol{\Pi}^1_2$.

Recall our scheme for getting ultrafilters:

Define $a(M)$ somehow.

Set $C_M = \{a(P) \mid P \text{ is an iterate of } M\}$.

Use the C_Ms to generate an ultrafilter.

Here we want an ultrafilter on $\mathcal{P}_{\omega_1}(\aleph_\omega)$. So we need $a(M) \in \mathcal{P}_{\omega_1}(\aleph_\omega)$.

Natural attempt: set $a(M) = \pi_{M,\infty}{}''\tau(M)$.

Then $a(M) \in \mathcal{P}_{\omega_1}(\aleph_\omega)$ and $C_M \subset \mathcal{P}_{\omega_1}(\aleph_\omega)$.

As before the sets C_M generate an ultrafilter.* It's the supercompactness measure on $\mathcal{P}_{\omega_1}(\aleph_\omega)$.

*The finite intersection property takes more work here than in the case of ω_1. More on this in the next talk.

Proof of normality:

Fix $f \in \mathsf{L}(\mathbb{R})$ on $\mathcal{P}_{\omega_1}(\aleph_\omega)$ such that $f(X) \in X$ for all X.

Wlog, f is definable from a real u. Fix φ so that $f(X) = \alpha$ iff $\mathsf{L}(\mathbb{R}) \models \varphi[u, X, \alpha]$.

Take M satisfying L.C. assumption with $u \in M$. Let τ be least so that $\mathsf{L}(M\|\,\tau) \models$ "τ is Woodin."

Look at $\alpha = f(a(M))$.

α belongs to $a(M) = \pi_{M,\infty}{}''\tau$.

Have $\bar{\alpha} < \tau$ in M so that

$$f(\pi_{M,\infty}{}''(\tau)) = \pi_{M,\infty}(\bar{\alpha}).$$

This statement (about M, τ, and $\bar{\alpha}$) is true in $\mathsf{L}(\mathbb{R})$; hence true in the symmetric collapse of M; hence true in the symmetric collapse of every iterate P of M, about P, $\pi_{M,P}(\tau)$, and $\pi_{M,P}(\bar{\alpha})$; hence true in $\mathsf{L}(\mathbb{R})$ about P, $\pi_{M,P}(\tau)$, and $\pi_{M,P}(\bar{\alpha})$.

So

$$\begin{aligned} f(a(P)) &= f(\pi_{P,\infty}{}''\pi_{M,P}(\tau)) \\ &=^* \pi_{P,\infty}(\pi_{M,P}(\bar{\alpha})) \\ &= \pi_{M,\infty}(\bar{\alpha}) \\ &= \alpha \end{aligned}$$

for every iterate P of M.

In other words, $f(X) = \alpha$ for all $X \in C_M$.

An ultrafilter on $[\mathcal{P}_{\omega_1}(\aleph_\omega)]^{<\omega_1}$:

Let M be an iterable fine-structural model over a real u.

Say that $\tau \in M$ is **good** if $M \| \tau \models$ "τ is Woodin."

Suppose M has a measurable limit of good cardinals, and let $\kappa = \kappa(M)$ be the least such.

Suppose κ is countable in V.

Let $\langle \tau_\xi \mid \xi < \gamma \rangle$ list the good cardinals of M below κ, in increasing order.

For each $\alpha < \gamma$ let g_α be generic over M for collapsing $\sup\{\tau_\xi \mid \xi < \alpha\}$. Let M_α denote $M[g_\alpha]$.

τ_α is the first good cardinal of $M_\alpha = M[g_\alpha]$.

Set $a_\alpha = \pi_{M_\alpha,\infty}{}''\tau_\alpha$, and $a(M) = \langle a_\alpha \mid \alpha < \gamma \rangle$.

Then each a_α belongs to $\mathcal{P}_{\omega_1}(\aleph_\omega)$, and $a(M)$ belongs to $[\mathcal{P}_{\omega_1}(\aleph_\omega)]^{<\omega_1}$.

Set $C_M = \{a(P) \mid P \text{ is an iterate of } M\}$.

Our earlier proofs all carry over to the current settings.

The sets C_M generate an ultrafilter on $[\mathcal{P}_{\omega_1}(\aleph_\omega)]^{<\omega_1}$, call it $\mathcal{F}$.

The ultrafilter concentrates on long sequences.

The projection of $\mathcal{F}$ to $[\mathcal{P}_{\omega_1}(\aleph_\omega)]^1$ is precisely our earlier filter, namely the supercompactness measure, on $\mathcal{P}_{\omega_1}(\aleph_\omega)$

The projection of $\mathcal{F}$ to $[\mathcal{P}_{\omega_1}(\aleph_\omega)]^\alpha$ is the α-length iteration of the supercompactness measure.

The proof of boundedness for the filter on $[\omega_1]^{<\omega_1}$ also carries over to current settings.

Recall that in that proof we defined E to be the set of reals x so that:

$$(\exists \text{ an iterate } P \text{ of } M)(\psi[a(P), x, u] \text{ holds in a symmetric collapse of } P).$$

E was $\boldsymbol{\Sigma}^1_2$, and this allowed proving boundedness for functions into $\omega_2 = \boldsymbol{\delta}^1_2$.

In the current settings being a (correct) iterate is $\boldsymbol{\Pi}^1_2$. E is therefore $\boldsymbol{\Sigma}^1_3$, and the proof of boundedness works for $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$. We get:

Claim: Let $g\colon [\mathcal{P}_{\omega_1}(\aleph_\omega)]^{<\omega_1} \to \aleph_{\omega+1}$. Then there is a set $X \in \mathcal{F}$ so that $g \restriction X$ is *bounded* below $\aleph_{\omega+1}$.

For the s.c. measure on $\mathcal{P}_{\omega_1}(\aleph_\omega)$ (as opposed to the iterated measure on $[\mathcal{P}_{\omega_1}(\aleph_1)]^{<\omega_1}$) boundedness is due to Becker (1979) by classical methods.

An application to forcing over L($\mathbb{R}$):

Recall: can use $\mathcal{F}$ to define a forcing notion.

Conditions are pairs (t, Y) where: t belongs to $[\mathcal{P}_{\omega_1}(\aleph_\omega)]^{<\omega_1}$; Y is a set of extensions of t; and $\{s \mid t^\frown s \in Y\}$ is nice.

($X \subset [\mathcal{P}_{\omega_1}(\aleph_\omega)]^{<\omega_1}$ is **nice** if: $X \in \mathcal{F}$; X is ctbly closed; and $\{r \mid s^\frown r \in X\} \in \mathcal{F}$ for each $s \in X$.)

The order on conditions is defined in the natural way: $(t', Y') < (t, Y)$ if t' extends t, $Y' \subset Y$, *and* $t' \in Y$.

Let $\mathbb{A}$ be this poset. Let H be $\mathbb{A}$–generic over L($\mathbb{R}$).

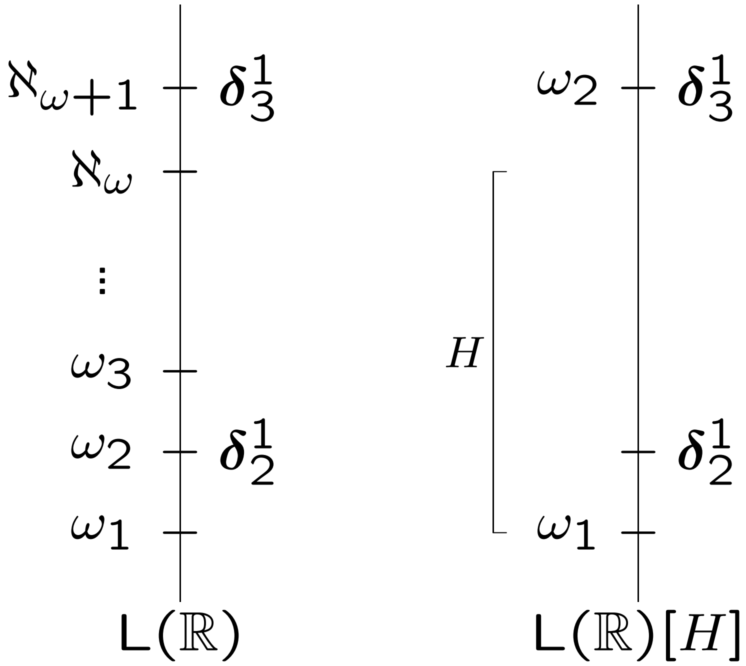
$\mathbb{A}$ is countably closed. So it does not add reals. It follows ω_1 is not collapsed by $\mathbb{A}$, and that $\boldsymbol{\delta}^1_3$ is not changed.

H introduces a sequence $\langle a_\xi \mid \xi < \omega_1 \rangle$, with each a_ξ a countable subset of $\aleph_\omega$.

The genericity of H implies that $\bigcup_{\xi<\omega_1} a_\xi = \aleph_\omega$. Thus, H collapses $\aleph_\omega$ to ω_1.

Boundedness implies that $\aleph_{\omega+1}$ is not collapsed.

So $\aleph_{\omega+1}$ becomes ω_2 in the generic extension.



Since $\boldsymbol{\delta}^1_3$ does not change, we have:

$$\mathsf{L}(\mathbb{R})[H] \models \text{“}\boldsymbol{\delta}^1_3 = \omega_2.\text{”}$$

Steel–VanWesep–Woodin ($\approx$1980) show how to force over $\mathsf{L}(\mathbb{R})$ and introduce the axiom of choice without collapsing ω_2. Their methods adapt to forcing over $\mathsf{L}(\mathbb{R})[H]$, giving:

Theorem (N., Woodin independently): It is consistent with ZFC (and $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$) that $\boldsymbol{\delta}^1_3 = \omega_2$.

Same argument works for higher levels.

Can get the s.c. measure on $\mathcal{P}_{\omega_1}(\lambda)$ for any $\lambda \leq \boldsymbol{\delta}^2_1$.

Can collapse $\alpha < \boldsymbol{\delta}^1_n$ to ω_1 without collapsing $\boldsymbol{\delta}^1_n$. Get the consistency of ZFC + $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$ + $\boldsymbol{\delta}^1_n = \omega_2$.

With a modification, can recover results by Becker–Jackson on the supercompactness of the $\boldsymbol{\delta}^1_n$s.

For example, to get the supercompactness measure on $\mathcal{P}_{\omega_2}(\aleph_\omega)$:

Let M be a model with a cardinal τ so that $\mathsf{L}(M\|\,\tau) \models$ "τ is Woodin." Define

$$a(M) = \bigcup_{\substack{Q \text{ an iterate of } M \text{ via a} \\ \text{tree in } \mathsf{L}(M\|\,\tau) \text{ (except} \\ \text{for final branch).}}} \pi_{Q,\infty}{}''\pi_{M,Q}(\tau).$$

Then define C_M as before.

$a(M)$ here has size $\aleph_1$. Get an ultrafilter on $\mathcal{P}_{\omega_2}(\aleph_\omega)$.